Measurable selectors and set-valued Pettis integral in non-separable Banach spaces

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Univ. of Murcia – Kharkov National Univ. – Polytechnical Univ. of Valencia

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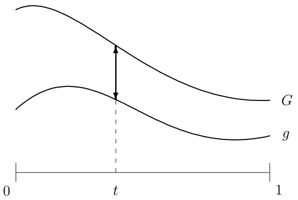
$$F(t) = [g(t), G(t)]$$

for some real-valued functions $g \leq G$.



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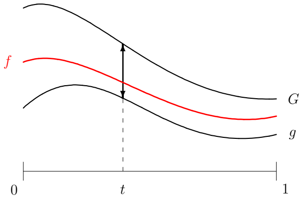

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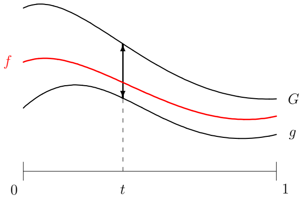

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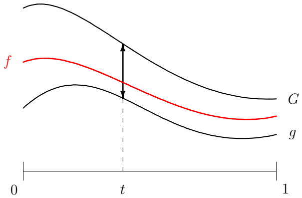

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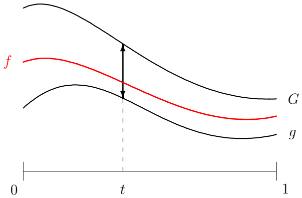

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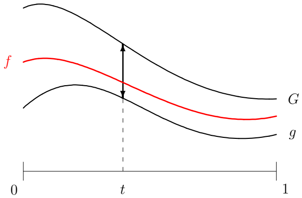

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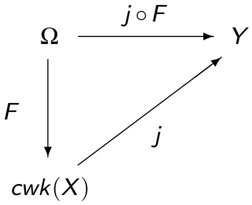

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- Corson's property (C) $\Rightarrow$ μ-PIP (for any μ) (Talagrand, 1984).

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Moreover, j is an **isometry** into $\ell_\infty(B_{X^*})$
when $cwk(X)$ is equipped with the Hausdorff distance.

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Studied by: **Castaing-Valadier** (1977), **Di Piazza-Musial** (2005-06), **El Amri-Hess** (2000), **Ziat** (1997-2000), etc.

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Then F admits **Borel measurable selectors**.

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Suppose X is separable.
Then every scalarly measurable multi-function $F:\Omega \to cwk(X)$ admits **strongly measurable selectors**.

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In this case:

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for all $A \in \Sigma$.

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- The formula

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Set-valued Pettis integral and selectors

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Remark

The “closure” can be removed if X^* is w^*-separable.

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▶ We **do not know** an example of a scalarly measurable multi-function without scalarly measurable selectors !!

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$$\Downarrow$$
$$F \text{ admits } \textit{scalarly measurable} \text{ selectors.}$$

THANKS FOR YOUR ATTENTION !!

http://personales.upv.es/jorodrui