On vector measures with separable range

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neither relatively **norm** compact nor **separable**, even for indefinite Pettis integrals (Fremlin-Talagrand, 1979).

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the mapping $\nu_f : \Sigma \to Y$ (the **indefinite integral** of f) is a vector measure (Pettis, 1938) with σ-finite variation (Rybakov, 1968).

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Question (Musial, 1991)

Which Banach spaces have the Pettis Separability Property?

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- *Under CH*, there is a weakly Lindelöf determined X such that (B_{X^*}, w^*) fails property (M) (Kalenda-Plebanek, 2002).

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Idea used in the proof of (i)⇒(ii):

- For each Radon probability μ on (B_{X^*}, w^*), the indefinite Gelfand integral of the "identity" $I : B_{X^*} \to X^*$ has norm separable range.

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Suppose the following statement holds:

> *Every X^*-valued vector measure with σ-finite variation has norm separable range.*

Then $X \not\supset \ell^1(\omega_1)$.

http://www.um.es/docencia/joserr/