Absolutely summing operators and integration of vector-valued functions

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Question

Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a Dunford integrable function.

Is $u \circ f$ Bochner integrable?

The answer is **"yes"** in either of the following cases:

- f is strongly measurable and Pettis integrable. (Diestel, 1972)
- (Ω, Σ, μ) is a compact Radon probability space and f is McShane integrable. (Marraffa, 2004)

Question

Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a Dunford integrable function.

Is $u \circ f$ Bochner integrable?

- If μ is perfect and f is bounded and Pettis integrable, then $u \circ f$ is **scalarly equivalent** to a Bochner integrable function. (Belanger-Dowling, 1988)
- Heiliö (1988) studied the problem when μ is a Baire measure on (X, w) and f is the identity on X.

Lemma

Let $u : X \longrightarrow Y$ be an absolutely summing operator, $f : \Omega \longrightarrow X$ a Dunford integrable function and $g : \Omega \longrightarrow Y$ a function that is scalarly equivalent to $u \circ f$. Then

$$g \text{ is strongly measurable } \iff g \text{ is Bochner integrable.}$$

In particular,

$$u \circ f \text{ is strongly measurable } \iff u \circ f \text{ is Bochner integrable.}$$

Theorem

Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a Dunford integrable function. Then $u \circ f$ is scalarly equivalent to some Bochner integrable function $u_f : \Omega \longrightarrow Y$.

Proposition

Let $u : X \longrightarrow Y$ be an absolutely summing operator. Then the linear map

$$\tilde{u} : (D(\mu, X), \| \ \|_{Pe}) \longrightarrow (L^1(\mu, Y), \| \ \|_1), \quad \tilde{u}(f) := u_f,$$

is continuous.

Application of Pietsch's factorization theorem

For a compact space K TFAE:

(1) each absolutely summing operator defined on $C(K)$ with values in another Banach space has separable range;

(2) $L^1(\nu)$ is separable for each Radon probability ν on K.

Some examples of compacta with the properties above:

- Gul'ko compacta (e.g. Eberlein compacta).
- Rosenthal compacta.
- Linearly ordered compacta.
- Radon-Nikodým compacta (e.g. scattered compacta).
- (Under MA + ¬CH.) Compacta with countable tightness.

Definition

A Banach space X belongs to the class MS iff X is isomorphic to a subspace of $C(K)$ for some compact space K such that $L^1(\nu)$ is separable for each Radon probability ν on K.

Some examples of Banach spaces in MS:

- Weakly countably $\mathscr{K}$-determined (e.g. WCG) spaces.
- Asplund generated (e.g. Asplund) spaces and their subspaces.
- (Under MA + $\neg$CH.) Banach spaces X such that (B_{X^*}, w^*) has countable tightness.

Theorem

If X is a Banach space that belongs to the class MS, then each absolutely summing operator defined on X with values in another Banach space has separable range.

Definition (Fremlin-Talagrand)

A family $\mathscr{H} \subset \mathbb{R}^{\Omega}$ is **stable** iff for each $A \in \Sigma$ with $\mu(A) > 0$ and each $\alpha < \beta$ in $\mathbb{R}$ there exist $k, l \in \mathbb{N}$ such that

$$\mu^*_{k+l}\Big(\bigcup_{h \in \mathscr{H}} (\{h < \alpha\}^k \times \{h > \beta\}^l) \cap A^{k+l} \Big) < \mu(A)^{k+l},$$

where μ_{k+l} is the product of $k+l$ copies of μ.

Definition (Fremlin-Talagrand)

A function $f : \Omega \longrightarrow X$ is **properly measurable** iff the family $\{x^* \circ f : \; x^* \in B_{X^*}\}$ is stable.

Theorem

Let $f : \Omega \longrightarrow X$ be a Pettis integrable function. Let us consider the following statements:

(1) there is a w^*-compact norming set $K \subset B_{X^*}$ such that for each Radon probability ν on K the family $\{x^* \circ f : \ x^* \in \mathrm{supp}(\nu)\}$ is stable;

(2) for each absolutely summing operator u defined on X with values in another Banach space the composition $u \circ f$ is Bochner integrable.

Then (1)⇒(2). Moreover, under CH, if μ is perfect then (1) and (2) are equivalent.

Corollary

Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a properly measurable function. Then $u \circ f$ is strongly measurable. If, in addition, f is Dunford integrable, then $u \circ f$ is Bochner integrable.

Examples of properly measurable Dunford integrable functions:

- Talagrand integrable functions.
- Birkhoff integrable functions.
- (Under CH.) Dunford integrable functions defined on a perfect probability space with values in a Banach space with w^*-separable dual unit ball (i.e. a space that is isometric to a subspace of ℓ_∞).

Proposition

Let X be a Banach space that is isomorphic to a subspace of a weakly Lindelöf space of the form $C(K)$. Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a Dunford integrable function. Then $u \circ f$ is Bochner integrable.

Question

Let $u : X \longrightarrow Y$ be an absolutely summing operator and $f : \Omega \longrightarrow X$ a Dunford integrable function.

Is $u \circ f$ Bochner integrable?

- The answer is **"no"** in general.
- The answer is **"yes"** if (Ω, Σ, μ) is a quasi-Radon probability space and f is McShane integrable.

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