Limits of Birkhoff integrable vector-valued functions

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Summary of the talk

Integration of functions $f : \Omega \to X$ where

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✓ X separable $\Longrightarrow$ Birkhoff $\equiv$ Pettis.

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- We consider on X either the **norm** or the **weak** topology.

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- $\varphi, \psi : \mathfrak{c} \to \mathfrak{c}$ one-to-one mappings such that $\varphi(\mathfrak{c}) \cap \psi(\mathfrak{c}) = \emptyset$.
  Set $f(t) := e_{\varphi(\alpha)}$ if $t \in A_\alpha$,
  set $f(t) := e_{\psi(\alpha)}$ if $t \in B_\alpha$, and $f(t) := 0$ otherwise.

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- Write $A_\alpha = \{a_{\alpha 1}, a_{\alpha 2}, \dots\}$ and $B_\alpha = \{b_{\alpha 1}, b_{\alpha 2}, \dots\}$.
  For $n \in \mathbb{N}$, set $f_n(t) := f(t)$ if $t \in \{a_{\alpha 1}, \dots, a_{\alpha n}\} \cup \{b_{\alpha 1}, \dots, b_{\alpha n}\}$, and $f_n(t) := 0$ otherwise.

More Counterexamples

Theorem

Suppose X admits a **uniformly convex** equivalent norm and has density character $\geq \mathfrak{c}$ (for instance, $X = \ell^p(\mathfrak{c})$, $1 < p < \infty$).

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Then there is a uniformly bounded sequence of Birkhoff integrable functions $f_n : [0,1] \to X$ converging pointwise to a function $f : [0,1] \to X$ which is **not** Birkhoff integrable.

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Let $f_n : \Omega \to X$ be a sequence of Birkhoff integrable functions and $f : \Omega \to X$ be a function such that:

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Then f is Birkhoff integrable and

$$\int f_n \, d\mu \to \int f \, d\mu \text{ weakly (resp. in norm).}$$

Equi-Birkhoff Integrability

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A sequence (f_n) of Birkhoff integrable functions is **equi-Birkhoff integrable** iff for every $\varepsilon > 0$ there is a countable partition (A_i) of Ω in Σ such that, for any choice of points $t_i \in A_i$, one has

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Let $f_n : \Omega \to X$ be a sequence of functions converging pointwise *in norm* to $f : \Omega \to X$. Then (f_n) is equi-Birkhoff integrable if and only if the function $F : \Omega \to X_c$ given by $F(t) := (f_n(t))$ is Birkhoff integrable.

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- In this case, f is Birkhoff integrable and $\int f_n \, d\mu \to \int f \, d\mu$ *in norm*.

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- equi-Birkhoff integrability $\implies$ uniform integrability
- <u>But</u>

$$\text{equi-Birkhoff int.} \equiv \text{uniform int. } (\forall \text{ sequences})$$
$$\Downarrow$$
X has the **Schur property**

Convergence Theorems for the Pettis Integral

Weak Convergence Theorem (Musiał)

Let $f_n : \Omega \to X$ be a sequence of **Pettis integrable** functions and $f : \Omega \to X$ a function such that:

- $\forall\, x^* \in X^*$, $\boxed{x^* \circ f_n \to x^* \circ f \;\; \mu\text{-a.e.}}$
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(i) $\{x^* \circ f_n\}_{x^* \in B_{X^*},\, n \in \mathbb{N}}$ is uniformly integrable.

(ii) f is Pettis integrable and $\int_A f_n \, d\mu \to \int_A f \, d\mu$ *in norm* $\forall\ A \in \Sigma$.

THANKS FOR YOUR ATTENTION !!

http://personales.upv.es/jorodrui

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- J. Rodríguez, *Pointwise limits of Birkhoff integrable functions*, Proc. AMS, to appear.