On the structure of L^1 of a vector measure via its integration operator

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Domination theorems for bilinear maps

- Let (Ω, Σ, μ) and $(\Delta, \mathscr{S}, \nu)$ be finite measure spaces.
- Let $X(\mu)$ and $Y(\nu)$ be Banach function spaces.
- Let Z be a Banach space.
- Consider a bilinear map $B : X(\mu) \times Y(\nu) \to Z$.

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We provide a general scheme
of what a domination theorem for B is.

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Definition

A **Banach function space** is a Banach space $X(\mu) \subset L^0(\mu)$ with norm $\|\cdot\|_{X(\mu)}$ satisfying that:

1. If $f \in L^0(\mu)$ and $g \in X(\mu)$ with $|f| \leq |g|$ μ-a.e. then $f \in X(\mu)$ and $\|f\|_{X(\mu)} \leq \|g\|_{X(\mu)}$.
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▶ A Banach function space is called **order continuous** if order bounded increasing sequences are convergent in norm.

p-convexity and p-concavity

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A Banach lattice E is p-**convex** if there is a constant $K > 0$ such that for each $x_1, \ldots, x_n \in E$,

$$\|(\sum_{i=1}^{n} |x_i|^p)^{1/p}\| \leq K(\sum_{i=1}^{n} \|x_i\|^p)^{1/p}.$$

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An operator $T : E \to F$ (where E and F are Banach lattices) is p-**concave** if there is a constant K such that for each $x_1, \ldots, x_n \in E$,

$$(\sum_{i=1}^{n} \|T(x_i)\|^p)^{1/p} \leq K\|(\sum_{i=1}^{n} |x_i|^p)^{1/p}\|.$$

The best constant in these inequalities is denoted by $M_{(p)}(T)$.

Homogeneicity

- We call a set U **homogeneous** whenever it carries a multiplication with positive scalars:

$$U \times [0, \infty) \to U, \quad (x, \lambda) \to \lambda x.$$

- If there is a homogeneous set U, a quasi Köthe function space X and a homogeneous mapping $\varphi : U \to X$, then we say that φ **represents** U **in** X **homogeneously**.

- For two homogeneous sets U_1, U_2 a form $u : U_1 \times U_2 \to \mathbb{K}$ is said to be **homogeneous** if

$$u(\lambda x, y) = u(x, \lambda y) = \lambda u(x, y) \quad \text{for all } \lambda \geq 0.$$

Theorem (Defant)

For $\ell = 1, 2$, let $0 < r_\ell < \infty$ and $1/t = 1/r_1 + 1/r_2$.
Let $u : U_1 \times U_2 \to \mathbb{K}$ a homogeneous form on homogeneous sets such that U_ℓ via φ_ℓ can be represented homogeneously in an quasi-Banach function space $X_\ell(\mu_\ell)$.
If u satisfies

$$\left(\sum_{i=1}^{n} |u(x_i, y_i)|^t\right)^{\frac{1}{t}} \leq K \left\| \left(\sum_{i=1}^{n} |\varphi_1(x_i)|^{r_1}\right)^{\frac{1}{r_1}} \right\|_{X_1} \left\| \left(\sum_{i=1}^{n} |\varphi_2(y_i)|^{r_2}\right)^{\frac{1}{r_2}} \right\|_{X_2}.$$

for all $x_1, \ldots, x_n \in U_1$ and $y_1, \ldots, y_n \in U_2$,
then there are two positive linear functionals $\Phi_\ell : X_{[r_\ell]}(\mu_\ell) \to \mathbb{R}$ such that for all $x \in U_1$, $y \in U_2$,

$$|u(x, y)| \leq \Phi_1(|\varphi_1(x)|^{r_1})^{\frac{1}{r_1}} \Phi_2(|\varphi_2(y)|^{r_2})^{\frac{1}{r_2}}.$$

If $X_\ell(\mu_\ell)$ is order continuous, then Φ_ℓ can be chosen to be a function in the Köthe dual of $(X_\ell)_{[r_\ell]}$.

Examples

- Domination Theorem for p-summing operators and (p, q)-dominated operators.
- Maurey-Rosenthal Theorem for p-convex spaces.

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- μ is a Rybakov control measure of m
- If E and F are Banach function spaces, we write

$$\boxed{E \hookrightarrow F}$$

whenever the 'identity' mapping from E to F is a well-defined one-to-one operator.

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for every $f_1, \ldots, f_n \in L^p(m)$ and $g_1, \ldots, g_n \in L^q(m)$, $n \in \mathbb{N}$.

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(b) There exist a constant $K > 0$ and $u, v \in B^+_{L^1(m)'}$ such that

$$\Big\|\int fg\ dm\Big\| \leq K\Big(\int |f|^p u\, d\mu\Big)^{\frac{1}{p}}\Big(\int |g|^q v\, d\mu\Big)^{\frac{1}{q}}$$

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$$L^r(m) \hookrightarrow L^r(\nu) \hookrightarrow L^1(m).$$

(d) The integration operator $I_m^r : L^r(m) \to X$ is r-concave.

Proof of (b)⇒(c), I

We want to prove . . .

$$\boxed{\|\int fg\ dm\| \leq K\left(\int |f|^p u\,d\mu\right)^{1/p}\left(\int |g|^q v\,d\mu\right)^{1/q} \quad \forall f \in L^p(m),\ \forall g \in L^q(m)\ (*)} \implies \boxed{L^r(m) \hookrightarrow L^r(v) \hookrightarrow L^1(m)}$$

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- *Take* $A \in \Sigma$ *with* $(u\, d\mu)(A) = 0$*, that is,* $\int \chi_A u\, d\mu = 0$*.*

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- *For each* $B \subset A$ *with* $B \in \Sigma$, *we apply* (*) *to* $f = \chi_B$ *and* $g = 1$ *to get* $m(B) = 0$. *Hence* $\|m\|(A) = 0$.

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(2) The bilinear map $\mathscr{P} : L^p(u\, d\mu) \times L^q(v\, d\mu) \to L^1(m)$ given by $(f, g) \rightsquigarrow fg$ is well-defined and continuous. Indeed:

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$\|\int fg\ dm\| \le K(\int |f|^p u\,d\mu)^{1/p}(\int |g|^q v\,d\mu)^{1/q}$ $\forall f \in L^p(m),\ \forall g \in L^q(m)$ (*) $\Longrightarrow$ $L^r(m) \hookrightarrow L^r(\nu) \hookrightarrow L^1(m)$

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So the restriction of $\mathscr{P}$ to simple functions is well-defined and continuous.

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- *Now we apply a "density argument".*

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(5) $\nu := h_0\,d\mu$ is a control measure of m and $L^r(m) \hookrightarrow L^r(h_0\,d\mu)$. $\square$

Our domination theorem (again)

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Let $r \geq 1$ and $p, q > 1$ be such that $1/r = 1/p + 1/q$. TFAE:

(a) There is a constant $K > 0$ such that

$$\left(\sum_{i=1}^{n} \left\| \int f_i g_i \, dm \right\|^r\right)^{\frac{1}{r}} \leq K \left\| \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \right\|_{L^p(m)} \left\| \left(\sum_{i=1}^{n} |g_i|^q\right)^{\frac{1}{q}} \right\|_{L^q(m)}$$

for every $f_1, \ldots, f_n \in L^p(m)$ and $g_1, \ldots, g_n \in L^q(m)$, $n \in \mathbb{N}$.

(b) There exist a constant $K > 0$ and $u, v \in B^+_{L^1(m)'}$ such that

$$\left\| \int fg \, dm \right\| \leq K \left(\int |f|^p u \, d\mu \right)^{\frac{1}{p}} \left(\int |g|^q v \, d\mu \right)^{\frac{1}{q}}$$

for every $f \in L^p(m)$ and $g \in L^q(m)$.

(c) There is a control measure ν of m such that

$$L^r(m) \hookrightarrow L^r(\nu) \hookrightarrow L^1(m).$$

(d) The integration operator $I^r_m : L^r(m) \to X$ is r-concave.

p-concavity and L^1 of a scalar measure

p-concavity and L^1 of a scalar measure

Definition

A B.f.s. E is p-**concave** (resp. p-**convex**), where $1 \leq p < \infty$, if there is a constant $K > 0$ such that

$$\left(\sum_{i=1}^{n} \|z_i\|^p\right)^{\frac{1}{p}} \leq K \left\| \left(\sum_{i=1}^{n} |z_i|^p\right)^{\frac{1}{p}} \right\| \quad \text{(resp. the reverse one)}$$

for every $z_1, \ldots, z_n \in E$, $n \in \mathbb{N}$.

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for every $z_1, \ldots, z_n \in E$, $n \in \mathbb{N}$.

Corollary

TFAE:

(a) $L^p(m)$ is p-concave for some/every $1 \le p < \infty$.

(b) The integration operator $I_m^1 : L^1(m) \to X$ is 1-concave.

(c) $L^1(m)$ is order isomorphic to $L^1(\nu)$ for some control measure ν of m.

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Theorem (Maurey-Rosenthal)

A B.f.s. E is order isomorphic to the L^p space of a non-negative scalar measure (where $1 \leq p < \infty$) if and only if it is p-concave and p-convex.

p-summability and L^1 of a scalar measure

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Definition

An operator T from a B.f.s. E to X is **positive p-summing** (where $1 \leq p < \infty$) if there is a constant $K > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tz_i\|^p\right)^{\frac{1}{p}} \leq K \sup_{z' \in B_{E'}} \left(\sum_{i=1}^{n} |\langle z_i, z' \rangle|^p\right)^{\frac{1}{p}}$$

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(c) E is order isomorphic to the L^1 space of a non-negative scalar measure.

Proof of (a)$\Rightarrow$(c), I

We want to prove . . .

$I_m^1 : L^1(m) \to X$ is **positive** p-**summing** (for some $1 \leq p < \infty$) $\Longrightarrow$ $L^p(m)$ is p-**concave**

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(1) If $p = 1$ then I_m^1 is 1-concave and $L^1(m)$ is 1-concave.

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$I_m^1 : L^1(m) \to X$ is **positive** p**-summing** (for some $1 \le p < \infty$) $\Longrightarrow$ $L^p(m)$ is p-**concave**

(1) If $p = 1$ then I_m^1 is 1-concave and $L^1(m)$ is 1-concave.

(2) Now suppose $p > 1$ and let $1/p + 1/q = 1$. Fix $f_1, \dots, f_n \in L^p(m)$. Then:

$$\Big(\sum_{i=1}^n \Big\| \int f_i g_i \, dm \Big\|^p \Big)^{\frac{1}{p}} \le K \Big\| \Big(\sum_{i=1}^n |f_i|^p \Big)^{\frac{1}{p}} \Big\|_{L^p(m)} \quad \text{for all } g_1, \dots, g_n \in B_{L^q(m)}.$$

Indeed:

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Indeed:

- Since I_m^1 is positive p-summing we have:

$$\Big(\sum_{i=1}^n \Big\|\int f_i g_i \, dm\Big\|^p\Big)^{1/p} \le K \sup_{h \in B_{L^1(m)'}} \Big(\sum_{i=1}^n \Big(\int |f_i g_i||h| \, d\mu\Big)^p\Big)^{1/p}.$$

Proof of (a)⇒(c), I

We want to prove . . .

$I^1_m : L^1(m) \to X$ is **positive** p**-summing** (for some $1 \le p < \infty$) $\Longrightarrow L^p(m)$ is p**-concave**

(1) If $p = 1$ then I^1_m is 1-concave and $L^1(m)$ is 1-concave.

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- By Hölder's inequality:

$$\int |f_i g_i||h| \, d\mu \le \Big(\int |f_i|^p |h| \, d\mu\Big)^{1/p} \Big(\int |g_i|^q |h| \, d\mu\Big)^{1/q} \le \Big(\int |f_i|^p |h| \, d\mu\Big)^{1/p} \quad \text{for all } h \in B_{L^1(m)'}.$$

Proof of (a)⇒(c), II

We want to prove . . .

$I_m^1 : L^1(m) \to X$ is **positive** p**-summing** (for some $1 \le p < \infty$) $\Longrightarrow$ $L^p(m)$ is p**-concave**

We already known (for $p > 1$ and $1/p + 1/q = 1$) . . .

(2) Given $f_1, \dots, f_n \in L^p(m)$, we have:

$$\Big(\sum_{i=1}^n \Big\|\int f_i g_i \, dm\Big\|^p\Big)^{\frac{1}{p}} \le K \Big\|\Big(\sum_{i=1}^n |f_i|^p\Big)^{\frac{1}{p}}\Big\|_{L^p(m)} \quad \text{for all } g_1, \dots, g_n \in B_{L^q(m)}.$$

Proof of (a)⇒(c), II

We want to prove . . .

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(2) Given $f_1, \dots, f_n \in L^p(m)$, we have:

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(3) Since

$$\|f_i\|_{L^p(m)} = \sup_{g \in B_{L^q(m)}} \Big\| \int f_i g \, dm \Big\|$$

we infer:

$$\boxed{\Big(\sum_{i=1}^{n} \|f_i\|_{L^p(m)}^p\Big)^{\frac{1}{p}} \le K \Big\| \Big(\sum_{i=1}^{n} |f_i|^p\Big)^{\frac{1}{p}} \Big\|_{L^p(m)}}.$$

□

p-summability and L^1 of a scalar measure (again)

Theorem

Let E be an order continuous B.f.s. having weak order unit. TFAE:

(a) For **some/every** vector measure m representing E and **some/every** $1 \leq p < \infty$, the integration operator I_m^1 is **positive** p-summing.

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In general, the previous statements are not equivalent to ...

(b') For **every** vector measure m representing E there is some $1 \leq p < \infty$ such that I_m^1 is **absolutely** p-summing.

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