The Gelfand integral for multi-valued functions

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Functional Analysis Valencia 2010

The co-authors

- **B. Cascales**, **V. Kadets**, and J. Rodríguez, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces*, J. Funct. Anal. **256** (2009), 673–699.
- **B. Cascales**, **V. Kadets**, and J. Rodríguez, *Measurability and selections of multi-functions in Banach spaces*, J. Convex Anal. **17** (2010), 229–240.
- **B. Cascales**, **V. Kadets**, and J. Rodríguez, *The Gelfand integral for multi-valued functions*, submitted.

Classical measurable selection theorems

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Theorem (Kuratowski and Ryll-Nardzewski, 1965)

Let (Ω, Σ) be a measurable space and X a **Polish** space.
Let $F : \Omega \to 2^X$ be a multi-function having non-empty closed values such that

$$\boxed{\{\omega \in \Omega : \; F(\omega) \cap U \neq \emptyset\} \in \Sigma} \qquad \text{for every open set } U \subset X.$$

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Theorem (Aumann, 1969)

Let (Ω, Σ, μ) be a complete probability space and X a **Polish** space.
Let $F : \Omega \to 2^X$ be a multi-function having non-empty values such that

$$\{(\omega, x) \in \Omega \times X : \ x \in F(\omega)\} \in \Sigma \otimes \mathrm{Borel}(X).$$

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$$\delta_*(x^*, C) := \inf\{x^*(x) : x \in C\},$$

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Well-known fact

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Why?

If X is **separable**, then $F:\Omega\to cwk(X)$ is scalarly measurable iff

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Lemma (Valadier, 1971)

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3. Each F_n is scalarly measurable and $F_0 \supset F_1 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$
4. Then $\bigcap F_n$ is a **single-valued** scalarly measurable selector of F.

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(1) We assume *wlog* that there is $M > 0$ such that, for each $x^* \in S_{X^*}$, we have $|\delta^*(x^*, F)| \leq M$ μ-a.e.

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(2) It suffices to prove that for every $\varepsilon > 0$ there is a scalarly measurable $G : \Omega \to wk(X)$ such that $G \subset F$ and $\Delta G \leq \varepsilon$.

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(6) For each $\omega\in\Omega$, we find a $\overline{\mathrm{co}}(F(\omega))$-valued *martingale* (g_n) such that

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(7) This **contradicts** Lebesgue's dominated convergence theorem.

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2. F is $w^*k(X^*)$-valued and $\overline{\mathrm{co}}^{w^*}(F(\omega))$ has the RNP for all $\omega \in \Omega$. (Cascales-Kadets-R.)

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GENERAL PROBLEM

When do w^*-scalarly measurable multi-functions $F : \Omega \to 2^{X^*}$ admit w^*-scalarly measurable selectors?

Positive answers . . .

The w^*-scalarly measurable multi-function $F : \Omega \to 2^{X^*}$ admits w^*-scalarly measurable selectors in each of the following cases:

1. F is $cw^*k(X^*)$-valued and X is separable. (Valadier, 1971)
2. F is $w^*k(X^*)$-valued and $\overline{\text{co}}^{w^*}(F(\omega))$ has the RNP for all $\omega \in \Omega$. (Cascales-Kadets-R.)
3. Ω is a compact metric space and μ is Radon, F is $cw^*k(X^*)$-valued and $\delta^*(x, F)$ is continuous for all $x \in X$. (Cascales-Kadets-R.)

Weak*-scalarly measurable selectors

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▶▶▶ OPEN PROBLEM ◀◀◀

Does every w^*-scalarly measurable multi-function $F:\Omega\to w^*k(X^*)$ admit w^*-scalarly measurable selectors?

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$f : \Omega \to X^*$ is a w^*-**almost selector** of $F : \Omega \to 2^{X^*}$ iff for every $x \in X$ we have

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- The proof uses the existence of *liftings* on (Ω, Σ, μ).

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In this case, for each $A \in \Sigma$ there is a vector $\int_A f \, d\mu \in X^*$ satisfying

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I.M. Gelfand (1913–2009)

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Corollary

Suppose X is **separable**. If $F : \Omega \to cw^*k(X^*)$ is Gelfand integrable, then

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Compactness of Gelfand integral

QUESTIONS

For a Gelfand integrable multi-function $F : \Omega \longrightarrow cw^*k(X^*)$, one might ask:

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THANKS FOR YOUR ATTENTION !!

http://webs.um.es/joserr