The McShane integral in weakly compactly generated spaces

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McShane's approach to Lebesgue integration

Theorem (McShane, 1969)

$f : [0,1] \to \mathbb{R}$ is **Lebesgue integrable** if and only if there is $I \in \mathbb{R}$ such that:

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$$\left| \sum_{i=1}^{n} \lambda(A_i) f(t_i) - I \right| < \varepsilon$$

for every finite partition $A_1, \ldots, A_n$ of $[0,1]$ into intervals and every choice of points $t_1, \ldots, t_n \in [0,1]$ satisfying

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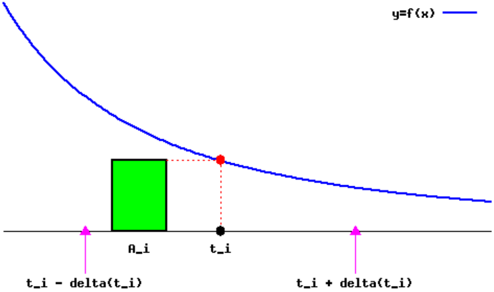

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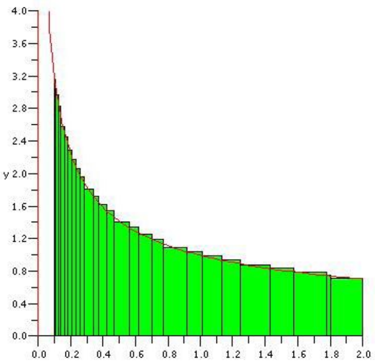

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2. for each measurable set $A \subset [0,1]$ there is a vector $\int_A f \in X$ such that

$$\boxed{x^*\left(\int_A f\right) = \int_A x^* f} \quad \forall x^* \in X^*.$$

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Theorem (Gordon 1990, Fremlin-Mendoza 1994)

If X is **separable**, then for any $f : [0,1] \to X$ we have:

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▶ Key: strong measurability $\equiv$ scalar measurability if X is **separable**.

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Problem (Di Piazza-Preiss, 2003)

Suppose X is **weakly compactly generated (WCG)**.
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   i.e. there is an operator $T : \ell_2(\Gamma) \to X$ with $\overline{T[\ell_2(\Gamma)]} = X$.

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The answer to Musiał's problem is **NO** *under CH* (Di Piazza-Preiss, R.).

Our main result

Theorem (Avilés-Plebanek-R.)

There exists a **WCG** Banach space X and a **scalarly null** function $f : [0,1] \to X$ that is **not** McShane integrable.

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X can be taken **reflexive**, using the Davis-Figiel-Johnson-Pelczynski theorem.

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Are there measure-filling hereditary compact families on $[0,1]$?

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Open Problem DU (Fremlin)

Are there ε-filling compact families on **uncountable** sets?

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There exist measure-filling hereditary compact families on $[0,1]$.

Sketch of proof:

- Consider a partition $[0,1] = \bigcup_{s \in S} Z_s$ with $|S| = \mathfrak{c}$ and $\lambda^*(Z_s) = 1$.
- Let $\mathscr{G}$ be a **log-filling** compact family of finite subsets of S.
- Let $\mathscr{F}$ be the family of all finite sets $A \subset [0,1]$ such that
  - $|A \cap Z_s| \leq 1$ for every s,
  - $\{s \in S : A \cap Z_s \neq \emptyset\} \in \mathscr{G}$.

Some ideas of the proof III: **log-filling** families

Definition

A family $\mathscr{G}$ of finite subsets of a set S is **log-filling** on S iff it is hereditary and for every finite set $A \subset S$ there exists $B \subset A$, $B \in \mathscr{G}$, such that $|B| \geq \log |A|$.

Theorem (Fremlin)

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  - $|A \cap Z_s| \leq 1$ for every s,
  - $\{s \in S : A \cap Z_s \neq \emptyset\} \in \mathscr{G}$.
- Then $\mathscr{F}$ is **measure-filling**, hereditary and compact.