BIRKHOFF INTEGRAL FOR MULTI-VALUED FUNCTIONS

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$(X, \| \ \|) \equiv$ real Banach space

$B_X = \{x \in X : \ \|x\| \leq 1\}$

$X^* \equiv$ (topological) dual of X

$cwk(X) \equiv$ family of all non-empty convex weakly compact subsets of X

$ck(X) \equiv$ family of all non-empty convex norm compact subsets of X

$(\Omega, \Sigma, \mu) \equiv$ complete finite measure space

For $B \subset X$ bounded and $x^* \in X^*$, we write

$$\delta^*(x^*, B) := \sup\{x^*(x) : x \in B\}.$$

Lem. *The map* $j : cwk(X) \longrightarrow \ell_\infty(B_{X^*})$ *given by* $j(B) := \delta^*(\cdot, B)$ *satisfies:*

(i) $j(B + C) = j(B) + j(C)$

(ii) $j(\lambda B) = \lambda j(B)$

(iii)

$$h(B, C) := \inf\{\eta > 0 : \ B \subset C + \eta B_X, \ C \subset B + \eta B_X\} = \|j(B) - j(C)\|_\infty$$

for every $B, C \in cwk(X)$ *and every* $\lambda \geq 0$.

Def. *A multifunction* $F : \Omega \longrightarrow cwk(X)$ *is said to be Debreu integrable if the single-valued function* $j \circ F : \Omega \longrightarrow \ell_\infty(B_{X^*})$ *is Bochner integrable. The Debreu integral of* F *is the unique element* $(D)\int_\Omega F \ d\mu \in cwk(X)$ *such that*

$$j\Big((D)\int_\Omega F \ d\mu\Big) = \int_\Omega j \circ F \ d\mu.$$

Def. *A multifunction* $F : \Omega \longrightarrow cwk(X)$ *is said to be Birkhoff integrable if the single-valued function* $j \circ F : \Omega \longrightarrow \ell_\infty(B_{X^*})$ *is Birkhoff integrable. The Birkhoff integral of* F *is the unique element* $(B)\int_\Omega F \ d\mu \in cwk(X)$ *such that*

$$j\Big((B)\int_\Omega F \ d\mu\Big) = \int_\Omega j \circ F \ d\mu.$$

Prop. *Let Y be a Banach space. A function $f : \Omega \longrightarrow Y$ is Birkhoff integrable if, and only if, there is $y \in Y$ with the following property:*

*for every $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable partition $\Gamma = (A_n)$ of Ω in Σ finer than Γ_0 and any choice $T = (t_n)$ in Γ (*i.e.*, $t_n \in A_n$ for every n)*

(i) the series $\sum_n \mu(A_n) f(t_n)$ converges unconditionally in Y

(ii)

$$\left\| \sum_n \mu(A_n) f(t_n) - y \right\| \leq \varepsilon.$$

In this case, $y = \int_\Omega f \, d\mu$.

Prop. *A multifunction $F : \Omega \longrightarrow cwk(X)$ is Birkhoff integrable if, and only if, there exists $B \in cwk(X)$ with the following property:*

*for every $\varepsilon > 0$ there is a countable partition Γ_0 of Ω in Σ such that for every countable partition $\Gamma = (A_n)$ of Ω in Σ finer than Γ_0 and any choice $T = (t_n)$ in Γ (*i.e., $t_n \in A_n$ *for every n)*

(i) the series $\sum_n \mu(A_n)F(t_n)$ is unconditionally convergent

(ii)

$$h\Big(\sum_n \mu(A_n)F(t_n), B\Big) \leq \varepsilon.$$

In this case, $B = (B)\int_\Omega F \, d\mu$.

Lem. *Let* (B_n) *be a sequence in* $cwk(X)$. *TFAE:*

(i) $\sum_n B_n$ *is unconditionally convergent,* i.e., *for every choice* $b_n \in B_n$, $n \in \mathbb{N}$, *the series* $\sum_n b_n$ *is unconditionally convergent in* X;

(ii) there is $B \in cwk(X)$ *with the following property: for every* $\varepsilon > 0$ *there is a finite set* $P \subset \mathbb{N}$ *such that* $h(\sum_{n\in Q} B_n, B) \le \varepsilon$ *for every finite set* $Q \subset \mathbb{N}$ *such that* $P \subset Q$;

(iii) $\sum_n j(B_n)$ *is unconditionally convergent in* $\ell_\infty(B_{X^*})$.

In this case,

$$\sum_n B_n := \Big\{ \sum_{n=1}^{\infty} b_n : \ b_n \in B_n, \ n \in \mathbb{N} \Big\} = B$$

$$j(\sum_n B_n) = \sum_n j(B_n).$$

(For X separable) F is Pettis integrable if $\delta^*(x^*, F) \in \mathcal{L}^1(\mu)$ for every $x^* \in X^*$ and for every $A \in \Sigma$ there is $(P) \int_A F \, d\mu \in cwk(X)$ such that

$$\delta^*\left(x^*, (P) \int_A F \, d\mu\right) = \int_A \delta^*(x^*, F) \, d\mu$$

for every $x^* \in X^*$.