On vector measures with separable range

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Theorem (Bartle-Dunford-Schwartz, 1955)

The range of a countably additive Y-valued measure is **weakly** relatively compact.

In general, the range is neither **norm relatively compact** nor **separable**, even for indefinite Pettis integrals (Fremlin-Talagrand, 1979).

The indefinite integral of any Y-valued Pettis integrable function has norm relatively compact range if ...

- $Y \not\supset \ell^1(\omega_1)$ (Talagrand, 1984).
- $Y = X^*$ and $X \not\supset \ell^1$ (Rybakov, 1977).

Definition (Musial, 1991)

Y has the **Pettis Separability Property** if the indefinite integral of any Y-valued Pettis integrable function has **separable** range.

Problem (Musial, 1991)

Which Banach spaces have the Pettis Separability Property?

Y has the Pettis Separability Property if ...

(1) Every weakly compact subset of Y is separable. Examples:

- (B_{Y^*}, w^*) is separable (Amir-Lindenstrauss, 1968).
- $Y = C(K)$ and K is the support of a Radon probability measure (Rosenthal, 1970).

(2) $Y = C(K)$ and every sequentially continuous function $f : K \longrightarrow \mathbb{R}$ is continuous (Plebanek, 1993).

Our aim

We study the **norm separability of the range** of a

countably additive X^*-valued measure with σ-finite variation,

when X admits a **projectional generator**.

Examples of spaces with a projectional generator

- Weakly Lindelöf determined (Valdivia, 1988).
- Duals of Asplund spaces (Orihuela-Valdivia, 1990).

Main Theorem

Suppose that X admits a projectional generator.
The following statements are equivalent:

(i) (B_{X^*}, w^*) has **property (M)**, i.e. every Radon probability measure on (B_{X^*}, w^*) has w^*-separable support.

(ii) Every countably additive X^*-valued measure with σ-finite variation has **norm separable range**.

- If X is weakly $\mathscr{K}$-countably determined, then (B_{X^*}, w^*) has property (M) (Argyros-Negrepontis, 1983).
- Under CH, there is a weakly Lindelöf determined X such that (B_{X^*}, w^*) does not have property (M) (Kalenda, 2002).
- Under MA+$\neg$CH, (B_{X^*}, w^*) has property (M) for every weakly Lindelöf determined X (Argyros-Mercourakis-Negrepontis, 1982).

Applications

A partial answer to Musial's question:

Corollary 1

If

- X admits a projectional generator and
- (B_{X^*}, w^*) has property (M),

then X^* has the Pettis Separability Property.

Applications

Theorem (Rybakov, 1977)

The following statements are equivalent:

(i) $X \not\supset \ell^1$.

(ii) Every countably additive X^*-valued measure with σ-finite variation has norm relatively compact range.

Corollary 2

Suppose that X admits a projectional generator. Then

$$X \not\supset \ell^1 \quad \Longrightarrow \quad (B_{X^*}, w^*) \text{ has property (M).}$$

(i)⇒(ii) Suppose that (B_{X^*}, w^*) has property (M).

Fix a measurable space (Ω, Σ) and a countably additive measure $\nu : \Sigma \to X^*$ with σ-finite variation.

(A) $\exists$ countably additive measure $\mu : \Sigma \to [0, \infty)$ such that $\lim_{\mu(A)\to 0} \|\nu(A)\| = 0$. Assume w.l.o.g. that μ is complete.

(B) *Particular case.*- Suppose that $|\nu|(A) \leq \mu(A)$ $\forall A \in \Sigma$.

- Fix a lifting ρ on (Ω, Σ, μ).
- $\exists$ Gelfand integrable function $f : \Omega \to B_{X^*}$ such that:
  1. $\rho(\langle f, x\rangle) = \langle f, x\rangle$ $\forall x \in X$.
  2. $\langle \nu(A), x\rangle = \int_A \langle f, x\rangle \, d\mu$ $\forall x \in X$, $\forall A \in \Sigma$.
- f is Σ-Borel(B_{X^*}, w^*)-measurable and the image probability measure μf^{-1} is Radon.
- $\exists$ w^*-separable set $T \in \mathrm{Borel}(B_{X^*}, w^*)$ such that $\mu(\Omega \setminus f^{-1}(T)) = 0$.

Therefore

$$\nu(\Sigma) \subset \overline{\mathrm{span}}^{w^*}(T), \text{ which is } w^*\text{-separable.}$$

(C) *General case.-* $\exists$ partition of Ω into countably many measurable sets $A_1, A_2, \dots$ and $\exists$ positive constants $C_1, C_2, \dots$ such that $|\nu|(A) \leq C_n \mu(A)$ $\forall$ measurable set $A \subset A_n$. By the Particular case,

$\nu(\Sigma)$ is contained in a w^*-separable subset of X^*.

Lemma

Suppose that X admits a projectional generator.
Let $S \subset X^*$ be w^*-separable.
Then $\exists$ subspaces $X_0, X_1 \subset X$ such that $X = X_0 \oplus X_1$, X_0 is **separable** and $\langle x^*, x \rangle = 0$ $\forall x^* \in S$, $\forall x \in X_1$.
In particular, S is w^*-metrizable.

(D) $\exists$ separable subspace $X_0 \subset X$ and $\exists$ subspace $Z \subset X^*$ isomorphic to X_0^* such that $\nu(\Sigma) \subset Z$.

Since $(B_{X_0^{**}}, w^*)$ is separable, $\nu(\Sigma)$ is norm separable. $\square$

(ii)$\Rightarrow$(i) Suppose that every countably additive X^*-valued measure with σ-finite variation has norm separable range.

Fix a Radon probability measure μ on (B_{X^*}, w^*).

(A) The 'identity' mapping $I : B_{X^*} \longrightarrow X^*$ is bounded and Gelfand integrable with respect to μ.

Then there is a countably additive measure $\nu : \mathrm{Borel}(B_{X^*}, w^*) \longrightarrow X^*$ with finite variation such that

$$\langle \nu(A), x\rangle = \int_A \langle I, x\rangle \, d\mu \quad \forall x \in X, \ \forall A \in \mathrm{Borel}(B_{X^*}, w^*).$$

(B) We have $\mathrm{supp}(\mu) \subset S := \overline{\mathrm{span}}^{w^*}\left(\nu(\mathrm{Borel}(B_{X^*}, w^*))\right)$.

(C) Since ν has norm separable range, S is w^*-separable.

By the Lemma, $\mathrm{supp}(\mu)$ is w^*-metrizable.

It follows that $\mathrm{supp}(\mu)$ is w^*-separable. $\square$

References

D. Amir and J. Lindenstrauss, *The structure of weakly compact sets in Banach spaces*, Ann. of Math. (2) **88** (1968).

S. Argyros, S. Mercourakis, and S. Negrepontis, *Analytic properties of Corson-compact spaces*, Proc. Fifth Prague Topological Symposium, August 1981, Heldermann-Verlag, Berlin, 1982.

S. Argyros, S. Mercourakis, and S. Negrepontis, *Functional-analytic properties of Corson-compact spaces*, Studia Math. **89** (1988).

S. Argyros and S. Negrepontis, *On weakly $\mathscr{K}$-countably determined spaces of continuous functions*, Proc. Amer. Math. Soc. **87** (1983).

R. G. Bartle, N. Dunford, and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955).

J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

N. Dinculeanu, *Vector measures*, International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Oxford, 1967.

D. van Dulst, *Characterizations of Banach spaces not containing ℓ^1*, CWI Tract, vol. 59, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

M. Fabian, *Gâteaux differentiability of convex functions and topology. Weak Asplund spaces*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1997.

References

- D. H. Fremlin and M. Talagrand, *A decomposition theorem for additive set-functions, with applications to Pettis integrals and ergodic means*, Math. Z. **168** (1979).
- I. M. Gelfand, *Sur un lemme de la théorie des espaces linéaires*, Comm. Inst. Sci. Math. Méc. Univ. de Kharkoff et Soc. Math. Kharkoff **13** (1936).
- K. Musiał, *Topics in the theory of Pettis integration*, Rend. Istit. Mat. Univ. Trieste **23** (1991).
- J. Orihuela and M. Valdivia, *Projective generators and resolutions of identity in Banach spaces*, Rev. Mat. Univ. Complut. Madrid **2** (1990), Special Issue, suppl.
- G. Plebanek, *On Pettis integrals with separable range*, Colloq. Math. **64** (1993).
- G. Plebanek, *Convex Corson compacta and Radon measures*, Fund. Math. **175** (2002).
- H. P. Rosenthal, *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measure μ*, Acta Math. **124** (1970).
- V. I. Rybakov, *Certain properties of measures on a normed space possessing the property RN*, Math. Notes **21** (1977).
- D. Sentilles, *Decomposition of weakly measurable functions*, Indiana Univ. Math. J. **32** (1983).
- M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. **51** (1984), no. 307.
- M. Valdivia, *Resolution of the identity in certain Banach spaces*, Collect. Math. **39** (1988).