Weak Baire measurability of the balls in a Banach space

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defines an **equivalent norm** on $\ell^\infty(\mathscr{B})$.

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Recall . . .

Theorem B

There is an equivalent norm $\|\cdot\|$ on $\ell^\infty(\mathbb{N})$ such that:

- $B(\ell^\infty(\mathbb{N}), \|\cdot\|)$ belongs to $\mathrm{Baire}(\ell^\infty(\mathbb{N}), \mathrm{weak})$.
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Let $\varphi : \{0,1\}^{\mathbb{N}} \to [1, \infty)$ be a bounded function. TFAE:

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