Categories of Modules for Idempotent Rings and Morita Equivalences

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En resolución, él se enfrascó tanto en su lectura, que se le pasaban las noches leyendo de claro en claro, y los días de turbio en turbio; y así, del poco dormir y del mucho leer se le secó el celebro[1] de manera, que vino a perder el juicio.

MIGUEL DE CERVANTES SAAVEDRA:
El Ingenioso Hidalgo don Quijote de la Mancha.

[1] In modern spanish this word is written "cerebro"

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CHAPTER 1

Introduction

In the following all rings are associative rings. It is not assumed they have an identity unless it is mentioned explicitly, but they will be idempotent ($R^2 = R$).

One of the most powerfull techniques that is used in the study of rings with identity, is to associate to each ring R, its category of unitary modules Mod-R and relate properties of R with properties of Mod-R and vice versa. There are a lot of examples of this, but we shall mention only the following one

Definition 1.1. Let R be a ring. R is called von Neumann regular if and only if for all $r \in R$ there exists $s \in R$ such that $r = rsr$.

This definition, in the case of rings with identity, has a well known characterization

Proposition 1.2. *Let R be a ring with identity. The following conditions are equivalent*

1. *R is von Neumann regular.*
2. *All modules in* Mod-R *are flat.*

In the case of rings with identity, a module is flat if and only if it is a direct limit of projective modules. The definition of projectivity is a categorical definition and the direct limit is also a categorical concept, therefore, flat modules are transferd by category equivalences and the property of being von Neumann regular also.

If we try to generalize this property for rings without identity, we have several difficulties. First of all, we have to choose a category of modules for the ring R.

Consider the category of all right R-modules, which we shall denote MOD-R. This category is the category of unital right $R \times \mathbb{Z}$-modules, where $R \times \mathbb{Z}$ is the Dorroh's extension of R (this ring consists of the pairs $(r, z) \in R \times \mathbb{Z}$ with the sum defined componentwise and the product given by $(r, z)(r', z') = (rr' + z'r + zr', zz')$, see Theorem 2.20). This category is not the best choice for this kind of study, because although R is von Neumann regular, $R \times \mathbb{Z}$ can never be so (look at the elements (r, z) with $z \neq 0, 1, -1$). In the case of rings with identity, this problem is solved by choosing the full subcategory of MOD-R with the modules M such that $MR = M$, i.e. Mod-R. This solution can be generalized for other rings and this is one of the topics we are going to study here. We have the following categories for an idempotent ring R

DEFINITION 1.3. Let R be an idempotent ring and A a ring with identity such that R is a two-sided ideal of it.

1. CMod-R is the full subcategory of Mod-A with the modules M such that the canonical homomorphism $\lambda : M \to \mathrm{Hom}_A(R, M)$, $(\lambda_m(r) = mr)$, is an isomorphism.
2. Mod-R is the full subcategory of Mod-A with the modules M such that $MR = M$ and $\{m \in M : mR = 0\} = 0$.
3. DMod-R is the full subcategory of Mod-A with the modules M such that the canonical homomorphism $\mu : M \otimes_A R \to M$, $(\mu(m \otimes r) = mr)$, is an isomorphism.

These categories have been considered in several papers, even for rings without the assumption of being idempotent, see [**10, 11, 12, 16**], but in the case of idempotent rings is is proved that they are equivalent. We give a direct proof of this fact in Theorem 2.45, although this result is known in more general terms, see [**9**, Proposition 1.15].

We are going to study many general points of these categories. For example projectivity, injectivity, generators, monomorphisms, epimorphisms, direct and inverse limits, etc. Some of these things are adaptations of the concepts that are given in Grothendieck categories and other are generalizations of concepts given for categories of modules over a ring with identity.

There are some properties that cannot be extended to these categories. For instance, it is possible to build an idempotent ring R such that the category CMod-R (and then the other) has no projective object different from 0. This can be found in [**8**, Example 3.4(i)]. This has a particular importance in the case of flat modules that cannot be considered as a direct limit of projective modules.

These problems make necessary to study some particular classes of idempotent rings that are closer to rings with identity. This will be done in Chapter 5. If we assume that R and R' are rings of a particular type (rings with local units, see Definition 5.7), it is proved in [**4**, Proposition 3.1] that if Mod-R and Mod-R' are equivalent, R is von Neuman regular if and only if R' is.

Chapter 4 is going to study the equivalences between the categories for two idempotent rings R and R'. These results have been proved in several steps by different people. Apart from the classical case of Morita Theorems for rings with identity, we can find this study for rings with local units in [**1, 2, 4**]. In the more general case of idempotent rings, our results are taken from [**7**], although the proofs will not be exactly the same. There are generalizations for some of these results for Grothendieck categories in [**5, 6**].

What is the original part of this work?. First of all, the point of view. Usually these categories have been considered as categories related with a Morita context, defined for the trace ideals. We look at these categories by themselves. We even obtain in Proposition 2.46

that the definition of these categories is not dependent on the choice of the ring A. This ring could be chosen, for example, to be the Dorroh's extension of R or any other ring with identity such that R is a two-sided ideal of it.

Secondly, we obtain a general theory of noncommutative localization for these categories. This study has been made by several authors for Grothendieck categories, but there are many results that cannot be generalized because in Grothendieck categories, the ring disappears. Using idempotent rings we obtain a generalization of results that hold for rings with identity.

Another thing we generalize, is the concept of the Picard group of a ring. We define this group for idempotent rings.

We define a ring to be coclosed if $R \otimes_A R \simeq R$ in the canonical way. We obtain in Chapter 5 many results for this kind of ring. For example, we prove that the study of idempotent rings can be reduced to the study of coclosed rings because the categories for R and $R \otimes_A R$ are the same and $R \otimes_A R$ is a coclosed ring. We also generalize for these rings, facts known for rings with local units and but which cannot be generalized for idempotent rings.

In Chapter 4 we study the bimodules that define functors between the categories for idempotent rings R and R'. Using this study we can simplify some proofs.

CHAPTER 2

Categories of Modules for Rings I

1. Noncommutative Localization

In this section we shall recall some results about torsion theories and localization in the category of unitary modules for a ring with identity A. All these things can be found in [**14**] and we shall reference this book for the proofs.

DEFINITION 2.1. A *preradical* r of Mod-A is a functor $r : \text{Mod-}A \to \text{Mod-}A$ that assigns to each object M of Mod-A a subobject $r(M)$ in such way that every morphism $f : M \to N$ in Mod-A induces $r(f) : r(M) \to r(N)$ by restriction. In other words, a preradical is a subfunctor of the identity functor of Mod-A.

A preradical r is called *idempotent* in case $r \circ r = r$ and it is called a *radical* in case $r(M/r(M)) = 0$ for all $M \in \text{Mod-}A$.

To a preradical r, one can associate two classes of objects of Mod-A, namely:

$\mathcal{T}_r$: the class of modules M such that $r(M) = M$.
$\mathcal{F}_r$: the class of modules M such that $r(M) = 0$.

DEFINITION 2.2. A *torsion theory* of Mod-A is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules of Mod-A such that

1. $\text{Hom}_A(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. If $\text{Hom}_A(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$.
3. If $\text{Hom}_A(T, M) = 0$ for all $T \in \mathcal{T}$, then $M \in \mathcal{F}$.

$\mathcal{T}$ is called the *torsion class* and its objects are called *torsion objects* while $\mathcal{F}$ is called the *torsion-free class* and its objects, the *torsion free objects*.

DEFINITION 2.3. Let $\mathcal{C}$ be a class of objects in an abelian category $\mathbb{A}$. We shall say:

1. $\mathcal{C}$ is *closed under subobjects* if and only if for every $M \in \mathcal{C}$ and every monomorphism $\mu : N \to M$ in $\mathbb{A}$, $N \in \mathcal{C}$.
2. $\mathcal{C}$ is *closed under quotient objects* if and only if for every $M \in \mathcal{C}$ and every epimorphism $\eta : M \to N$ in $\mathbb{A}$, $N \in \mathcal{C}$.
3. $\mathcal{C}$ is *closed under products* if and only if for every $\{M_i : i \in I\}$ contained in $\mathcal{C}$ if $\prod_{i \in I} M_i$ is a product of the family $\{M_i : i \in I\}$ in $\mathbb{A}$, then $\prod_{i \in I} \in \mathcal{C}$.

4. $\mathcal{C}$ is *closed under coproducts* if and only if for every $\{M_i : i \in I\}$ contained in $\mathcal{C}$ if $\coprod_{i\in I} M_i$ is a coproduct of the family $\{M_i : i \in I\}$ in $\mathbb{A}$, then $\coprod_{i\in I} \in \mathcal{C}$.
5. $\mathcal{C}$ is *closed under extensions* if and only if for every short exact sequence $0 \to L \to M \to N \to 0$ in $\mathbb{A}$ with L and N in $\mathcal{C}$ then, M is in $\mathcal{C}$.

PROPOSITION 2.4. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in* Mod-A*. Then*

1. *$\mathcal{T}$ is closed under quotient objects, coproducts and extensions.*
2. *$\mathcal{F}$ is closed under subobjects, products and extensions.*

PROOF. See [**14**, Proposition VI.2.1] and [**14**, Proposition VI.2.2]. □

If $\mathcal{T}$ is a class of modules in Mod-A closed under quotient objects, coproducts and extensions, then we can build the corresponding class $\mathcal{F}$ with the property $F \in \mathcal{F}$ if and only if $\mathrm{Hom}_A(T, F) = 0$ for all $T \in \mathcal{T}$. With this definition $(\mathcal{T}, \mathcal{F})$ is a torsion theory. In the other direction, if $\mathcal{F}$ is a class of modules in Mod-A closed under subobjects, products and extensions, we can define the corresponding class $\mathcal{T}$ with the property $T \in \mathcal{T}$ if and only if $\mathrm{Hom}_A(T, F) = 0$ for all $F \in \mathcal{F}$ and with this definition $(\mathcal{T}, \mathcal{F})$ is a torsion theory. These operations are inverses of each other. Therefore, in order to define a torsion theory, we need only one of the classes $\mathcal{T}$ or $\mathcal{F}$.

PROPOSITION 2.5. *There is a bijective correspondence between torsion theories in* Mod-A *and idempotent radicals in* Mod-A*.*

PROOF. For the proof see [**14**, Proposition VI.2.3]. We shall only say how the idempotent radical is built. If $(\mathcal{T}, \mathcal{F})$ is a torsion theory and M is an object in Mod-A, we can define $r(M)$ as the largest subobject N of M such that $N \in \mathcal{T}$. Conversely, given an idempotent radical r, $(\mathcal{T}_r, \mathcal{F}_r)$ is the corresponding torsion theory. □

DEFINITION 2.6. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called *hereditary* if $\mathcal{T}$ is closed under submodules.

DEFINITION 2.7. A *(right) Gabriel topology* is a family $\mathcal{G}$ of right ideals of A satisfying the following axioms.

T1 If $\mathfrak{a} \in \mathcal{G}$, $\mathfrak{b} \leq A_A$ and $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{b} \in \mathcal{G}$.
T2 If $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathcal{G}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{G}$.
T3 If $\mathfrak{a} \in \mathcal{G}$ and $a \in A$, then $(\mathfrak{a} : a) \in \mathcal{G}$.
T4 If for some $\mathfrak{a} \leq A_A$ there exists a $\mathfrak{b} \in \mathcal{G}$ such that $(\mathfrak{a} : b) := \{r \in R : ar \in \mathfrak{a}\} \in \mathcal{G}$ for all $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathcal{G}$.

THEOREM 2.8. *There is a bijective correspondence between:*

1. *Right Gabriel topologies on A.*
2. *Hereditary torsion theories for* Mod-A*.*
3. *Left exact radicals of* Mod-A*.*

PROOF. For the proof see [**14**, Theorem VI.5.1]. We are going to give here only the constructions but not the complete proof.

If $(\mathcal{T},\mathcal{F})$ is an hereditary torsion theory in Mod-A, the corresponding right Gabriel topology $\mathcal{G}$ on A is $\mathcal{G}$ is $\{\mathfrak{a} \leq A_A : A/\mathfrak{a} \in \mathcal{T}\}$.

Conversely, if $\mathcal{G}$ is a Gabriel topology on A, the corresponding torsion theory $(\mathcal{T},\mathcal{F})$ is as follows: $M \in \mathcal{T}$ if and only if $\mathrm{r.ann}(m) \in \mathcal{G}$ for every $m \in M$. A module $M \in \mathcal{F}$ if and only if $\mathrm{Hom}_A(T, M) = 0$ for all $T \in \mathcal{T}$.

The correspondence between hereditary torsion theories and left exact radicals is the same as in Proposition 2.5. ☐

DEFINITION 2.9. A torsion class $\mathcal{T}$ is called a *TTF-class* (TTF stands for "torsion torsion-free") if it is a torsion class and a torsion-free class. Therefore we can build a torsion class $\mathcal{U}$ and a torsion-free class $\mathcal{F}$ such that $(\mathcal{U},\mathcal{T})$ is a torsion theory and $(\mathcal{T},\mathcal{F})$ is another torsion theory. The triple $(\mathcal{U},\mathcal{T},\mathcal{F})$ is called a TTF-theory.

PROPOSITION 2.10. *A torsion class $\mathcal{T}$ is a TTF-class if and only if there exists an idempotent two-sided ideal I in the corresponding right Gabriel topology $\mathcal{G}$.*

PROOF. For the proof see [**14**, Proposition VI.6.12] and [**14**, Proposition VI.8.1]. We shall give here only the definition of I. If $\mathcal{T}$ is a TTF-class, $\prod_{\mathfrak{a}\in\mathcal{G}} A/\mathfrak{a}$ is a torsion object and then, the kernel of the canonical homomorphism $\alpha : A \to \prod_{\mathfrak{a}\in\mathcal{G}} A/\mathfrak{a}$ is in $\mathcal{G}$. This is the ideal, $I = \mathrm{Ker}(\alpha) = \cap_{\mathfrak{a}\in\mathcal{G}}\mathfrak{a}$. The class $\mathcal{U}$ can be defined to be the class of modules M such that $MI = M$. For this fact see [**14**, Proposition VI.8.2]. ☐

From now on, unless stated otherwise, $(\mathcal{T},\mathcal{F})$ will be a torsion theory, $\mathcal{G}$ will be the corresponding Gabriel topology on A and the left exact preradical will be denoted by t.

For each module $M \in \mathcal{F}$ we shall define

$$\mathbf{a}(M) = \varinjlim_{\mathfrak{a}\in\mathcal{G}} \mathrm{Hom}_A(\mathfrak{a}, M)$$

where this direct limit is taken over the downwards directed family of right ideals $\mathcal{G}$. Every element in $\mathbf{a}(M)$ is thus represented by a homomorphism $\xi : \mathfrak{a} \to M$ for some $\mathfrak{a} \in \mathcal{G}$, with the understanding that ξ represents the same element in $\mathbf{a}(M)$ as does $\zeta : \mathfrak{b} \to M$ if and only if ξ and ζ coincide on some $\mathfrak{c} \in \mathcal{G}$, such that $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$.

There is a canonical A-homomorphism $\iota_{\mathfrak{a}} : M \to \mathrm{Hom}_A(\mathfrak{a}, M)$ given by

$$\begin{array}{rccc} \iota_{\mathfrak{a}}[m] : & \mathfrak{a} & \to & M \\ & a & \mapsto & ma \end{array}$$

The family of A-homomorphisms $(\iota_{\mathfrak{a}})_{\mathfrak{a}\in\mathcal{G}}$ define

$$\iota = \varinjlim_{\mathfrak{a} \in \mathcal{G}} \iota_{\mathfrak{a}} : M \to \varinjlim_{\mathfrak{a} \in \mathcal{G}} \mathrm{Hom}_A(\mathfrak{a}, M) = \mathbf{a}(M)$$

In the general case, we define

$$\mathbf{a}(M) = \mathbf{a}(M/t(M))$$

and the homomorphisms $\iota_{\mathfrak{a}}$ are the compositions of the already defined $\iota_{\mathfrak{a}}$ with the canonical projection $M \to M/t(M)$.

Definition 2.11. An A-module M is *$\mathcal{G}$-closed* if the canonical homomorphisms

$$\iota_{\mathfrak{a}} : M \to \mathrm{Hom}_A(\mathfrak{a}, M)$$

are isomorphisms for all $\mathfrak{a} \in \mathcal{G}$.

In this case the morphisms $\iota : M \to \mathbf{a}(M)$ is an isomorphism. The converse is also true.

Proposition 2.12. *For every A-module M, $\mathbf{a}(M)$ is $\mathcal{G}$-closed.*

Proof. See [**14**, Proposition IX.1.8]. □

We shall denote by Mod-$(A, \mathcal{G})$ the full subcategory of Mod-A consisting of all $\mathcal{G}$-closed modules. This is called the *quotient category* of Mod-A with respect to $\mathcal{G}$ (or the torsion theory $(\mathcal{T}, \mathcal{F})$).

Proposition 2.13. *The functor $\mathbf{a}$: Mod-$A \to$ Mod-$(A, \mathcal{G})$ is a left adjoint of the inclusion functor $\mathbf{i}$: Mod-$(A, \mathcal{G}) \to$ Mod-A.*

Proof. See [**14**, Proposition X.1.11]. □

Definition 2.14. A full subcategory $\mathbb{A}$ of Mod-A is *reflective* if the inclusion functor $\mathbf{i} : \mathbb{A} \to$ Mod-A has a left adjoint $\mathbf{a}$.

Definition 2.15. A reflective subcategory of Mod-A is called a *Giraud subcategory* if the left adjoint of the inclusion functor preserves kernels.

Theorem 2.16. *The category* Mod-$(A, \mathcal{G})$ *is a Giraud subcategory of* Mod-A.

Proof. See [**14**, Theorem X.1.6]. □

Theorem 2.17. *If $\mathbb{A}$ is a Giraud subcategory of* Mod-A*, then $\mathbb{A}$ is a Grothendieck category, and the left adjoint $\mathbf{a}$: Mod-$A \to$ Mod-$(A, \mathcal{G})$ of $\mathbf{i}$: Mod-$(A, \mathcal{G}) \to$ Mod-A is an exact functor.*

Proof. See [**14**, Theorem X.1.2] and [**14**, Theorem X.1.3]. □

It is important to notice that the inclusion functor $\mathbf{i}$ is, in general, not exact.

COROLLARY 2.18. *The category* Mod-$(A, \mathcal{G})$ *is a Grothendieck category.*

This corollary has a converse in some sense. This is the Gabriel-Popescu Theorem.

THEOREM 2.19. *Let* $\mathbb{A}$ *be a Grothendieck category with a generator* U. *Put* $A = \mathrm{End}_{\mathbb{A}}(U, U)$ *and let* $T : \mathbb{A} \to$ Mod-A *be the functor* $T(C) = \mathrm{Hom}_{\mathbb{A}}(U, C)$. *Then*

1. T *is full and faithful.*
2. T *induces an equivalence between* $\mathbb{A}$ *and the category* Mod-$(A, \mathcal{G})$ *where* $\mathcal{G}$ *is the strongest Gabriel topology on* A *for which all modules* $T(M)$ *are* $\mathcal{G}$*-closed.*

PROOF. See [**14**, X.4.1]. □

2. The Construction of the Categories

2.1. The Category MOD-R**.** We shall denote by MOD-R the category of all right R-modules and R-homomorphisms.

The following theorem is a well known result that states that this category is in fact a category of unitary modules over a ring with identity.

THEOREM 2.20. *Let* R *be a ring, and* $R \times \mathbb{Z}$ *be the Dorroh's extension of* R. *Then the category* MOD-R *is equivalent to the category* Mod-$R \times \mathbb{Z}$ *of unitary modules over the ring with identity* $R \times \mathbb{Z}$.

PROOF. This result is well known, and we shall only give some remarks about the proof. First we have to recall the definition of the Dorroh's extension of a ring R. The elements of this ring are the pairs $(r, n) \in R \times \mathbb{Z}$ with the sum defined componentwise and the product defined as follows:

$$(r, n) \cdot (s, m) = (rs + mr + ns, nm) \quad \forall r, s \in R \, \forall n, m \in \mathbb{Z}$$

The ring R can be identified inside $R \times \mathbb{Z}$ as the two-sided ideal

$$R \times 0 = \{(r, 0) \in R \times \mathbb{Z} : r \in R\}$$

The identity of the ring $R \times \mathbb{Z}$ is the element $(0, 1)$.

We are going to prove that any right R-module is a unitary right $R \times \mathbb{Z}$-module, and conversely.

Let M be a right R-module. We can define an operation $m \cdot (r, n) = m \cdot r + n \cdot m$ for all $m \in M$, $r \in R$ and $n \in \mathbb{Z}$. We can multiply m by the elements of $\mathbb{Z}$ because of the abelian group structure. With this operation M is a unitary $R \times \mathbb{Z}$-module ($m(0, 1) = m0 + 1m = m$).

In the other direction, if M is a unitary $R \times \mathbb{Z}$-module, because R is a two-sided ideal of $R \times \mathbb{Z}$, M has an R-module structure and the forgetful functor is inverse to the one that we have considered previously. □

One of our main objectives is the study of different possible categories that can be associated to a ring in order to relate the properties of the category with the properties of the ring. The first possibility is this category, but we are going to give some reasons that show us that this is not a very good choice.

Consider for instance the following well known statements for a ring with identity R:

1. R is right noetherian if and only if every direct sum of injective unitary right R-modules is injective.
2. R is right artinian if and only if every injective unitary right R-module is a direct sum of injective envelopes of simple modules.
3. R is von Neumann regular if and only if every right unitary R-module is flat.

If we try to extend these properties for the case of MOD-R we can see that this is impossible, because $R \times \mathbb{Z}$ can never be artinian nor von Neumann regular.

In the case of rings with identity the problem can be easily identified. Suppose R is a ring with identity and $M \in$ MOD-R. If we define $M' = \{m \in M : mR = 0\}$ and $M'' = M/M'$ then the map

$$\begin{array}{rccl} \epsilon : & M & \rightarrow & M' \times M'' \\ & m & \mapsto & (m - m \cdot 1_R, m \cdot 1_R + M') \end{array}$$

is an abelian group isomorphism. We have to check some things:

$$(m - m \cdot 1_R) \cdot r = m \cdot r - (m \cdot 1_R) \cdot r = m \cdot r - m \cdot (1_R r) = m \cdot r - m \cdot r = 0$$

therefore $m - m \cdot 1_R \in M'$. Suppose $\epsilon(m) = 0$; then $m = m \cdot 1_R$ and $m \cdot 1_R \in M'$. Therefore $m = m \cdot 1_R = m(\cdot 1_R 1_R) = (m \cdot 1_R) \cdot 1_R = 0$ because $m \cdot 1_R \in M'$. To prove that ϵ is surjective, let $(m_1, m_2 + M') \in M' \times M''$; then $(m_1, m_2 + M') = \epsilon(m_1 + m_2 \cdot 1_R)$, because

$$\begin{aligned} (m_1 + m_2 \cdot 1_R) - (m_1 + m_2 \cdot 1_R) \cdot 1_R &= m_1 + m_2 \cdot 1_R - \underbrace{m_1 \cdot 1_R}_{0} - \underbrace{m_2 \cdot 1_R \cdot 1_R}_{m_2 \cdot 1_R} \\ &= m_1 + m_2 \cdot 1_R - m_2 \cdot 1_R = m_1, \text{ and} \end{aligned}$$

$$(m_1 + m_2 \cdot 1_R) + M' = m_2 \cdot 1_R + M' = m_2 + M' \quad \text{because } (m_2 - m_2 \cdot 1_R) \in M'$$

Every R-homomorphism $f : M \rightarrow N$ can be decomposed into $f' = f|_{M'} : M' \rightarrow N'$ and $f'' : M'' \rightarrow N''$.

The module M' is a special module. It is an abelian group having a trivial multiplication with the elements of R, $M'R = 0$. On the other hand, M'' is a unitary R-module, ($m - m \cdot 1_R + M' = 0$ for all $m \in M$), therefore $M'' \in$ Mod-R.

This proves that, for a ring with identity R, the category MOD-R is composed of two categories, the category of abelian groups $\mathcal{A}b$ and the category of unitary R-modules Mod-R. In fact, the part that give us the information we are looking for, is Mod-R, i.e. the usual category that is associated with R when R has an identity, the other part $\mathcal{A}b$ comes from the factor $\mathbb{Z}$ that we have added to form $R \times \mathbb{Z}$.

In the general case we can always find the category $\mathcal{A}b$ of abelian groups with trivial multiplication inside MOD-R, but it is not so easy to avoid these modules as we have done in the case of rings with identity because they are in general not direct summands. What we have to do is to use localization techniques to avoid such modules. This is what we are going to do now.

2.2. The Category CMod-R. Let R be an idempotent ring. From now on, A will be a fixed ring with identity such that R is a two-sided ideal of A. We shall use this ring to construct the categories CMod-R, DMod-R and Mod-R, but we shall not include this ring A in the notation because we shall prove that this constructions will be independent on this choice. This will be proved in the Section 4, Proposition 2.46. This kind of ring always exists; we could use for instance the Dorroh's extension of R.

DEFINITION 2.21. Given $M \in$ Mod-A, we shall say that M is *torsion* if and only if $MR = 0$. The class of torsion modules will be denoted by $\mathcal{T}$.

PROPOSITION 2.22. *The class $\mathcal{T}$ is a TTF-class.*

PROOF. If $M \in \mathcal{T}$ and $N \leq M$, then $NR \subseteq MR = 0$ and therefore $N \in \mathcal{T}$.

If $M \in \mathcal{T}$ and $\eta : M \to N$ is an epimorphism, then $N = M/\mathrm{Ker}(\eta)$ and for all $m + \mathrm{Ker}(\eta) \in N$ and $r \in R$, $(m + \mathrm{Ker}(\eta))r = mr + \mathrm{Ker}(\eta) = 0$. This proves that $\mathcal{T}$ is closed under quotients.

Let $\{M_i : i \in I\}$ be a family of modules in $\mathcal{T}$. If $(m_i)_{i \in I} \in \prod_{i \in I} M_i$ and $r \in R$, $(m_i)_{i \in I} r = (m_i r)_{i \in I} = 0$.

Let $\{M_i : i \in I\}$ be a family of modules in $\mathcal{T}$. As $\coprod_{i \in I} M_i \leq \prod_{i \in I} M_i$ we have that $(\coprod_{i \in I} M_i)R = 0$.

Let $0 \to K \to L \to L/K \to 0$ be a short exact sequence in Mod-A, with K and L/K in $\mathcal{T}$. If $l \in L$ and $r, s \in R$, $(l + K)r = 0 + K$ therefore $lr \in K$ and $lrs = 0$, then $LR = LR^2 = 0$. □

This TTF-class define two new classes, the class $\mathcal{U}$ and the class $\mathcal{F}$, such that $(\mathcal{U}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion theories. We shall give a description of these classes.

DEFINITION 2.23. Let $M \in$ Mod-A. We shall denote by $\mathbf{t}\,(M)$ the submodule

$$\mathbf{t}\,(M) = \{m \in M : mR = 0\}.$$

If $f \in \mathrm{Hom}_A(M,N)$, we shall denote by $\mathbf{t}\,(f)$ the induced A-homomorphism from $\mathbf{t}\,(M)$ to $\mathbf{t}\,(N)$ by restriction of f. (If $m \in \mathbf{t}\,(M)$, $f(m)r = f(mr) = f(0) = 0\ \forall r \in R$, therefore $f(\mathbf{t}\,(M)) \subseteq \mathbf{t}\,(N)$).

Proposition 2.24. $\mathbf{t}$ *is an idempotent radical in* Mod-A, *and the corresponding torsion class for* $\mathbf{t}$ *is* $\mathcal{T}$.

Proof. A module M is in $\mathcal{T}$ if and only if $\mathbf{t}\,(M) = M$. □

We have therefore a description of the class $\mathcal{F}$. A module $M \in$ Mod-A is in $\mathcal{F}$ if and only if $\mathbf{t}\,(M) = 0$, i.e.,

$$M \in \mathcal{F} \text{ if and only if } \forall m \in M, mR = 0 \Rightarrow m = 0$$

Let $\mathcal{G} = \{\mathfrak{a} \le A_A : A/\mathfrak{a} \in \mathcal{T}\} = \{\mathfrak{a} \le A_A : R \le \mathfrak{a}\}$

Definition 2.25. The category CMod-R is the quotient category Mod-$(A, \mathcal{G})$.

We shall give some other descriptions of this category and the localization functor. For that we need some more definitions.

Definition 2.26. Let $M \in$ Mod-A. We shall say that M is $\mathbf{t}$-*injective* if and only if for every short exact sequence in Mod-A

$$0 \to X \to Y \to Z \to 0$$

such that $Z \in \mathcal{T}$ and for every A-homomorphism $f : X \to M$, there exists an A-homomorphism $g : Y \to M$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & & & & & \\ & & M & & & & & & \end{array}$$

is commutative.

Proposition 2.27. *Let* $M \in$ Mod-A. *The following conditions are equivalent:*

1. $M \in$ CMod-R.
2. *The canonical homomorphism*

$$\begin{array}{rccl} \lambda : & M & \to & \mathrm{Hom}_A(R, M) \\ & m & \mapsto & \begin{array}{rccc} \lambda_m : & R & \to & M \\ & r & \mapsto & mr \end{array} \end{array}$$

   is an isomorphism.
3. $M \in \mathcal{F}$ *and* M *is* $\mathbf{t}$-*injective.*

Proof. $(1 \Rightarrow 2)$. Let $M \in$ CMod-$R =$ Mod-$(A, \mathcal{G})$. Then for all $\mathfrak{a} \in \mathcal{G}$, $\iota_{\mathfrak{a}} : M \to \operatorname{Hom}_A(\mathfrak{a}, M)$ is an isomorphism. As $R \in \mathcal{G}$ we deduce that $\lambda = \iota_R$ is an isomorphism.

$(2 \Rightarrow 3)$. Note that

$$\operatorname{Ker}(\lambda) = \{m \in M : \lambda_m = 0\} = \{m \in M : \lambda_m(r) = 0 \ \forall r \in R\} = \\ = \{m \in M : mR = 0\} = \mathbf{t}\,(M).$$

Therefore, if λ is injective, $\mathbf{t}\,(M) = 0$ and then $M \in \mathcal{F}$. In order to prove that M is $\mathbf{t}$-injective, let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in Mod-A with $ZR = 0$ and $f \in \operatorname{Hom}_A(X, M)$.

The condition $ZR = 0$ implies that for all $y \in Y$, $yR \subseteq X$ and we can define $\bar{f} : Y \to \operatorname{Hom}_A(R, M)$ as $\bar{f}(y)(r) = f(yr)$. If we compose $\bar{f}$ with λ^{-1} we get $g = \lambda^{-1} \circ \bar{f}$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & f \downarrow & \swarrow g & & & & & \\ & & M & & & & & & \end{array}$$

is commutative. For, if $x \in X$ and $r \in R$, then

$$(g(x) - f(x))r = \lambda_{g(x)}(r) - f(x)r = \lambda_{\lambda^{-1} \circ f(x)}(r) - f(x)r = f(x)r - f(x)r = 0,$$

therefore $g(x) - f(x) \in \mathbf{t}\,(M) = 0$.

$(3 \Rightarrow 1)$. Let $\mathfrak{a} \in \mathcal{G}$, i.e. $R \subseteq \mathfrak{a}$.

$$\operatorname{Ker}(\iota_{\mathfrak{a}}) = \{m \in M : m\mathfrak{a} = 0\} \subseteq \{m \in M : mR = 0\} = \mathbf{t}\,(M) = 0$$

If $\mathfrak{a} \in \mathcal{G}$ the short exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ satisfies $A/\mathfrak{a} \in \mathcal{T}$. If $f \in \operatorname{Hom}_A(\mathfrak{a}, M)$, we know that there exists $g : A \to M$ that extends f, so that $f = \iota_{\mathfrak{a}}(g(1_A))$. This proves that $\iota_{\mathfrak{a}}$ is an epimorphism. □

Proposition 2.28. *The following functors are equivalent*

1. *The localization functor*
$$\mathbf{c}\,(M) = \varinjlim_{\mathfrak{a} \in \mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M)).$$
2. *The functor* $M \mapsto \operatorname{Hom}_A(R, M/\mathbf{t}\,(M))$.
3. *The functor* $M \mapsto \operatorname{Hom}_A(R \otimes_A R, M)$.

Proof. $(1 = 2)$. As $R \in \mathcal{G}$, there is a canonical homomorphism

$$j_R : \operatorname{Hom}_A(R, M/\mathbf{t}\,(M)) \to \varinjlim_{\mathfrak{a} \in \mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M)).$$

We have to prove that this is in fact an isomorphism.

Consider an element in $\varinjlim_{\mathfrak{a}\in\mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M))$. This element can be represented as a homomorphism $f : \mathfrak{a} \to M/\mathbf{t}\,(M)$ for some $\mathfrak{a} \in \mathcal{G}$. We define $\varphi(f) = f|_R : R \to M/\mathbf{t}\,(M)$. We want to prove that φ and j_R are inverse to each other, but first we have to prove that the definition of φ is not dependent on the choice of f.

Suppose we have $f : \mathfrak{a} \to M/\mathbf{t}\,(M)$ and $\bar{f} : \mathfrak{b} \to M/\mathbf{t}\,(M)$ such that they both represent the same element in $\varinjlim_{\mathfrak{a}\in\mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M))$; then there exists $\mathfrak{c} \subseteq \mathfrak{a} \cap \mathfrak{b}$ with $\mathfrak{c} \in \mathcal{G}$, such that $f|_{\mathfrak{c}} = \bar{f}|_{\mathfrak{c}}$. But, $\mathfrak{c} \in \mathcal{G} \Leftrightarrow R \subseteq \mathfrak{c}$, therefore $f|_R = \bar{f}|_R$. This proves that the definition is good.

Clearly $\varphi \circ j_R = \mathrm{id}_{\operatorname{Hom}_A(R, M/\mathbf{t}(M))}$. On the other hand, let $f : \mathfrak{a} \to M/\mathbf{t}\,(M)$ represent an element in $\varinjlim_{\mathfrak{a}\in\mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M))$. We have to prove that f and $f|_R$ represents the same element in $\varinjlim_{\mathfrak{a}\in\mathcal{G}} \operatorname{Hom}_A(\mathfrak{a}, M/\mathbf{t}\,(M))$; but this is clear because $R \in \mathcal{G}$.

(2 = 3). We shall use the canonical isomorphism (see [**3**, Lemma 19.11])

$$\operatorname{Hom}_A(R, \operatorname{Hom}_A(R, M)) \simeq \operatorname{Hom}_A(R \otimes_A R, M)$$

If we define $\lambda : M \to \operatorname{Hom}_A(R, M)$ to be the left multiplication, ($\lambda_m(r) = mr$), then $\operatorname{Ker}(\lambda) = \mathbf{t}\,(M)$. Therefore we can define the induced monomorphism $\bar{\lambda} : M/\mathbf{t}\,(M) \to \operatorname{Hom}_A(R, M)$. This monomorphism induces

$$\begin{array}{rrcl} \operatorname{Hom}_A(R, \bar{\lambda}) : & \operatorname{Hom}_A(R, M/\mathbf{t}\,(M)) & \to & \operatorname{Hom}_A(\operatorname{Hom}_A(R, M)) \\ & f & \mapsto & \bar{\lambda} \circ f \end{array}$$

Moreover

$$\begin{aligned} \operatorname{Ker}(\operatorname{Hom}_A(R, \bar{\lambda})) &= \{f : R \to M/\mathbf{t}\,(M) : \bar{\lambda} \circ f = 0\} \\ &= \{f : R \to M/\mathbf{t}\,(M) : (\bar{\lambda} \circ f)(r) = 0\ \forall r \in R\} \\ &= \{f : R \to M/\mathbf{t}\,(M) : \bar{\lambda}_{f(r)}(s) = 0\ \forall r, s \in R\} \\ &= \{f : R \to M/\mathbf{t}\,(M) : f(r)s = 0\ \forall r, s \in R\} \\ &= \{f : R \to M/\mathbf{t}\,(M) : f(r) = 0\ \forall r \in R\} = 0. \end{aligned}$$

To prove that $\operatorname{Hom}_A(R, \bar{\lambda})$ is an epimorphism, consider a homomorphism $h : R \otimes_A R \to M$. The kernel of the canonical homomorphism $\mu : R \otimes_A R \to R$ is in $\mathcal{T}$, therefore $h(\operatorname{Ker}(\mu)) \subseteq \mathbf{t}\,(M)$ and we can induce a homomorphism $\bar{h} : R \to M/\mathbf{t}\,(M)$. Using the isomorphism

$$\operatorname{Hom}_A(R, \operatorname{Hom}_A(R, M)) \simeq \operatorname{Hom}_A(R \otimes_A R, M)$$

it is straight forward to prove that $\bar{h}$ is the inverse image of the corresponding $h \in \operatorname{Hom}_A(R, \operatorname{Hom}_A(R, M))$. ☐

PROPOSITION 2.29. *The functor* $\mathbf{c}(M) = \mathrm{Hom}_A(R, M/\mathbf{t}(M))$ *has the following properties:*

1. $\forall M \in \text{Mod-}A$, $\mathbf{c}(M) \in \text{CMod-}R$.
2. $\forall M \in \text{Mod-}A$, *the canonical homomorphism*

$$\begin{array}{rcll} \iota : & M & \to & \mathbf{c}(M) = \mathrm{Hom}_A(R, M/\mathbf{t}(M)) \\ & m & \mapsto & \iota_m \\ & & & \iota_m(r) = mr + \mathbf{t}(M) \end{array}$$

satisfies $\mathrm{Ker}(\iota), \mathrm{Coker}(\iota) \in \mathcal{T}$.

Suppose $\bar{\mathbf{c}} : \text{Mod-}A \to \text{CMod-}R$ *is a functor such that for all* $M \in \text{Mod-}A$ *there exists a natural homomorphism* $\bar{\iota} : M \to \bar{\mathbf{c}}(M)$ *with* $\mathrm{Ker}(\bar{\iota}), \mathrm{Coker}(\bar{\iota}) \in \mathcal{T}$. *Then* $\bar{\mathbf{c}}$ *is equivalent to* $\mathbf{c}$.

PROOF. The conditions 1 and 2 can be checked directly.

If $\mathrm{Im}(\bar{\iota}) \subseteq \bar{\mathbf{c}}(M) \in \text{CMod-}R$, then $\mathrm{Im}(\bar{\iota})$ is torsion free and then $\mathbf{t}(M) \subseteq \mathrm{Ker}(\bar{\iota})$. But $\mathrm{Ker}(\bar{\iota}) \in \mathcal{T}$; therefore $\mathrm{Ker}(\bar{\iota}) = \mathbf{t}(M)$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{M}{\mathbf{t}(M)} & \longrightarrow & \mathbf{c}(M) & \longrightarrow & \frac{\mathbf{c}(M)}{M/\mathbf{t}(M)} & \longrightarrow & 0 \\ & & {\scriptstyle \bar{\iota}'}\downarrow & \swarrow {\scriptstyle g} & & & & & \\ & & \bar{\mathbf{c}}(M) & & & & & & \end{array}$$

where $\bar{\iota}'$ is the induced morphism. Using the fact that $\bar{\mathbf{c}}(M)$ is $\mathbf{t}$-injective, we can find a homomorphism g that makes the diagram commutative. This morphism is unique. In the same fashion we can find a homomorphism f such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \frac{M}{\mathbf{t}(M)} & \longrightarrow & \bar{\mathbf{c}}(M) & \longrightarrow & \frac{\bar{\mathbf{c}}(\mathbf{M})}{M/\mathbf{t}(M)} & \longrightarrow & 0 \\ & & {\scriptstyle \iota'}\downarrow & \swarrow {\scriptstyle f} & & & & & \\ & & \mathbf{c}(M) & & & & & & \end{array}$$

The homomorphisms $f \circ g$ and $g \circ f$ fix the elements in $M/\mathbf{t}(M)$; therefore they have to be the identity morphisms in $\mathbf{c}(M)$ and $\bar{\mathbf{c}}(M)$, respectively because $\mathrm{Coker}(\iota)$ and $\mathrm{Coker}(\bar{\iota})$ are in $\mathcal{T}$. This proves that $\mathbf{c}(M) \simeq \bar{\mathbf{c}}(M)$.

Using the uniqueness of the morphisms, it is not difficult to prove that this isomorphism is natural. □

PROPOSITION 2.30. *Let* $M \in \text{CMod-}R$ *and let* $N \subseteq M$ *be an* A-*submodule. Then* M/N *is torsion-free if and only if* $N \in \text{CMod-}R$.

PROOF. Suppose first that M/N is torsion-free and let $h : R \to N$ be an A-homomorphism. If we compose h with the canonical inclusion

$j : N \to M$ we obtain $j \circ h : R \to M$ and using the fact that $M \in$ CMod-R we deduce that there exists $m \in M$ such that $j(h(r)) = mr$ for all $r \in R$, and then $h(r) = mr$ for all $r \in R$. What we have to prove is that $m \in N$, but this is clear because if $mr \in N$ for all $r \in R$, then $(m+N)R = 0$ in M/N and applying that M/N is torsion-free we would obtain $m \in N$.

On the other hand suppose $N \in$ CMod-R and let $m+N \in \mathbf{t}(M/N)$. Then we define $h : R \to M$ by $h(r) = mr$ for all $r \in R$. As $m + N \in \mathbf{t}(M/N)$ we deduce that in fact $\mathrm{Im}(h) \subseteq N$ and applying $N \in$ CMod-R we deduce that there exists $n \in N$ such that $h(r) = nr$ for all $r \in R$. But then $(m - n)R = 0$ and from this we deduce that $n = m$ and $m + N = 0$. □

2.3. The Category DMod-R**.** Following the idea of "eliminating" the modules with trivial multiplication by elements of R, there are other ways of proceeding. In this subsection we shall explore another construction based on properties dual to the previous ones.

Definition 2.31. Let $M \in$ Mod-A. We shall denote by $\mathbf{u}(M)$ the following submodule of M

$$\mathbf{u}(M) = MR = \left\{ \sum_{\text{finite}} m_i r_i : m_i \in M, r_i \in R \right\}$$

Proposition 2.32. *The functor* $\mathbf{u}$ *is the idempotent radical corresponding to the torsion theory* $(\mathcal{U}, \mathcal{T})$ *given above.*

Proof. The class $\mathcal{T}$ is the torsion free class for $\mathbf{u}$ because $M \in \mathcal{T}$ if and only if $\mathbf{u}(M) = MR = 0$. This property defines completely the corresponding idempotent radical. □

Definition 2.33. Let $M \in$ Mod-A. We shall say that M is *unitary* if and only if $M \in \mathcal{U}$, i.e., $MR = \mathbf{u}(M) = M$.

Definition 2.34. Let $M \in$ Mod-A. We shall say that M is **u**-*codivisible* if and only if for every short exact sequence in Mod-A

$$0 \to X \to Y \to Z \to 0$$

with $\mathbf{u}(X) = 0$ and every homomorphism $f : M \to Z$, there exists a homomorphism $g : M \to Y$ such that the diagram

$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & \overset{g}{\swarrow} & \downarrow f & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

is commutative.

PROPOSITION 2.35. *Let $M \in$ Mod-A. The following conditions are equivalent:*

1. *The canonical homomorphism*

$$\begin{array}{rrcl} \mu: & M \otimes_A R & \to & M \\ & m \otimes r & \mapsto & mr \end{array}$$

   is an isomorphism.

2. *M is unitary and $\mathbf{u}$-codivisible.*

A module that satisfies these properties is called coclosed.

PROOF. Suppose first that M is unitary and $\mathbf{u}$-codivisible. The condition $MR = M$ is equivalent to the surjectivity of μ. Consider the diagram given by

$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & & \downarrow {\scriptstyle id_M} & & \\ 0 & \longrightarrow & \mathrm{Ker}(\mu) & \longrightarrow & M \otimes_A R & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

If $\sum_{i=1}^{k} m_i \otimes r_i \in \mathrm{Ker}(\mu)$ and $r \in R$,

$$(\sum_{i=1}^{k} m_i \otimes r_i)r = \sum_{i=1}^{k} m_i r_i \otimes r = 0 \otimes r = 0.$$

This proves that $\mathbf{u}\,(\mathrm{Ker}(\mu)) = \mathrm{Ker}(\mu)R = 0$, and using that M is $\mathbf{u}$-codivisible we can find a homomorphism $g : M \to M \otimes_A R$ such that $\mu \circ g = \mathrm{id}_M$. This proves that $\mathrm{Ker}(\mu)$ is a direct summand of $M \otimes_A R \in \mathcal{U}$. Therefore, $\mathrm{Ker}(\mu) \in \mathcal{U}$ because it is a quotient of $M \otimes_A R$. Then $\mathrm{Ker}(\mu) = \mathrm{Ker}(\mu)R = 0$ and μ is an isomorphism as we claimed.

Conversely, suppose μ is an isomorphism. Then $MR = \mathrm{Im}(\mu) = M$. To prove that M is $\mathbf{u}$-codivisible consider the following diagram

$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & & \downarrow {\scriptstyle h} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

where the row is exact and $\mathbf{u}\,(X) = XR = 0$. By applying the functor $- \otimes_A R$ we get a new commutative diagram with exact rows and with the canonical morphisms in the columns

$$\begin{array}{ccccccccc}
 & & X \otimes_A R & \longrightarrow & Y \otimes_A R & \longrightarrow & Z \otimes_A R & \longrightarrow & 0 \\
 & f & \downarrow & \overset{g}{\searrow} & \downarrow & & \downarrow & k & \\
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0
\end{array}$$

By our hypothesis about X, we know that $f = 0$. Hence, $g = 0$ and we deduce from this fact that k factors through Y, giving a monomorphism $j : Z \otimes R \to Y$. Now, we get in this way a homomorphism $j \circ (h \otimes \mathrm{id}) : M \otimes R \to Y$, which, composed with the assumed isomorphism μ, gives a morphism $M \to Y$ that clearly is a lifting of the given morphism h. $\square$

Definition 2.36. We shall define DMod-R as the full subcategory of Mod-A that contains all the coclosed modules.

Lemma 2.37. *Let $M \in$ Mod-A and $D \in A$-Mod such that $RD = D$. Then, for all $m \in \mathbf{t}(M)$ and $d \in D$, $m \otimes d = 0 \in M \otimes_A D$.*

Proof. Clear. $\square$

Proposition 2.38. *The functor $\mathbf{d} = \mathbf{u}(-) \otimes_A R$ has the following properties:*

1. *$\forall M \in$ Mod-A, $\mathbf{d}(M) \in$ DMod-R.*
2. *$\forall M \in$ Mod-A, the canonical homomorphism*

$$\begin{array}{rrcl} \mu : & \mathbf{d}(M) = \mathbf{u}(M) \otimes_A R & \to & M \\ & mr \otimes s & \mapsto & mrs \end{array}$$

satisfies $\mathrm{Ker}(\mu), \mathrm{Coker}(\mu) \in \mathcal{T}$.

Suppose $\bar{\mathbf{d}} :$ Mod-$A \to$ DMod-R is a functor such that for all $M \in$ Mod-A there exists a natural homomorphism $\bar{\mu} : \bar{\mathbf{d}}(M) \to M$ with $\mathrm{Ker}(\bar{\mu}), \mathrm{Coker}(\bar{\mu}) \in \mathcal{T}$. Then $\bar{\mathbf{d}}$ is equivalent to $\mathbf{d}$.

Proof. First we shall prove that $\mathrm{Ker}(\mu) \in \mathcal{T}$. Suppose $\sum_i m_i r_i \otimes s_i \in \mathrm{Ker}(\mu)$, i.e., $\sum_i m_i r_i s_i = 0$, and let $t \in R$.

$$(\sum_i m_i r_i \otimes s_i)t = \sum_i m_i r_i s_i \otimes t = 0 \otimes t = 0.$$

Also $\mathrm{Coker}(\mu) = M/MR$; therefore $\mathrm{Coker}(\mu)R = 0$.

Consider the short exact sequence given by

$$0 \to \mathrm{Ker}(\mu) \to \mathbf{d}(M) \to MR \to 0.$$

If we apply the tensor functor $- \otimes_A R$ we get the exact sequence

$$\mathrm{Ker}(\mu) \otimes_A R \to \mathbf{d}(M) \otimes_A R \to \underbrace{MR \otimes_A R}_{=\mathbf{d}(M)} \to 0$$

But the kernel of the morphism $\mathbf{d}(M)\otimes_A R \to MR\otimes_A R$ is formed with the elements $\sum_i k_i \otimes r_i \in \mathbf{d}(M)\otimes_A R$ with $k_i \in \mathrm{Ker}(\mu)$ and $r_i \in R$. But $k_i \otimes r_i = 0$ because of the previous lemma; therefore $\mathbf{d}(M)\otimes_A R \to MR\otimes_A R$ is an isomorphism.

If $\bar{\mathbf{d}}(M) \in$ DMod-R then $\bar{\mathbf{d}}(M)R = \bar{\mathbf{d}}(M)$ and $\mathrm{Im}(\bar{\mu})R = \mathrm{Im}(\bar{\mu})$. If $\mathrm{Coker}(\bar{\mu})R = 0$ then $MR \subseteq \mathrm{Im}(\bar{\mu})$ and therefore $MR = \mathrm{Im}(\bar{\mu})$.

Consider the short exact sequence given by

$$0 \to \mathrm{Ker}(\bar{\mu}) \to \bar{\mathbf{d}}(M) \to MR \to 0.$$

By applying the tensor functor $-\otimes_A R$ we obtain the exact sequence

$$\mathrm{Ker}(\bar{\mu})\otimes_A R \to \underbrace{\bar{\mathbf{d}}(M)\otimes_A R}_{=\bar{\mathbf{d}}(M)} \to MR\otimes_A R \to 0$$

Suppose $\sum_{i=1}^n k_i \otimes r_i \in \bar{\mathbf{d}}(M)\otimes_A R$ with $k_i \in \mathrm{Ker}(\bar{\mu})$ for all $i = 1,\cdots,n$. For every $r_i \in R = R^2$ we can find elements $s_{ij}, t_{ij} \in R$ such that $r_i = \sum_j s_{ij}t_{ij}$. Then, using the fact that $\mathrm{Ker}(\bar{\mu})R = 0$ we obtain that

$$\sum_i k_i \otimes r_i = \sum_{i,j} k_i \otimes s_{ij}t_{ij} = \sum_{i,j} k_i s_{ij} \otimes t_{ij} = 0$$

This proves that $\bar{\mathbf{d}}(M)\otimes_A R \to MR\otimes_A R$ is an isomorphism. Using the fact that $\mathbf{d}(M) \in$ DMod-R and $\bar{\mathbf{d}}(M) \in$ DMod-R (i.e. $\mathbf{d}(M)\otimes_A R \simeq \mathbf{d}(M)$ and $\bar{\mathbf{d}}(M)\otimes_A R \simeq \bar{\mathbf{d}}(M)$) we conclude

$$\bar{\mathbf{d}}(M) \simeq \bar{\mathbf{d}}(M)\otimes_A R \simeq \mathbf{d}(M)\otimes_A R \simeq \mathbf{d}(M)\otimes_A R \simeq \mathbf{d}(M).$$

Because of the way we have defined this isomorphism, it is not difficult to prove that this isomorphism is natural. □

Corollary 2.39. *The following functors are equivalent*

1. $\mathbf{u}(-)\otimes_A R$
2. $-\otimes_A R\otimes_A R$

Proof. For every module $M \in$ Mod-A, $M\otimes_A R$ is unitary and therefore $M\otimes_A R\otimes_A R \in$ DMod-R. The canonical homomorphism

$$\begin{array}{rccc} \bar{\mu}: & M\otimes_A R\otimes_A R & \to & M \\ & m\otimes r\otimes s & \mapsto & mrs \end{array}$$

satisfies $\mathrm{Coker}(\bar{\mu}) = M/MR \in \mathcal{T}$ and it is easy to check that $\mathrm{Ker}(\bar{\mu})$ is also in $\mathcal{T}$. Then, the uniqueness of the previous proposition makes us deduce our claim. □

For the next result we have to use a technical result about tensor products.

LEMMA 2.40. *Let A be a ring with identity, $\{n_\lambda : \lambda \in \Lambda\}$ a generating set of the module ${}_AN \in A\text{-Mod}$ and $\{m_\lambda : \lambda \in \Lambda\}$ a family of elements in the module $M_A \in \text{Mod-}A$, with $\{\lambda \in \Lambda : m_\lambda \neq 0\}$ finite.*

Then $\sum_{\lambda\in\Lambda} m_\lambda \otimes n_\lambda = 0$ in $M \otimes_A N$ if and only if there exist elements $\{z_\omega \in M : \omega \in \Omega\}$ and $\{a_{\lambda\omega} \in A : \lambda \in \Lambda, \omega \in \Omega\}$ such that

1. *$\{(\lambda, \omega) \in \Lambda \times \Omega : a_{\lambda\omega} \neq 0\}$ is finite.*
2. *$\sum_{\lambda\in\Lambda} a_{\lambda\omega} n_\lambda = 0$ for all $\omega \in \Omega$.*
3. *$m_\lambda = \sum_{\omega\in\Omega} z_\omega a_{\lambda\omega}$.*

PROOF. See [**15**, Kapitel 2, 12.10]. □

PROPOSITION 2.41. *Let $M \in \text{DMod-}R$, let $K \subseteq M$ be an A-submodule of M, and let $p : M \to M/K$ be the canonical projection. Then K is unitary if and only if $M/K \in \text{DMod-}R$.*

PROOF. First suppose $M/K \in \text{DMod-}R$. Let us denote by η_M and $\eta_{M/K}$ the canonical isomorphisms $\eta_M : M \otimes R \to M$ and $\eta_{M/K} : M/K \otimes R \to M/K$. Let $k \in K$. As in particular, $k \in M$ we can find elements $m_i \in M$ and $r_i \in R$ such that $k = \sum_i m_i r_i$. As $k \in K$ $p(k) = \sum_i p(m_i) r_i = 0$ and using the fact that $\eta_{M/K}$ is an isomorphism, we obtain that $\sum_i p(m_i) \otimes r_i = 0$, and then $\sum_i m_i \otimes r_i \in \text{Ker}(p \otimes \text{id}_R)$, i.e. we can find elements $\bar{k}_j \in K$ and $\bar{r}_j \in R$ such that $\sum_i m_i \otimes r_i = \sum_j \bar{k}_j \otimes \bar{r}_j$. But then we deduce $k = \sum_i m_i r_i = \sum_j \bar{k}_j \bar{r}_j \in KR$, an that $KR = K$.

On the other hand suppose K is unitary; then M/K is also unitary because it is a quotient object of a unitary object. What we have to prove is that the morphism $\mu : M/K \otimes_A R \to M/K$ is a monomorphism. For that, suppose that $\sum_{i=1}^n (m_i + K) r_i = 0$. Then $\sum_i m_i r_i \in K$ and as K is unitary we can find elements $m_i \in K$ and $r_i \in R$ with $i = n+1, \cdots, t$ such that $0 = \sum_{i=1}^t m_i r_i$. If we apply that $\sum_{i=1}^t m_i \otimes r_i = 0$ in $M \otimes_A R$, we are in the situation of Lemma 2.40 but we have to extend the set $\{r_1, \cdots, r_t\}$ to a generating set of R over A on the left, say $\{r_i : i \in I\}$, and we can do it defining $m_i = 0$ for the the values $i \in I \setminus \{1, \cdots, t\}$. Using Lemma 2.40 we can find elements $a_{ki} \in A$ with $k = 1, \cdots, l$, almost all of them zero, and $\bar{m}_1, \cdots, \bar{m}_l \in M$ such that

1. $\sum_{i\in I} a_{ki} r_i = 0$ for all $k = 1, \cdots, l$.
2. $\sum_{k=1}^l \bar{m}_k a_{ki} = m_i$ for all $i \in I$.

From this we deduce

$$\sum_{i=1}^n (m_i + K) \otimes r_i = \sum_{i\in I} (m_i + K) \otimes r_i =$$
$$\sum_{i\in I} \sum_{k=1}^l (m_k a_{ki} + K) \otimes r_i = \sum_{k=1}^l (m_k + K) \otimes \sum_{i\in I} a_{ki} r_i = 0.$$

□

2.4. The Category Mod-R.

DEFINITION 2.42. We shall define the category Mod-R as the full subcategory of Mod-A which contains the modules M that satisfy $\mathbf{u}(M) = M$ and $\mathbf{t}(M) = 0$, i.e., $\mathcal{U} \cap \mathcal{F}$.

This category can be considered as a category between CMod-R and DMod-R. This category will be very useful in order to study properties like finite generatedness.

Given a module $M \in$ Mod-A, there are two different ways of defining a module in Mod-R associated to it; they are $MR/\mathbf{t}(MR)$ and $(M/\mathbf{t}(M))R$. We are going to prove that this modules are equal.

(We shall denote by $\mathbf{t}^{-1}$ the functor given by $\mathbf{t}^{-1}(M) = M/\mathbf{t}(M)$ for any module M, and defined for morphisms in the natural way).

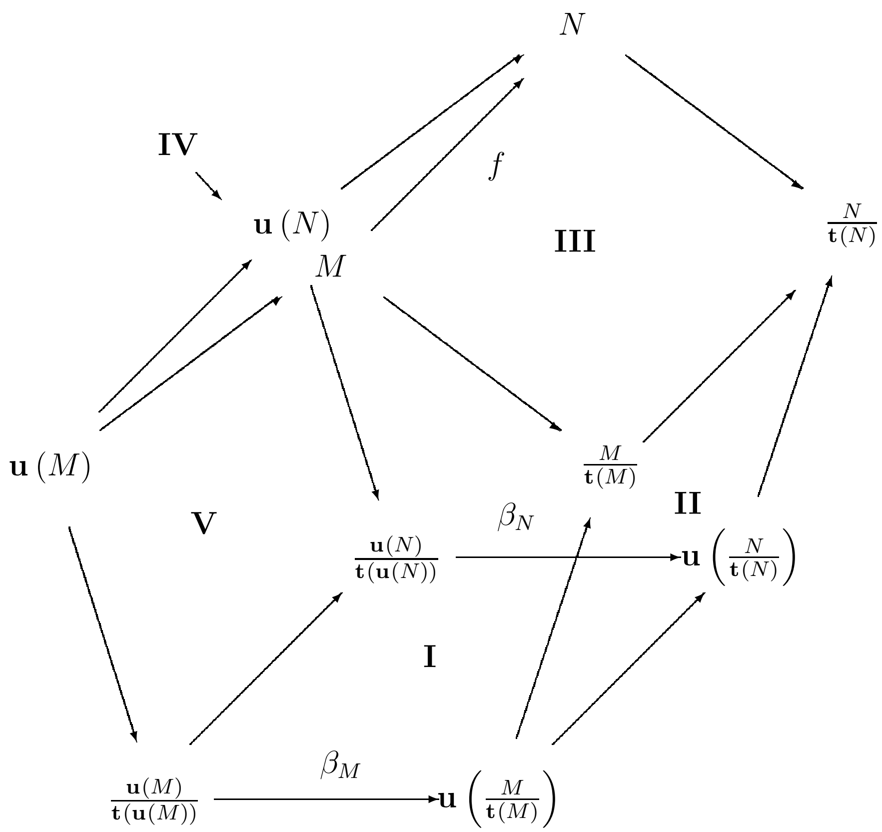
PROPOSITION 2.43. *There exists a natural equivalence between the functors* $\mathbf{u} \circ \mathbf{t}^{-1}$ *and* $\mathbf{t}^{-1} \circ \mathbf{u}$.

PROOF. Consider the canonical homomorphism $\alpha = \mathbf{u}(M) \to M \to M/\mathbf{t}(M)$. The kernel of α is $\mathbf{u}(M) \cap \mathbf{t}(M) = \mathbf{t}(\mathbf{u}(M))$, this equality comes from the left exactness of the functor $\mathbf{t}$. The image of α is a unitary module, because it is a quotient of $\mathbf{u}(M)$. Therefore $\mathrm{Im}(\alpha) \subseteq \mathbf{u}(M/\mathbf{t}(M))$. We have then the morphism

$$\begin{array}{rrcl} \beta_M : & \mathbf{u}(M)/\mathbf{t}(\mathbf{u}(M)) & \to & \mathbf{u}(M/\mathbf{t}(M)) \\ & \sum m_i r_i + \mathbf{t}(\mathbf{u}(M)) & \mapsto & \sum (m_i + \mathbf{t}(M)) r_i \end{array}$$

We have to prove that β_M is an isomorphism. The injectivity is clear because $\mathrm{Ker}(\alpha) = \mathbf{t}(\mathbf{u}(M))$. The surjectivity is also clear, if we have an element $\sum (m_i + \mathbf{t}(M)) r_i$, we can take $\sum m_i r_i + \mathbf{t}(M) \in \beta_M^{-1}(\sum (m_i + \mathbf{t}(M)) r_i)$.

In order to prove the naturality of β_M, suppose $f : M \to N$ and consider the following diagram:



We have to prove that the square **I** commutes and we know that the squares **II**,**III**,**IV**, **V** and the pentagons commute. From this we deduce that the following morphisms are equal

$$\mathbf{u}(M) \to \frac{\mathbf{u}(M)}{\mathbf{t}(\mathbf{u}(M))} \to \frac{\mathbf{u}(N)}{\mathbf{t}(\mathbf{u}(N))} \to \mathbf{u}\left(\frac{N}{\mathbf{t}(N)}\right) \to \frac{N}{\mathbf{t}(N)}$$

$$\mathbf{u}(M) \to \frac{\mathbf{u}(M)}{\mathbf{t}(\mathbf{u}(M))} \to \mathbf{u}\left(\frac{M}{\mathbf{t}(M)}\right) \to \mathbf{u}\left(\frac{N}{\mathbf{t}(N)}\right) \to \frac{N}{\mathbf{t}(N)}$$

And using the fact that $\mathbf{u}(M) \to \frac{\mathbf{u}(M)}{\mathbf{t}(\mathbf{u}(M))}$ is an epimorphism and $\mathbf{u}\left(\frac{N}{\mathbf{t}(N)}\right) \to \frac{N}{\mathbf{t}(N)}$ is a monomorphism we obtain the commutativity of the square **I**. $\square$

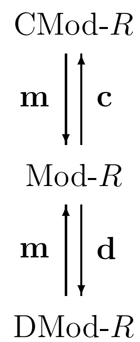
Definition 2.44. The functor $\mathbf{u} \circ \mathbf{t}^{-1}$ will be denoted by $\mathbf{m}$: Mod-$A \to$ Mod-R. This functor is also $\mathbf{t}^{-1} \circ \mathbf{u}$ up to natural isomorphism.

3. The Equivalence of the Categories

In this section we shall prove that the three categories that we consider, are in fact equivalent.

THEOREM 2.45. *Let R be an idempotent ring. Then, the categories* CMod-R*,* Mod-R *and* DMod-R *are equivalent.*

PROOF. Consider the following diagram of categories and functors.



The functor $\mathbf{m}$ on the modules in CMod-R that are torsion-free is the same as the functor $\mathbf{u}$, and on the modules in DMod-R that are unitary is the same as $\mathbf{t}^{-1}$. Because of this we shall use the notation $\mathbf{u}$ and $\mathbf{t}^{-1}$ instead of $\mathbf{m}$ in these cases.

We have to prove the following facts.

1. CMod-R and Mod-R are equivalent.
   (a) For every $M \in$ CMod-R there exists a natural isomorphism between M and $\mathbf{c}(\mathbf{u}(M))$.
   (b) For every $M \in$ Mod-R there exists a natural isomorphism between M and $\mathbf{u}(\mathbf{c}(M))$.
2. DMod-R and Mod-R are equivalent.
   (a) For every $M \in$ DMod-R there exists a natural isomorphism between M and $\mathbf{d}(\mathbf{t}^{-1}(M))$.
   (b) For every $M \in$ Mod-R there exists a natural isomorphism between M and $\mathbf{t}^{-1}(\mathbf{d}(M))$.

(**1**) CMod-R and Mod-R are equivalent.

(**1.a**) For every $M \in$ CMod-R there exists a natural isomorphism between M and $\mathbf{c}(\mathbf{u}(M))$.

If $M \in$ CMod-R, M is torsion free and then $\mathbf{u}(M) \subseteq M$ is also torsion free. Therefore $\mathbf{c}(\mathbf{u}(M)) = \mathrm{Hom}_A(R, \mathbf{u}(M))$. The isomorphism is defined as follows:

$$\begin{array}{rccl} \lambda: & M & \to & \mathrm{Hom}_A(R, \mathbf{u}(M)) \\ & m & \mapsto & \begin{array}{rccc} \lambda_m: & R & \to & \mathbf{u}(M) \\ & r & \mapsto & mr \end{array} \end{array}$$

$$\begin{aligned}\mathrm{Ker}(\lambda) &= \{m \in M : \lambda_m = 0\} \\ &= \{m \in M : \lambda_m(r) = 0 \ \forall r \in R\} \\ &= \{m \in M : mR = 0\} = \mathbf{t}(M) = 0\end{aligned}$$

Let $f : R \to \mathbf{u}(M)$ be any homomorphism and let $j : \mathbf{u}(M) \to M$ denote the canonical inclusion. As $M \in$ CMod-R, for the morphism $j \circ f : R \to M$ there exists $m \in M$ such that $(j \circ f)(r) = mr = j(mr)$ for all $r \in R$. If we apply the fact that j is a monomorphism, then $f(r) = mr \ \forall r \in R$ and therefore $\lambda(m) = \lambda_m = f$. This proves that λ is surjective.

In order to prove the naturality, let $h : M \to N$ be a homomorphism with M and N in CMod-R. We have to check that

$$\mathrm{Hom}_A(R, \mathbf{u}(h)) \circ \lambda = \lambda \circ h$$

This is equivalent to the property $\lambda_{h(m)} = h \circ \lambda_m$ for all $m \in M$; but this is true because

$$\begin{aligned}\lambda_{h(m)}(r) &= h(m)r = h(mr) = \\ &= h(\lambda_m(r)) = (h \circ \lambda_m)(r) \ \forall r \in R\end{aligned}$$

(**1.b**) For every $M \in$ Mod-R there exists a natural isomorphism between M and $\mathbf{u}(\mathbf{c}(M))$.

Let $M \in$ Mod-R. Then M is torsion free and therefore $\mathbf{c}(M) = \mathrm{Hom}_A(R, M)$. Consider the homomorphism $\lambda : M \to \mathrm{Hom}_A(R, M)$ given above. Note that $\mathrm{Ker}(\lambda) = \mathbf{t}(M) = 0$. Therefore λ is a monomorphism. The condition $MR = M$ implies $\mathrm{Im}(\lambda)R = \mathrm{Im}(\lambda)$,and therefore $\mathrm{Im}(\lambda) \subseteq \mathbf{u}(\mathbf{c}(M))$. We can consider the restriction of the canonical homomorphism $\lambda : M \to \mathbf{u}(\mathbf{c}(M))$ and we have proved that λ is injective. What we have to prove is that $\mathrm{Im}(\lambda) = \mathbf{u}(\mathbf{c}(M))$.

Let $\sum_i f_i r_i \in \mathbf{u}(\mathbf{c}(M))$ with $r_i \in R$ and $f_i : R \to M$ in $\mathbf{c}(M)$. What we are going to prove is that $\sum_i f_i r_i = \lambda_{\sum_i f_i(r_i)} \in \mathrm{Im}(\lambda)$. For any $r \in R$,

$$\sum_i f_i r_i(r) = \sum_i f_i(r_i r) = \sum_i f_i(r_i)r = \lambda_{\sum_i f_i(r_i)}(r)$$

and this proves our claim.

To prove the naturality of the isomorphism, let $M, N \in$ Mod-R and $h : M \to N$. We have to prove that $\mathbf{u}(\mathrm{Hom}_A(R, h)) \circ \lambda = \lambda \circ h$. Let $m \in M$ and $r \in R$ then

$$\begin{aligned}&(\mathbf{u}(\mathrm{Hom}_A(R, h)) \circ \lambda)(m)(r) \\ &= \mathbf{u}(\mathrm{Hom}_A(R, h))(\lambda_m(r)) \\ &= h(mr) = h(m)r = \lambda_{h(m)}(r).\end{aligned}$$

and therefore $(\mathbf{u}(\mathrm{Hom}_A(R, h)) \circ \lambda)(m) = \lambda_{h(m)}$. Then $\mathbf{u}(\mathrm{Hom}_A(R, h)) \circ \lambda = \lambda \circ h$, that is the naturality condition.

(**2**) DMod-R and Mod-R are equivalent.

(**2.a**) For every $M \in$ DMod-R there exists a natural isomorphism between M and $\mathbf{d}\left(\mathbf{t}^{-1}(M)\right)$.

If $M \in$ DMod-R, $M/\mathbf{t}(M)$ is unitary and then $\mathbf{d}(M/\mathbf{t}(M)) = M/\mathbf{t}(M) \otimes_A R$. Consider the short exact sequence

$$0 \to \mathbf{t}(M) \to M \to M/\mathbf{t}(M) \to 0$$

and apply the tensor functor $- \otimes_A R$ to obtain

$$\mathbf{t}(M) \otimes_A R \to \underbrace{M \otimes_A R}_{=M} \xrightarrow{\eta} \underbrace{M/\mathbf{t}(M) \otimes_A R}_{=\mathbf{d}\left(\mathbf{t}^{-1}(M)\right)} \to 0$$

The morphism we have to prove an isomorphism is η. Because of the definition, η is an epimorphism. To prove that η is a monomorphism let $k \in \operatorname{Ker}(\eta)$. Then $k = \sum_i t_i \otimes r_i \in M \otimes_A R$ with $t_i \in \mathbf{t}(M)$ and $r_i \in R$. But as $t_i R = 0$, $t_i \otimes r_i = 0$ for all i and therefore $k = 0$.

To prove the naturality of this isomorphism, let $M, N \in$ DMod-R and $h : M \to N$. Consider

$$\begin{array}{ccccccc}
M & \xrightarrow{\sim} & M \otimes_A R & \xrightarrow{\eta} & M/\mathbf{t}(M) \otimes_A R & = & \mathbf{d}\left(\mathbf{t}^{-1}(M)\right) \\
{\scriptstyle h}\downarrow & & {\scriptstyle h \otimes_A R}\downarrow & & \downarrow{\scriptstyle \mathbf{t}^{-1}(h) \otimes_A R} & & \downarrow{\scriptstyle \mathbf{d}\left(\mathbf{t}^{-1}(h)\right)} \\
N & \xrightarrow{\sim} & N \otimes_A R & \xrightarrow{\eta} & N/\mathbf{t}(N) \otimes_A R & = & \mathbf{d}\left(\mathbf{t}^{-1}(N)\right)
\end{array}$$

The commutativity of this diagram proves the naturality of the isomorphism.

(**2.b**) For every $M \in$ Mod-R there exists a natural isomorphism between M and $\mathbf{t}^{-1}(\mathbf{d}(M))$.

Let $M \in$ Mod-R. The condition $MR = M$ implies that $\mathbf{d}(M) = M \otimes_A R$. What we are going to prove is that the kernel of $\mu : M \otimes_A R \to M$, $(\mu(m \otimes r) = mr)$, is $\mathbf{t}(M)$. This would give us the isomorphism that we are looking for.

$\boxed{\mathbf{t}(M \otimes_A R) \supseteq \operatorname{Ker}(\mu)}$. Suppose $\sum_i m_i \otimes r_i \in \operatorname{Ker}(\mu)$. Then $\sum_i m_i r_i = 0$ and therefore, for all $r \in R$,

$$(\sum_i m_i \otimes r_i) r = \sum_i m_i r_i \otimes r = 0 \otimes r = 0$$

$\boxed{\mathbf{t}(M \otimes_A R) \subseteq \operatorname{Ker}(\mu)}$. Suppose $\sum_i m_i \otimes r_i \in \mathbf{t}(M \otimes_A R)$. Then $\mu(\sum_i m_i \otimes r_i) \in \mathbf{t}(M) = 0$ and therefore $\sum_i m_i \otimes r_i \in \operatorname{Ker}(\mu)$ as we claimed.

The morphism μ induces an isomorphism $\bar{\mu} : M \otimes_A R/\mathbf{t}(M \otimes_A R) \to M$. To prove the naturality, let $h : M \to N$ be a homomorphism between $M, N \in$ Mod-R.

Clearly the following diagram commutes

$$\begin{array}{ccccc}
\mathbf{t}\,(M \otimes_A R) & \longrightarrow & M \otimes_A R & \longrightarrow & M \\
\scriptstyle{\mathbf{t}(h\otimes_A R)}\downarrow & & \scriptstyle{h\otimes_A R}\downarrow & & \scriptstyle{h}\downarrow \\
\mathbf{t}\,(N \otimes_A R) & \longrightarrow & N \otimes_A R & \longrightarrow & N
\end{array}$$

And this is equivalent to the naturality of the isomorphism. □

4. The Independence of the Base Ring

In the previous sections we have made several constructions inside the category Mod-A where A is a ring with identity such that R is a two-sided ideal of it. We claimed that these constructions are not dependent on the ring A that we chose. This is not completely true. The classes $\mathcal{T}$, $\mathcal{F}$ and $\mathcal{U}$, and properties like being $\mathbf{t}$-injective or $\mathbf{u}$-codivisible are dependent on it. Nevertheless, the categories CMod-R, Mod-R and DMod-R are not dependent on this choice. This is what we are going to prove in this section.

When we studied the category MOD-R, we mentioned that one possible choice for the ring A could be the Dorroh's extension of R, $R \times \mathbb{Z}$. In order to prove the independence of the choice we shall suppose that we have made the constructions for the ring $R \times \mathbb{Z}$ and we shall obtain that if we choose another A, the result is the same.

Proposition 2.46. *Let B be the Dorroh's extension of R, i.e., $B = R \times \mathbb{Z}$. Form the categories* CMod-R*,* Mod-R *and* DMod-R *for the ring B and suppose that A is a ring with identity such that R is a two-sided ideal of it. Then*

1. CMod-R *is the full subcategory of* Mod-A *formed with the modules M_A such that $\mathrm{Hom}_A(R, M) \simeq M$ in the canonical way.*
2. DMod-R *is the full subcategory of* Mod-A *formed with the modules M_A such that $M \otimes_A R \simeq M$ in the canonical way.*
3. Mod-R *is the full subcategory of* Mod-A *formed with the modules M_A such that $MR = M$ and $\forall m \in M$, $mR = 0 \Rightarrow m = 0$.*

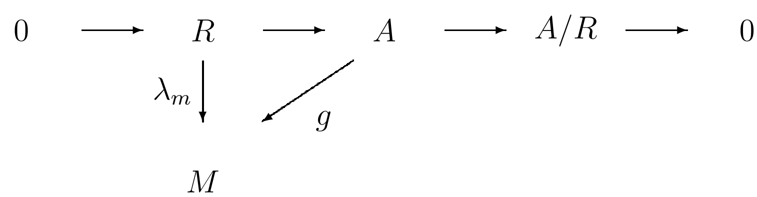
And the same holds for the corresponding categories on the left.

The functors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{m}$ does not depend either on the ring A.

Proof.

(**1**) CMod-R is the full subcategory of Mod-A formed with the modules M_A such that $\mathrm{Hom}_A(R, M) \simeq M$ in the canonical way.

Let $M \in$ CMod-R. We have to give M an A-module structure. For that, let $m \in M$ and $a \in A$. The ring A is an R-module and therefore a B-module and $(A/R)R = 0$. If we consider the diagram



There exists an R-homomorphism $g : A \to M$ that extends λ_m because of the $\mathbf{t}$-injectivity of M. We shall define $ma := g(a)$. This definition extends the product with elements of R. The definition is not dependent on the choice of g because g is unique ($\mathrm{Hom}_B(A/R, M) = 0$). We have to check the following points:

1. $m(a + a') = ma + ma'$ for all $m \in M$ and $a, a' \in A$.
   This is true because g is an abelian group homomorphism.
2. $m(aa') = (ma)a'$ for all $m \in M$ and $a, a' \in A$. Let $r \in R$, $g : A \to M$ that extends λ_m and $\tilde{g} : A \to M$ that extends $\lambda_{g(a)}$. Then

$$(m(aa') - (ma)a')r = g(aa')r - \tilde{g}(a')r$$
$$= g((aa')r) - \tilde{g}(a'r) = m((aa')r) - g(a)(a'r)$$
$$= m(a(a'r)) - g(a(a'r)) = m(a(a'r)) - m(a(a'r)) = 0.$$

   Therefore $m(aa') - (ma)a' \in \mathbf{t}\,(M) = 0$.
3. $(m + m')a = ma + m'a$ for all $m, m' \in M$ and $a \in A$.
   Let $r \in R$. Then

$$((m + m')a - ma - m'a)r = ((m + m')a)r - (ma)r - (m'a)r$$
$$= (m + m')(ar) - m(ar) - m'(ar) = 0$$

   and using the fact that $\mathbf{t}\,(M) = 0$, we prove the claim.
4. $m1_A = m$. Let $r \in R$. Then

$$(m1_A - m)r = m1_A r - mr = mr - mr = 0.$$

   Therefore $m1_A - m \in \mathbf{t}\,(M) = 0$.

This A-module structure is unique. Suppose there are two multiplications $\circ$ and $*$ such that $m \circ r = m * r = mr$ for all $m \in M$ and $r \in R$. Then

$$(m \circ a - m * a)r = (m \circ a)r - (m * a)r =$$
$$m \circ (ar) - m * (ar) = m(ar) - m(ar) = 0.$$

Therefore $m \circ a = m * a$.

We have to prove that with this A-module structure, the module M satisfies $\mathrm{Hom}_A(R, M) \simeq M$.

Let $\lambda : M \to \mathrm{Hom}_A(R, M)$ be the canonical homomorphism. Then

$$\mathrm{Ker}(\lambda) = \{m \in M : \lambda_m = 0\} = \{m \in M : mR = 0\} = \mathbf{t}\,(M) = 0.$$

This proves that λ is a monomorphism. In order to prove that it is an epimorphism, let $f \in \mathrm{Hom}_A(R, M)$. If $r \in R$ and $(s, n) \in B = R \times \mathbb{Z}$, then

$$f(r(s, n)) = f(rs + nr) = f(r)s + nf(r) = f(r)(s, n),$$

and therefore $f \in \mathrm{Hom}_B(R, M)$ and there exists $m \in M$ such that $f(r) = mr = \lambda_m(r)$ for all $r \in R$. This proves that λ is an epimorphism.

Conversely suppose that $M \in$ Mod-A satisfies that $\lambda : M \to \mathrm{Hom}_A(R, M)$ is an isomorphism. Let $\bar{\lambda} : M \to \mathrm{Hom}_B(R, M)$ be the canonical homomorphism. We have to prove that $\bar{\lambda}$ is an isomorphism. Now

$$0 = \mathrm{Ker}(\lambda) = \mathbf{t}\,(M) = \mathrm{Ker}(\bar{\lambda}),$$

and therefore $\bar{\lambda}$ is injective. To prove the surjectivity suppose $f \in \mathrm{Hom}_B(R, M)$, $a \in A$ and $r \in R$.

$$f(ra)s = f((ra)s) = f(r(as)) = f(r)(as) = (f(r)a)s \;\forall s \in R.$$

This proves that $f(ra) - f(r)a \in \mathbf{t}\,(M) = \mathrm{Ker}(\lambda) = 0$ and that f is also an A-homomorphism. If we apply the surjectivity of λ we can find $m \in M$ such that $f(r) = \lambda_m(r) = mr = \bar{\lambda}(r)$ for all $r \in R$ and this proves the surjectivity of $\bar{\lambda}$.

We have to prove also that if $M, N \in$ CMod-R, then $\mathrm{Hom}_B(M, N) = \mathrm{Hom}_A(M, N)$. Let $f \in \mathrm{Hom}_A(M, N)$, $m \in M$ and $(r, z) \in B = R \times \mathbb{Z}$. Then

$$f(m(r, z)) = f(mr + zm) = f(m)r + zf(m) = f(m)(r, z).$$

This proves that $\mathrm{Hom}_A(M, N) \subseteq \mathrm{Hom}_B(M, N)$. On the other hand, suppose $f \in \mathrm{Hom}_B(M, N)$, $m \in M$, $r \in R$ and $a \in A$. Then

$$f(ma)r = f((ma)r) = f(m(ar)) = f(m)(ar) = (f(m)a)r \;\forall r \in R.$$

This proves that $f(ma) - f(m)a \in \mathbf{t}\,(N) = 0$ and therefore $f \in \mathrm{Hom}_A(M, N)$.

(**2**) DMod-R is the full subcategory of Mod-A consisting of the modules M_A such that $M \otimes_A R \simeq M$ in the canonical way.

Let $M \in$ DMod-R. We have to give M an A-module structure. For that given $a \in A$ and $m \in M = MR$, we can find $m_i \in M$ and $r_i \in R$ such that $m = \sum_i m_i r_i$. Therefore $ma = \sum_i m_i(r_i a)$. The problem here is that this definition could depend on the choice of the m_i and r_i. We have to prove that this is not true, and for that it is sufficient to prove that $\sum_i m_i r_i = 0$ implies $\sum_i m_i(r_i a) = 0$ because $\sum_i m_i r_i = \sum_j n_j s_j$ if and only if $\sum_i m_i r_i - \sum_j n_j s_j = 0$. Suppose that $\sum_i m_i r_i = 0$ and $a \in A$. In order to applay Lemma 2.40 we can suppose that the elements $\{r_i : i \in I\}$ form a generating set of R over B on the left because, if it is not so, we can add elements $m_i = 0$ as long as we need.

If $\sum_i m_i r_i = 0$, then $\sum_i m_i \otimes r_i = 0 \in M \otimes_B R$ because $M \in$ DMod-R. Using Lemma 2.40 we can find elements $w_1, \cdots, w_k \in M$ and $b_{it} \in B$ with $t = 1, \cdots, k$ such that

1. $\{(i,t) \in I \times \{1, \cdots, k\} : b_{it} \neq 0\}$ is finite.
2. $\sum_i b_{it} r_i = 0$ for all $t \in \{1, \cdots, k\}$.
3. $\sum_{t=1}^k w_t b_{it} = m_i$ for all $i \in I$.

Then, we deduce that

$$\sum_i m_i(r_i a) = \sum_{it} w_t b_{it}(r_i a) =$$

$$= \sum_{i,t} w_t(b_{it} r_i a) = \sum_t w_t(\sum_i b_{it} r_i)a = 0.$$

This proves also that the multiplication we have defined between elements of M and A is the unique one that extends the multiplication with the elements of R. We have to check also the following points:

1. $m(a + a') = ma + ma'$ for all $m \in M$ and $a, a' \in A$.
   If $m = \sum_i m_i r_i$, then

$$m(a + a') = \sum_i m_i(r_i(a + a')) = \sum_i m_i(r_i a + r_i a')$$

$$= \sum_i m_i(r_i a) + \sum_i m_i(r_i a') = ma + ma'.$$

2. $m(aa') = (ma)a'$ for all $m \in M$ and $a, a' \in A$.
   If $m = \sum_i m_i r_i$, then

$$m(aa') = \sum_i m_i(r_i(aa')) = \sum_i m_i((r_i a)a') = (ma)a'$$

3. $(m + m')a = ma + m'a$ for all $m, m' \in M$ and $a \in A$.
   If $m = \sum_i m_i r_i$ and $m' = \sum_j m'_j r'_j$, then

$$(m + m')a = \sum_i m_i(r_i a) + \sum_j m'_j(r_j a) = ma + ma'$$

4. $m1_A = m$.
   If $m = \sum_i m_i r_i$, then

$$m1_A = \sum_i m_i(r_i 1_A) = \sum_i m_i r_i = m.$$

Suppose $M \in$ Mod-A and Mod-B such that for all $m \in M$ and $r \in R$, the multiplication between m and r is the same with the A-module structure and the B-module structure. Let $\mu : M \otimes_A R \to M$ and $\bar{\mu} : M \otimes_B R \to M$ be the canonical homomorphisms. What we have to prove is that μ is an isomorphism if and only if $\bar{\mu}$ is an isomorphism. It is clear that $\mathrm{Im}(\mu) = MR = \mathrm{Im}(\bar{\mu})$, so that μ is surjective if and only if $\bar{\mu}$ is surjective.

In the following proof, the roles of A and B are interchangable. Therefore we have to make only one direction.

Suppose $\bar{\mu}$ is an isomorphism. Then μ is epimorphism and $\bar{\mu}$ is also an epimorphism and $M = MR$. To prove that μ is a monomorphism, let $\sum_i m_i \otimes r_i \in \mathrm{Ker}(\mu)$ with $\{r_i : i \in I\}$ being a generating set of R over B on the left. Then $\sum_i m_i r_i = 0$ and $\sum_i m_i \otimes r_i \in \mathrm{Ker}(\bar{\mu}) = 0$. If we use Lemma 2.40 we can find elements $w_1, \cdots, w_k \in M$ and $b_{it} \in B$ with $t = 1, \cdots, k$ such that

1. $\{(i,t) \in I \times \{1, \cdots, k\} : b_{it} \neq 0\}$ is finite.
2. $\sum_i b_{it} r_i = 0$ for all $t \in \{1, \cdots, k\}$.
3. $\sum_{t=1}^{k} w_t b_{it} = m_i$ for all $i \in I$.

These elements w_t are in $M = MR$ and we can write $w_t = \sum_\lambda z_{t\lambda} s_{t\lambda}$ with $z_{t\lambda} \in M$ and $s_{t\lambda} \in R$. We have to prove that $\sum_i m_i \otimes r_i = 0$ in $M \otimes_A R$

$$\sum_i m_i \otimes r_i = \sum_{it} w_t b_{it} \otimes r_i =$$

$$\sum_{i,t,\lambda} z_{t\lambda}(s_{t\lambda} b_{it}) \otimes r_i = \sum_{i,t,\lambda} z_{t\lambda} \otimes (s_{t\lambda} b_{it}) r_i =^{1}$$

$$\sum_{i,t,\lambda} z_{t\lambda} s_{t\lambda} \otimes b_{it} r_i = \sum_{i,t} w_t \otimes b_{it} r_i =$$

$$\sum_t w_t \otimes \sum_i b_{i,t} r_i = \sum_t w_t \otimes 0 = 0$$

Let $M, N \in$ DMod-R. We have to check that $\mathrm{Hom}_A(M,N) = \mathrm{Hom}_B(M,N)$, and for that let $f \in \mathrm{Hom}_B(M,N)$ and $m \in M$. If $m = \sum_i m_i r_i$ and $a \in A$, then

$$f(ma) = \sum_i f(m_i(r_i a)) = \sum_i f(m_i)(r_i a) = (\sum_i f(m_i) r_i) a = f(m)a$$

Thus $\mathrm{Hom}_B(M,N) \subseteq \mathrm{Hom}_A(M,N)$. The proof of the reverse inclusion is similar.

[1] Here it is possible to say that $z_{t\lambda} s_{t\lambda} b_{it} \otimes r_i = z_{t\lambda} s_{t\lambda} \otimes b_{it} r_i$ because $- \otimes_B - = - \otimes_R -$, but if we interchange A and B it is not true that $- \otimes_A - = - \otimes_R -$, it is only true if one of the modules is unitary. As we are trying to make a proof in which the roles of A and B are interchangable, we use this trick

(**3**) Mod-R is the full subcategory of Mod-A consisting of the modules M_A such that $MR = M$ and $\forall m \in M$, $mR = 0 \Rightarrow m = 0$.

Suppose $M \in$ Mod-R. We have to define a multiplication $M \times A \to M$ that extends the multiplication with R. Suppose $m \in M$ and $a \in A$. Then $M = MR$ implies $m = \sum_i m_i r_i$, and we define $ma = \sum_i m_i(r_i a)$. We have to prove that this definition is not dependent on the choice of the m_i and r_i, and for that suppose $\sum_i m_i r_i = 0$, $a \in A$ and $r \in R$. Then

$$(\sum_i m_i(r_i a))r = \sum_i m_i((r_i a)r)$$
$$= \sum_i m_i(r_i(ar)) = (\sum_i m_i r_i)(ar) = 0$$

Therefore $\sum_i m_i(r_i a) \in \mathbf{t}\,(M) = 0$ and the definition is good. This definition is the unique that extend the multiplication by R. With this definition M acquires an A-module structure; the proof is the same as in the case of DMod-R.

Now what we have to prove is that $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_B(M, N)$ for all $M, N \in$ Mod-R.

Let $f \in \mathrm{Hom}_B(M, N)$, $m \in M$, $a \in A$ and $r \in R$. Then

$$f(ma)r = f((ma)r) = f(m(ar)) = f(m)(ar) = (f(m)a)r,$$

and therefore $f(ma) - f(m)a \in \mathbf{t}\,(N) = 0$. Thus $f \in \mathrm{Hom}_A(M, N)$. It follows that $\mathrm{Hom}_B(M, N) \subseteq \mathrm{Hom}_A(M, N)$ and similar proof shows that $\mathrm{Hom}_A(M, N) \subseteq \mathrm{Hom}_B(M, N)$. Thus $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_B(M, N)$, as we claimed.

The condition "R is a two-sided ideal of A" is left-right symmetric, and therefore we don't have to make the proof for the corresponding categories on the left.

We have to prove also that the functors $\mathbf{d}$, $\mathbf{c}$ and $\mathbf{m} = \mathbf{u} \circ \mathbf{t}^{-1} = \mathbf{t}^{-1} \circ \mathbf{u}$ do not depend on the ring A. The last functor clearly does not depend on it, because in its definition, the ring A does not appear. The problem is with the functors $\mathbf{c}$ and $\mathbf{d}$. Suppose M is a module that has two structures, an A-module structure and a B-module structure such that for all $m \in M$ and $r \in R$, mr is the same if we compute it with either of the structures.

For the ring B we have the functors $\mathbf{c}$ and $\mathbf{d}$, and for the ring A denote by $\bar{\mathbf{c}}$ and $\bar{\mathbf{d}}$ the corresponding functors. Associated with these functors there are mappings $\bar{\mu} : \bar{\mathbf{d}}(M) \to M$ and $\bar{\iota} : M \to \bar{\mathbf{c}}(M)$ such that $\mathrm{Ker}(\bar{\mu})R = 0, \mathrm{Coker}(\bar{\mu})R = 0$ and $\mathrm{Ker}(\bar{\iota})R = 0, \mathrm{Coker}(\bar{\iota}) = 0$. These functors satisfy the conditions of the Propositions 2.29 and 2.38 and we deduce that $\mathbf{c} \simeq \bar{\mathbf{c}}$ and $\mathbf{d} \simeq \bar{\mathbf{d}}$. ☐

CHAPTER 3

Categories of Modules for Rings II

In this chapter R is an idempotent ring and A a ring with identity such that R is a two-sided ideal of A.

In the previous chapter, we proved that CMod-R was a Giraud subcategory of Mod-A and therefore, a Grothendieck category. We proved also that the categories CMod-R, Mod-R and DMod-R are equivalent, and therefore all of them are Grothendieck categories. In the following sections we are going to investigate monomorphisms, epimorphisms, products, short exact sequences, etc, in these categories. Such results are rather useful because there are several curious differences between the case of rings with identity and other idempotent rings.

1. Epimorphisms and Monomorphisms

One of the first things we need to study are the subobjects and quotient objects of a given object. In order to do that we need to know the monomorphisms and epimorphisms in our categories. We shall recall the categorical definitions before doing anything else. These definitions are for more general categories, although we shall give them in Grothendieck categories in order to avoid the study of particular cases we are not interested in.

Definition 3.1. Let $\mathbb{G}$ be a Grothendieck category. A morphism $f : M \to N$ is called an *epimorphism* in case for every morphism $h : N \to K$ in $\mathbb{G}$ with $h \circ f = 0$, then h should be 0.

A morphism $f : M \to N$ is called a *split epimorphism* in case there exists $g : N \to M$ such that $f \circ g = \mathrm{id}$.

Every split epimorphism is an epimorphism. The converse of course is not true. In Grothendieck categories a morphism $f : M \to N$ is an epimorphism if and only if $\mathrm{Im}(f) = N$ or if $f = f^{kc}$ (the cokernel of the kernel of f). These categorical notions have the following problem, namely $\mathrm{Im}(f)$ need not to be the same thing, if we calculate it on our categories or we calculate it in Mod-A. For example, in CMod-R, $\mathrm{Im}^C(f) = \mathbf{c}(\mathrm{Im}(f))$, nevertheless in DMod-$R$, $\mathrm{Im}^D(f) = \mathrm{Im}(f)$. We shall study this general problem in the following sections generally by calculating direct and inverse limits in the categories and relating them with the calculations on Mod-A. To start with this matter, we shall study the case of monomorphisms and epimorphisms.

PROPOSITION 3.2. *Let R be an idempotent ideal of a ring A with identity.*

1. *A morphism $f : M \to N$ in* CMod-R *is an epimorphism if and only if $N/\mathrm{Im}(f) \in \mathfrak{T}$.*
2. *A morphism $f : M \to N$ in* Mod-R *is an epimorphism if and only if the mapping f is surjective.*
3. *A morphism $f : M \to N$ in* DMod-R *is an epimorphism if and only if the mapping f is surjective.*

PROOF. 1. The CMod-R case: Suppose $N/\mathrm{Im}(f) \in \mathfrak{T}$ and $h : N \to K$ satisfies $h \circ f = 0$ with $n \in N$ such that $h(n) \neq 0$. The element $h(n) \in K \in$ CMod-R, and therefore $h(n) \notin 0 = \mathbf{t}(K)$ and we can find an element $r \in R$ with $h(n)r \neq 0$. But $nr \in \mathrm{Im}(f)$ because $N/\mathrm{Im}(f) \in \mathfrak{T}$, and therefore we can find $m \in M$ with $f(m) = nr$. Now

$$0 \neq h(n)r = h(nr) = h(f(m)) = (h \circ f)(m) = 0,$$

a contradiction.

On the other hand, suppose $N/\mathrm{Im}(f) \notin \mathfrak{T}$, and consider $K = \mathbf{c}(N/\mathrm{Im}(f))$ and the canonical morphism $\iota : N/\mathrm{Im}(f) \to K$. This morphism ι is not 0 because $\mathrm{Ker}(\iota) = \mathbf{t}(N/\mathrm{Im}(f))$, but the morphism $M \to N \to N/\mathrm{Im}(f) \to K$ is 0 i.e. $\iota \circ f = 0$, $\iota \neq 0$, and hence f is not an epimorphism.

2. The Mod-R case: If f is surjective, it is clear that f is an epimorphism. On the other hand suppose $L = N/\mathrm{Im}(f) \neq 0$. As $LR = L$, L cannot be in $\mathfrak{T}$, and then $K = L/\mathbf{t}(L) \neq 0$ and is torsion free. This module is also unitary because $NR = R$. If we denote $p_1 : N \to L$ and $p_2 : L \to K$ the canonical projections, then $p_2 \circ p_1 \circ f = 0$ and $p_2 \circ p_1 \neq 0$ shows that f is not an epimorphism.
3. The DMod-R case: If f is surjective, it is clear that f is an epimorphism. On the other hand suppose $f : M \to N$ is an epimorphism and that $N/\mathrm{Im}(f) \neq 0$. This module is unitary, therefore $(N/\mathrm{Im}(f)) \otimes_A R \in$ DMod-R. Consider the exact sequence

$$M \otimes_A R \to N \otimes_A R \to (N/\mathrm{Im}(f)) \otimes_A R.$$

The first morphism is in fact f because $M \otimes_A R = M$ and $N \otimes_A R = N$ and composing with the other epimorphism $N \otimes_A R \to (N/\mathrm{Im}(f)) \otimes_A R$, gives the 0 morphism and this is not possible because $(N/\mathrm{Im}(f)) \otimes_A R \neq 0$.

□

The case of split epimorphisms is a bit different. In the three cases, if f is a split epimorphism then f is a surjective map, because for any element of n, the element $g(n) \in M$ satisfies $f(g(n)) = n$.

Definition 3.3. Let $\mathbb{G}$ be a Grothendieck category. A morphism $g : N \to M$ is called a *monomorphism* in case that for every morphism $h : K \to N$ in $\mathbb{G}$ with $g \circ h = 0$, then h should be 0.

A morphism $g : N \to M$ is called a *split monomorphism* in case there exists $f : M \to N$ such that $f \circ g = \text{id}$.

Proposition 3.4. *Let R be an idempotent ideal of a ring A with identity.*

1. *A morphism $g : N \to M$ in* CMod-R *is a monomorphism if and only if the mapping g is injective.*
2. *A morphism $g : N \to M$ in* Mod-R *is a monomorphism if and only if g is injective.*
3. *A morphism $g : N \to M$ in* DMod-R *is a monomorphism if and only if* $\text{Ker}(g) \in \mathcal{T}$.

Proof. 1. The CMod-R case:

It is clear that if g is an injective mapping, g is a monomorphism. On the other hand suppose $g : N \to M$ is a monomorphism.

Suppose $\alpha : R \to \text{Ker}(g)$ is any A-homomorphism. If we compose it with the inclusion $j : \text{Ker}(g) \subseteq N$, $j \circ \alpha : R \to N$ and there exists $n \in N$ such that $\alpha(r) = nr$ for all $r \in R$. What we want to prove is that $n \in \text{Ker}(g)$, for that suppose $g(n) \notin 0 = \mathbf{t}(M)$, then we can find an element $r \in R$ such that $g(n)r \neq 0$ and $g(nr) \neq 0$, but $nr = \alpha(r) \in \text{Ker}(g)$ and this is not possible.

We have proved that for any A-homomorphism $\alpha : R \to \text{Ker}(g)$ there exists a $n \in \text{Ker}(g)$ such that $\alpha(r) = nr$ for all $r \in R$. The module $\text{Ker}(g) \subseteq N$,and therefore , it is also torsion free and then $\text{Ker}(g) \in$ CMod-R. Using the fact that g is a monomorphism we deduce that the canonical inclusion $\text{Ker}(g) \subseteq N$ would be the 0 mapping and we obtain $\text{Ker}(g) = 0$.

2. The Mod-R case:

It is clear that if g is an injective mapping, it is a monomorphism. On the other hand, let $g : N \to M$ be a monomorphism with $\text{Ker}(g) \neq 0$. $\text{Ker}(g) \subseteq N$, therefore $\text{Ker}(g)$ is torsion free, and then $0 \neq \text{Ker}(g)R \in$ Mod-R. This is not possible because the canonical inclusion $j : \text{Ker}(g)R \to N$ composed with g is 0 but $j \neq 0$.

3. The DMod-R case:

Suppose $g : N \to M$ satisfies $\text{Ker}(g)R = 0$, and let $h : K \to N$ be a morphism such that $g \circ h = 0$. Then $\text{Im}(h) \subseteq \text{Ker}(g)$. If for some $k \in K$, $h(k) \neq 0$ where $k = \sum_i k_i r_i \in K = KR$, then $h(k) = \sum_i h(k_i) r_i \neq 0$. Clearly we can find $i \in I$ such that $h(k_i) r_i \neq 0$. But $h(k_i) r_i \in \text{Im}(h)R \subseteq \text{Ker}(g)R = 0$, and this is a contradiction.

Conversely, let $g : N \to M$ be a monomorphism. The module $\mathrm{Ker}(g)R \otimes_A R \in$ DMod-R and the morphism

$$\begin{array}{rrcl} h: & \mathrm{Ker}(g)R \otimes R & \to & N \\ & kr \otimes s & \mapsto & krs \end{array}$$

satisfies $g \circ h = 0$ and hence $h = 0$. But $\mathrm{Im}(h) = \mathrm{Ker}(g)R^2 = \mathrm{Ker}(g)R$. Thus $\mathrm{Ker}(g) \in \mathcal{T}$.

□

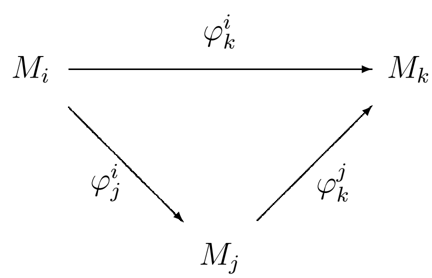
2. Limit and Colimit Calculi

In the previous section we have studied the case of monomorphisms and epimorphisms. As we are in a Grothendieck category that in particular is normal and conormal, monomorphisms and kernels are the same thing and epimorphisms and cokernels are also the same. Therefore we could consider the results in the previous section as a particular case of the results we are going to give here, because we are going to calculate all the inverse and direct limits in the categories CMod-R, Mod-R and DMod-R with respect to the ones calculated in Mod-A.

We shall adopt the notations and definitions given in [**13**, Chapter 2].

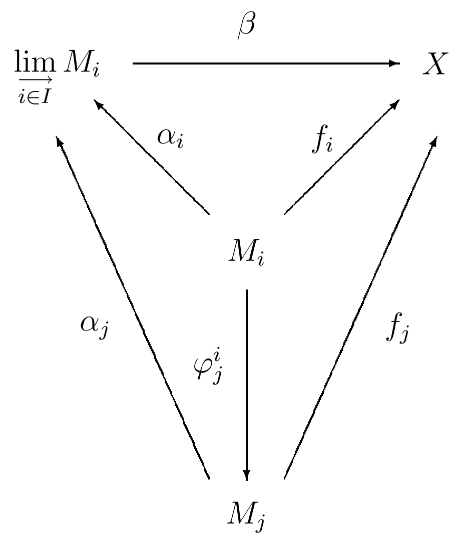
Definition 3.5. Let I be a quasi-ordered set and $\mathbb{G}$ a category. A *direct system* in $\mathbb{G}$ with index set I is a family of objects $\{M_i : i \in I\}$ and morphisms $\{\varphi_j^i : M_i \to M_j : i \leq j\}$ such that

1. $\varphi_i^i : M_i \to M_i$ is the identity morphism for every $i \in I$.
2. If $i \leq j \leq k$, there is a commutative diagram



Definition 3.6. Let $\{M_i, \varphi_j^i\}$ be a direct system in a category $\mathbb{G}$. The *direct limit* of this system, denoted $\varinjlim_{i \in I} M_i$, is an object and a family of morphisms $\alpha_i : M_i \to \varinjlim_{i \in I} M_i$ with $\alpha_i = \alpha_j \circ \varphi_j^i$ whenever $i \leq j$ satisfying the following universal mapping problem:

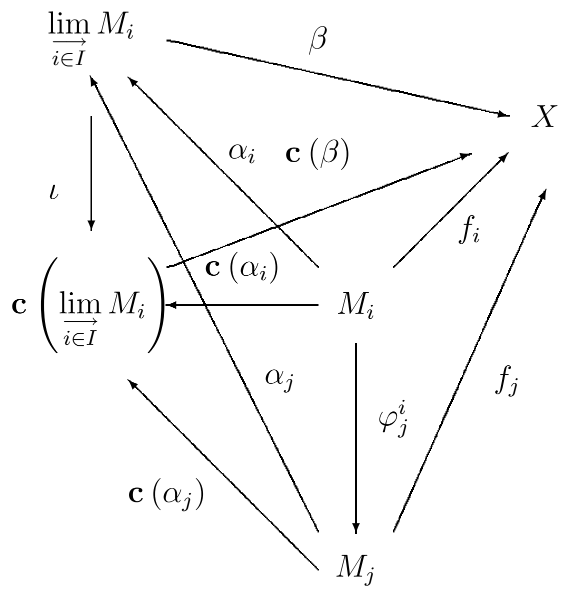
For every object X and every family of morphisms $f_i : M_i \to X$ with $f_i = f_j \circ \varphi^i_j$ whenever $i \le j$, there is a unique morphism $\beta : \varinjlim_{i\in I} M_i \to X$ making the following diagram commute.



Proposition 3.7. *Let* $\{M_i, \varphi^i_j\}$ *be a direct system in the category* CMod-R $\subseteq$ Mod-A*, and* $\left\{\varinjlim_{i\in I} M_i, \alpha_i\right\}$ *be the direct limit calculated in* Mod-A*. Then, the direct limit calculated in* CMod-R *is* $\left\{\mathbf{c}\left(\varinjlim_{i\in I} M_i\right), \mathbf{c}\left(\alpha_i\right)\right\}$.

Proof.

Suppose for some $X \in$ CMod-R we have morphisms $f_i : M_i \to X$ such that $f_j \circ \varphi^i_j = f_i$ for all $i \le j$. Therefore, using the universal property of $\varinjlim_{i\in I} M_i$ we can find a unique β such that $\beta \circ \alpha_i = f_i$ for all $i \in I$. Consider the following diagram

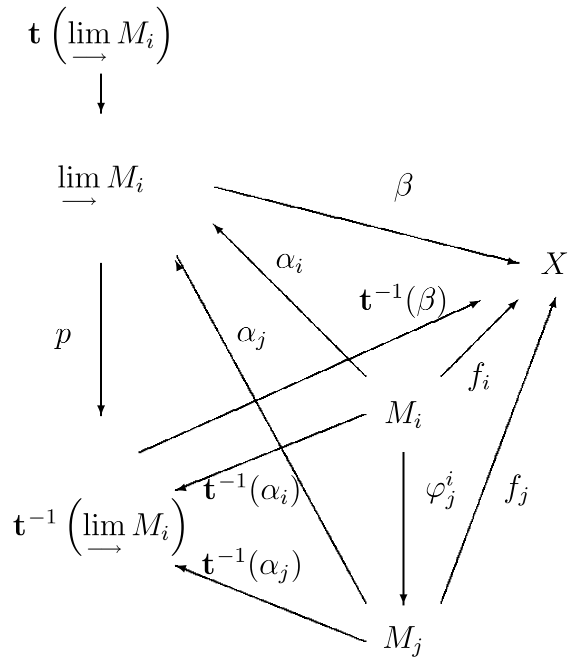


Then $\mathbf{c}\,(\beta)\circ\mathbf{c}\,(\alpha_i)=\mathbf{c}\,(\beta\circ\alpha_i)=\mathbf{c}\,(f_i)=f_i\ (i\in I)$ and then $\mathbf{c}\,(\beta)$ satisfies the corresponding property for $\left\{\mathbf{c}\left(\varinjlim M_i\right),\mathbf{c}\,(\alpha_i)\right\}$. We only have to check that this morphism is the unique one that satisfies this property. For that suppose $\tilde{\beta}:\mathbf{c}\left(\varinjlim M_i\right)\to X$ satisfies $\tilde{\beta}\circ\mathbf{c}\,(\alpha_i)=f_i$ for all $i\in I$. We know that $\mathbf{c}\,(\alpha_i)=\iota\circ\alpha_i$ and then $\tilde{\beta}\circ\iota\circ\alpha_i=f_i$ for all $i\in I$ and the universal property for $\varinjlim M_i$ implies $\tilde{\beta}\circ\iota=\beta=\mathbf{c}\,(\beta)\circ\iota$.

Suppose that $\beta\neq\tilde{\beta}$. It follows that for some $\omega\in\mathbf{c}\left(\varinjlim M_i\right)$, $(\tilde{\beta}-\mathbf{c}\,(\beta))(\omega)\notin 0=\mathbf{t}\,(X)$. Then we can find $r\in R$ such that $(\tilde{\beta}-\mathbf{c}\,(\beta))(\omega r)\neq 0$. But $\mathrm{Coker}(\iota)\in\mathcal{T}$ and therefore $\omega r\in\mathrm{Im}(\iota)$ and this contradicts $(\tilde{\beta}-\mathbf{c}\,(\beta))\circ\iota=0$. $\square$

PROPOSITION 3.8. *Let $\{M_i,\varphi_j^i\}$ be a direct system in the category* Mod-$R\subseteq$ Mod-A, *and* $\left\{\varinjlim_{i\in I} M_i,\alpha_i\right\}$ *be the direct limit calculated in* Mod-A. *Then, the direct limit calculated in* Mod-R *is* $\left\{\mathbf{t}^{-1}\left(\varinjlim_{i\in I} M_i\right),\mathbf{t}^{-1}(\alpha_i)\right\}$.

PROOF. Consider the following diagram



All the modules M_i are in Mod-R and therefore, they are unitary. Then $\varinjlim M_i$ is also unitary because $\mathcal{U}$ is closed under coproducts and quotients. This implies that the module

$$\frac{\varinjlim M_i}{\mathbf{t}\left(\varinjlim M_i\right)} = \mathbf{t}^{-1}\left(\varinjlim M_i\right) \in \text{Mod-}R.$$

Suppose that for some $X \in$ Mod-R we have morphisms $f_i : M_i \to X$ such that $f_j \circ \varphi_j^i = f_i$ for all $i \leq j$. Using the universal property of $\varinjlim M_i$ we can find $\beta : \varinjlim M_i \to X$ such that $\beta \circ \alpha_i = f_i$ for all $i \in I$. Then

$$\mathbf{t}^{-1}(\beta) \circ \mathbf{t}^{-1}(\alpha_i) = \mathbf{t}^{-1}(\beta \circ \alpha_i) = \mathbf{t}^{-1}(f_i) = f_i \quad (i \in I).$$

Then $\mathbf{t}^{-1}(\beta)$ satisfies the corresponding property for

$$\left\{\mathbf{t}^{-1}\left(\varinjlim M_i\right), \mathbf{t}^{-1}(\alpha_i)\right\}.$$

We only have to check that this morphism is the unique one that satisfies this property. For that, suppose $\tilde{\beta} : \mathbf{t}^{-1}\left(\varinjlim M_i\right) \to X$ satisfies $\tilde{\beta} \circ \mathbf{t}^{-1}(\alpha_i) = f_i$ for all $i \in I$. We know that $\mathbf{t}^{-1}(\alpha_i) = p \circ \alpha_i$ and then $\tilde{\beta} \circ p \circ \alpha_i = f_i$ for all $i \in I$ and the universal property for $\varinjlim M_i$ implies $\tilde{\beta} \circ p = \beta = \mathbf{t}^{-1}(\beta) \circ p$.

As X is torsion free, $\mathrm{Hom}_A\left(\mathbf{t}\left(\varinjlim M_i\right), X\right) = 0$, and therefore $\tilde{\beta} \circ p = \beta = \mathbf{t}^{-1}(\beta) \circ p$ factors through p in a unique way, $\tilde{\beta} = \mathbf{t}^{-1}(\beta)$, i.e. as we claimed. □

Proposition 3.9. *Let* $\{M_i, \varphi_j^i\}$ *be a direct system in the category* DMod-$R \subseteq$ Mod-A, *and* $\left\{\varinjlim_{i \in I} M_i, \alpha_i\right\}$ *be the direct limit calculated in* Mod-A. *Then* $\left\{\varinjlim_{i \in I} M_i, \alpha_i\right\}$ *is also in* DMod-R *and this is also the direct limit calculated in* DMod-R.

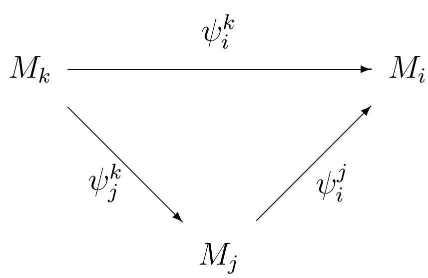
Proof. The functor $- \otimes_A R$ has a right adjoint[1], and therefore, it commutes with direct limits (see [**14**, Proposition IV.9.4]) and then

$$\left(\varinjlim M_i\right) \otimes_A R \simeq \varinjlim (M_i \otimes_A R) \simeq \varinjlim M_i.$$

This proves that $\varinjlim M_i \in$ DMod-R □

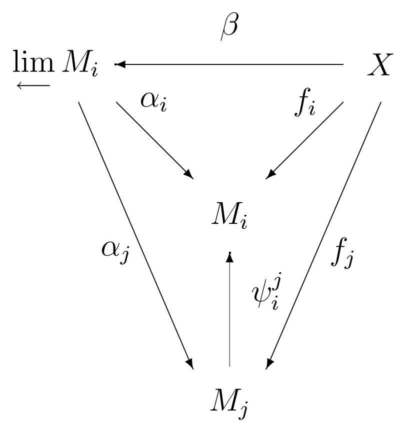
Definition 3.10. Let I be a quasi-ordered set and $\mathbb{G}$ a category. An *inverse system* in $\mathbb{G}$ with index set I is a family $\{M_i : i \in I\}$ of objects in $\mathbb{G}$ and a family $\{\psi_i^j : M_j \to M_i : i \leq j\}$ such that

1. $\psi_i^i : M_i \to M_i$ is the identity morphism for every $i \in I$.
2. If $i \leq j \leq k$ there is a commutative diagram



Definition 3.11. Let $\{M_i, \psi_i^j\}$ be an inverse system in $\mathbb{G}$. The *inverse limit* of this system, denoted by $\varprojlim_{i \in I} M_i$ is an object in $\mathbb{G}$ and a family of morphisms $\alpha_i : \varprojlim_{i \in I} M_i \to M_i$ with $\alpha_i = \psi_i^j \circ \alpha_j$ whenever $i \leq j$ satisfying the following universal mapping problem:
for every X and morphisms $f_i : X \to M_i$ with $\psi_i^j \circ f_j = f_i$ whenever $i \leq j$, there is a unique morphism $\beta : X \to \varprojlim_{i \in I} M_i$ making the following diagram commute.

[1] The right adjoint is $\mathrm{Hom}_A(R, -)$. This relation can be seen in [**?**, Lemma 19.11]



We shall give also the results for the inverse limits in our categories. The proof of these results are more or less dual to the proofs we have given in the case of direct limits.

Proposition 3.12. *Let* $\{M_i, \varphi_j^i\}$ *be an inverse system in the category* CMod-$R \subseteq$ Mod-A, *and* $\left\{\varprojlim_{i\in I} M_i, \alpha_i\right\}$ *be the inverse limit calculated in* Mod-A. *Then,* $\varprojlim_{i\in I} M_i$ *is in* CMod-R *and therefore, this is the inverse limit calculated in* CMod-R.

Proposition 3.13. *Let* $\{M_i, \varphi_j^i\}$ *be an inverse system in the category* Mod-$R \subseteq$ Mod-A, *and* $\left\{\varprojlim_{i\in I} M_i, \alpha_i\right\}$ *be the inverse limit calculated in* Mod-A. *Then, the inverse limit calculated in* Mod-R *is* $\left\{\mathbf{u}\left(\varprojlim_{i\in I} M_i\right), \mathbf{u}\left(\alpha_i\right)\right\}$.

Proposition 3.14. *Let* $\{M_i, \varphi_j^i\}$ *be an inverse system in the category* DMod-$R \subseteq$ Mod-A, *and* $\left\{\varprojlim_{i\in I} M_i, \alpha_i\right\}$ *be the inverse limit calculated in* Mod-A. *Then, the inverse limit calculated in* DMod-R *is* $\left\{\mathbf{d}\left(\varprojlim_{i\in I} M_i\right), \mathbf{d}\left(\alpha_i\right)\right\}$.

3. Special Kinds of Limits

We are used to concepts like intersections, inverse images, exact sequences, kernels, cokernels, and so on . These concepts are categorical, and they can be defined in categories that are not the category of modules over a ring with identity. What we are going to do in this section is to recall these categorical definitions and notice the differences that appear in Mod-A and in the categories CMod-R, Mod-R and DMod-R.

Objects in Grothendieck categories are considered up to isomorphisms. This is rather useful when we study limits and colimits, that are unique up to isomorphisms, and allow us to define concepts like $\mathrm{Ker}(f)$ without ambiguity.

Let M be an object in a Grothendieck category $\mathbb{G}$. A subobject of M is an object $N \in \mathbb{G}$ with a monomorphism $\mu : N \to M$. In the category CMod-R, the monomorphisms are the same as in Mod-A. Therefore if $M \in$ CMod-R, the subobjects of M are the A-submodules of M that are also in CMod-R. In the case of Mod-R and DMod-R, it is different because the monomorphisms are not the same as in Mod-A. Therefore if $M \in$ DMod-R, a subobject of M is an object $N \in$ DMod-R with a morphism $\mu : N \to M$ such that $\mathrm{Ker}(\mu)R = 0$. The same happens in Mod-R.

The kernel of a morphism $f : M \to N$ is an inverse limit, and therefore we have the following

1. If $f : M \to N$ is a morphism in CMod-R, then $\mathrm{Ker}(f)$ calculated in CMod-R is the same as in Mod-A.
2. If $f : M \to N$ is a morphism in Mod-R, then $\mathrm{Ker}(f)$ calculated in Mod-R is $\mathrm{Ker}(f)R$.
3. If $f : M \to N$ is a morphism in DMod-R, then $\mathrm{Ker}(f)$ calculated in DMod-R is $\mathrm{Ker}(f)R \otimes_A R$.

The cokernel of a morphism $f : M \to N$ is a direct limit, therefore we have the following.

1. If $f : M \to N$ is a morphism in CMod-R, then $\mathrm{Coker}(f)$ calculated in CMod-R is $\mathrm{Hom}_A(R, \mathrm{Im}(f)/\mathbf{t}\,(\mathrm{Im}(f)))$.
2. If $f : M \to N$ is a morphism in Mod-R, then $\mathrm{Coker}(f)$ calculated in Mod-R is $\mathrm{Im}(f)/\mathbf{t}\,(\mathrm{Im}(f))$.
3. If $f : M \to N$ is a morphism in DMod-R, then $\mathrm{Coker}(f)$ calculated in DMod-R is the same as in Mod-A.

In the case of exact sequences we have a condition that is similar for the three cases.

Consider the following sequence in

1. CMod-R
2. Mod-R
3. DMod-R

$$K \xrightarrow{\ f\ } L \xrightarrow{\ g\ } M$$

This sequence is exact at L if and only if $g \circ f = 0$ and $\mathrm{Ker}(g)/\mathrm{Im}(f) \in \mathcal{T}$.

We have to be careful with this. We shall give a list here some categorical definitions and remarks[2]

DEFINITION 3.15. Let $\mathbb{G}$ be a Grothendieck category.

1. Every *kernel* is a monomorphism. Every monomorphism is the kernel of its cokernel ($\mathbb{G}$ is normal).
2. Every *cokernel* is an epimorphism. Every epimorphism is the cokernel of its kernel ($\mathbb{G}$ is conormal).
3. The *image* of a morphism $f : M \to N$ is the kernel of the cokernel $f^c : N \to \mathrm{Coker}(f)$.
4. The *coimage* of a morphism $f : M \to N$ is the cokernel of the kernel $f^k : \mathrm{Ker}(f) \to M$.

   Note that every morphism $f : M \to N$, can be decomposed as $f = f^{kc} \circ f^{ck}$ where $f^{kc} = (f^c)^k$ and $f^{ck} = (f^k)^c$. (First Isomorphism Theorem).
5. Let $\mu_i : M_i \to M$ be a family of subobjects of M, the *sum* of these objects in M is the image of the induced morphism $\coprod \mu_i : \coprod M_i \to M$.
6. An object G is a generator in $\mathbb{G}$ if for every object M there exists an epimorphism in $\mathbb{G}$, $G^{(I)} \to M$, for some index set I.
7. Let $f : M \to N$ be a morphism and $\mu : K \to N$ be a subobject of N, the *inverse image* of N is the subobject of $\bar{\mu} : f^{-1}(K) \to M$ defined by the following pull-back diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \uparrow{\scriptstyle \bar{\mu}} & & \uparrow{\scriptstyle \mu} \\ f^{-1}(K) & \longrightarrow & K \end{array}$$

8. Consider the following sequence in $\mathbb{G}$

$$K \xrightarrow{f} L \xrightarrow{g} M.$$

   We shall say that this sequence is *exact at* L if $f^{kc} = g^k$ (or equivalently if $f^c = g^{ck}$).

4. The Exactness of Functors c, d and m

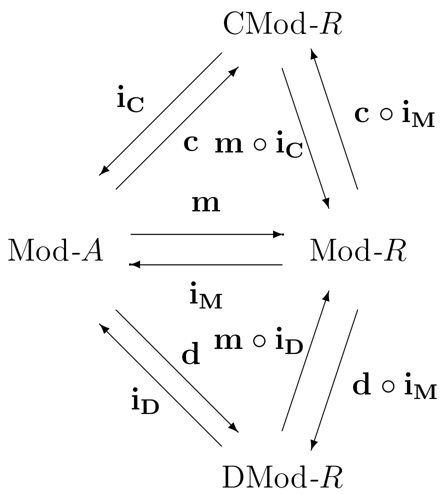
DEFINITION 3.16. We shall define

$$\begin{array}{rrcl} \mathbf{i_C} : & \text{CMod-}R & \to & \text{Mod-}A \\ \mathbf{i_M} : & \text{Mod-}R & \to & \text{Mod-}A \\ \mathbf{i_D} : & \text{DMod-}R & \to & \text{Mod-}A \end{array}$$

as the canonical inclusions of the categories CMod-R, Mod-R and DMod-R in Mod-A.

[2]We consider as known the categorical definitions of kernels, cokernels, products, coproducts and pull-backs. In case of doubt see [**14**, Sections IV.2, IV.3 and IV.5]

PROPOSITION 3.17. *Consider the following diagram of categories and functors:*



Then we have the following relations:

1. $\mathbf{m} \circ \mathbf{i_C} = \mathbf{u} \circ \mathbf{i_C}$
2. $\mathbf{m} \circ \mathbf{i_D} = \mathbf{t}^{-1} \circ \mathbf{i_D}$
3. $\mathbf{m} \circ \mathbf{i_C} \circ \mathbf{c} \circ \mathbf{i_M} = \mathrm{id}_{\text{Mod-}R}$
4. $\mathbf{c} \circ \mathbf{i_M} \circ \mathbf{m} \circ \mathbf{i_C} = \mathrm{id}_{\text{CMod-}R}$
5. $\mathbf{m} \circ \mathbf{i_D} \circ \mathbf{d} \circ \mathbf{i_M} = \mathrm{id}_{\text{Mod-}R}$
6. $\mathbf{d} \circ \mathbf{i_M} \circ \mathbf{m} \circ \mathbf{i_D} = \mathrm{id}_{\text{DMod-}R}$
7. $\mathbf{c} \circ \mathbf{i_C} = \mathrm{id}_{\text{CMod-}R}$
8. $\mathbf{m} \circ \mathbf{i_M} = \mathrm{id}_{\text{Mod-}R}$
9. $\mathbf{d} \circ \mathbf{i_D} = \mathrm{id}_{\text{DMod-}R}$
10. $\mathbf{c}$ *is a left adjoint of* $\mathbf{i_C}$
11. $\mathbf{d}$ *is a right adjoint of* $\mathbf{i_D}$
12. $\mathbf{m} \circ \mathbf{i_C} \circ \mathbf{c} = \mathbf{m}$
13. $\mathbf{c} \circ \mathbf{i_M} \circ \mathbf{m} = \mathbf{c}$
14. $\mathbf{m} \circ \mathbf{i_D} \circ \mathbf{d} = \mathbf{m}$
15. $\mathbf{d} \circ \mathbf{i_M} \circ \mathbf{m} = \mathbf{d}$
16. $\mathbf{c}$ *is an exact functor*
17. $\mathbf{m}$ *is an exact functor*
18. $\mathbf{d}$ *is an exact functor*

PROOF. Some of these facts are already known, but here we have made a list with all we are going to use.

The first two claims are true because the modules in CMod-R are torsion-free and in DMod-R are unitary.

Claims (3),(4),(5) and (6) are the category equivalences between CMod-R, Mod-R and DMod-R.

Claims (7),(8) and (9) are true because **c** leaves unchanged the modules in CMod-R as well as **m** does in Mod-R and **d** in DMod-R.

Claim (10) is a well known fact about the localization functors.

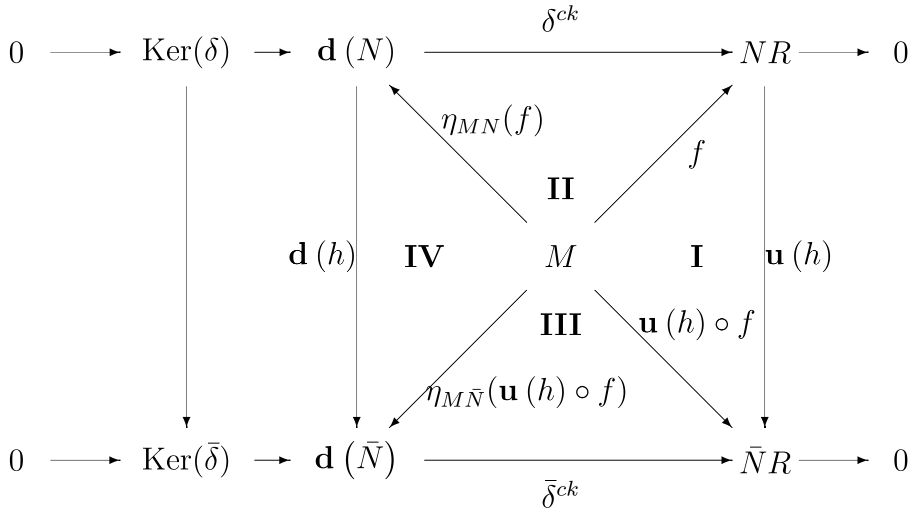
Claim (11). Let $M \in$ DMod-R and $N \in$ Mod-A. We have to prove that there exists a natural isomorphism

$$\eta_{MN} : \mathrm{Hom}_A(\mathbf{i_D}(M), N) \to \mathrm{Hom}_A(M, \mathbf{d}\,(N))$$

Let $f : M \to N$ be a morphism. $\mathrm{Im}(f)$ is a unitary module, therefore $\mathrm{Im}(f) \subseteq NR$. Then consider the following diagram

$$\begin{array}{ccccccccc} & & & & & & M & & \\ & & & & & \swarrow{\scriptstyle \eta_{MN}(f)} & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & \mathrm{Ker}(\delta) & \longrightarrow & \mathbf{d}\,(N) & \xrightarrow{\delta^{ck}} & NR & \longrightarrow & 0 \end{array}$$

The morphism $\delta : \mathbf{d}\,(N) \to N$ is canonical, and $\delta^{ck} : \mathbf{d}\,(N) \to NR$ is the induced epimorphism. We can define $\eta_{MN}(f)$ in this way because M is **u**-codivisible and if there are two morphisms $g_1, g_2 : M \to \mathbf{d}\,(N)$ such that $\delta^{ck} \circ g_i = f$ then $\delta^{ck} \circ (g_1 - g_2) = 0$ and $g_1 - g_2$ factors through $(\delta^{ck})^k = \delta^k$. But this proves that $g_1 - g_2 = 0$ because $\mathrm{Hom}_A(M, \mathrm{Ker}(\delta)) = 0$. The inverse of this homomorphism is $\mathrm{Hom}_A(M, \delta^{ck})$; this can be proved using the uniqueness. This proves also the naturality of this isomorphism in the variable M because $\mathrm{Hom}_A(M, \delta^{ck})$ is natural in this variable. To prove the naturality in the other variable let $h : N \to \bar{N}$ be a homomorphism, and consider the following diagram:



We know the commutativity of the triangles **I**, **II** and **III** and the big square, and we have to prove the commutativity of the triangle **IV**, i.e. $\mathbf{d}\,(h) \circ \eta_{MN}(f) = \eta_{M\bar{N}}(\mathbf{u}\,(h) \circ f)$. Because of the commutativity relations we obtain that $\bar{\delta}^{ck} \circ (\mathbf{d}\,(h) \circ \eta_{MN}(f) - \eta_{M\bar{N}}(\mathbf{u}\,(h) \circ f)) = 0$, therefore $\mathbf{d}\,(h) \circ \eta_{MN}(f) - \eta_{M\bar{N}}(\mathbf{u}\,(h) \circ f)$ factors through $(\bar{\delta}^{ck})^k = \bar{\delta}^k$ and then $\mathbf{d}\,(h) \circ \eta_{MN}(f) - \eta_{M\bar{N}}(\mathbf{u}\,(h) \circ f) = 0$ because $\mathrm{Hom}_A(M, \mathrm{Ker}(\bar{\delta})) = 0$.

Claim (12). Let $M \in$ Mod-A. We have to prove that $(\mathbf{m} \circ \mathbf{i_C} \circ \mathbf{c})(M)$ is naturally isomorphic to $\mathbf{m}\,(M)$. The first module is $\mathrm{Hom}_A(R, M/\mathbf{t}\,(M))R$ and the second is $(M/\mathbf{t}\,(M))R$. It is clear that if we prove it for torsion-free modules, we have proved for all of them because we only have to apply the result to $M/\mathbf{t}\,(M)$ and then suppose M torsion-free. We have to prove that $\mathrm{Hom}_A(R, M)R = MR$. As M is torsion free there is a canonical monomorphism $M \subseteq \mathrm{Hom}_A(R, M)$ and then $MR \subseteq \mathrm{Hom}_A(R, M)R$. On the other hand suppose $\sum f_i r_i \in \mathrm{Hom}_A(R, M)R$ with $f_i : R \to M$ and $r_i \in R$. As R is idempotent we can find elements $s_{ij}, t_{ij} \in R$ such that $r_i = \sum_j s_{ij} t_{ij}$. Then $\sum_i f_i r_i = \sum_{ij} f_i s_{ij} t_{ij}$. With the identification we are making, i.e. $M \subseteq \mathrm{Hom}_A(R, M)$, $f_i s_{ij} = f_i(s_{ij}) \in M$ and then $\sum_i f_i r_i = \sum_{ij} f_i(s_{ij}) t_{ij} \in MR$.

Claim (13). This is a consequence of Claim 12 and Claim 4.

Claim (14). Let $M \in$ Mod-A. We have to prove that $(\mathbf{m} \circ \mathbf{i_D} \circ \mathbf{d})(M) = \mathbf{m}\,(M)$. The first module is $(MR \otimes_A R)/\mathbf{t}\,(MR \otimes_A R)$ and the second is $MR/\mathbf{t}\,(MR)$. If we prove it for unitary modules, we have proved it for all of them because we only have to apply the result to MR and then suppose M is unitary. We have to prove that $\frac{M \otimes_A R}{\mathbf{t}(M \otimes_A R)} =$

$M/\mathbf{t}(M)$. Consider the canonical epimorphism

$$\begin{array}{rrcl}\eta : & \frac{M\otimes_A R}{\mathbf{t}(M\otimes_A R)} & \to & M/\mathbf{t}(M) \\ & m\otimes r+\mathbf{t}(M\otimes_A R) & \mapsto & mr+\mathbf{t}(M)\end{array}$$

(It is well defined because $M/\mathbf{t}(M)$ is torsion free and therefore $\mathbf{t}(M\otimes_A R)\subseteq \mathrm{Ker}(M\otimes_A R\to M/\mathbf{t}(M)))$

We have to prove that it is a monomorphism. Suppose $\sum_i m_i\otimes r_i+\mathbf{t}(M\otimes_A R)\in \mathrm{Ker}(\eta)$. Then $\sum_i m_i r_i\in \mathbf{t}(M)$. Let $s,t\in R$. Then $\sum_i m_i r_i s=0$ and

$$(\sum_i m_i\otimes r_i+\mathbf{t}(M\otimes_A R))st=\underbrace{\sum_i m_i r_i s}_{=0}\otimes t+\mathbf{t}(M\otimes_A R)=0$$

This proves our claim.

Claim (15). This is a consequence of Claim 4 and Claim 6.

Claims (16),(17) and (18).

$\mathbf{c}$ is a left adjoint of $\mathbf{i_C}$ because of claim 10.

$\mathbf{m}\circ\mathbf{i_C}$ is a left adjoint of $\mathbf{c}\circ\mathbf{i_M}$ because of the equivalence.

$\mathbf{d}\circ\mathbf{i_M}$ is a left adjoint of $\mathbf{m}\circ\mathbf{i_D}$ because of the equivalence.

Then $\mathbf{m}=\mathbf{m}\circ\mathbf{i_C}\circ\mathbf{c}$ is a left adjoint of $\mathbf{i_C}\circ\mathbf{c}\circ\mathbf{i_M}$ and $\mathbf{d}=\mathbf{d}\circ\mathbf{i_M}\circ\mathbf{m}$ is a left adjoint of $\mathbf{i_C}\circ\mathbf{c}\circ\mathbf{i_M}\circ\mathbf{m}\circ\mathbf{i_D}$.

We have proved that $\mathbf{c}$, $\mathbf{m}$ and $\mathbf{d}$ are left adjoints, therefore they are left exact.

$\mathbf{d}$ is a right adjoint of $\mathbf{i_D}$ because of claim 11.

$\mathbf{m}\circ\mathbf{i_D}$ is a right adjoint of $\mathbf{d}\circ\mathbf{i_M}$ because of the equivalence.

$\mathbf{c}\circ\mathbf{i_M}$ is a right adjoint of $\mathbf{m}\circ\mathbf{i_C}$ because of the equivalence.

Then $\mathbf{m}=\mathbf{m}\circ\mathbf{i_D}\circ\mathbf{d}$ is a right adjoint of $\mathbf{i_D}\circ\mathbf{d}\circ\mathbf{i_M}$ and $\mathbf{c}=\mathbf{c}\circ\mathbf{i_M}\circ\mathbf{m}$ is a right adjoint of $\mathbf{i_D}\circ\mathbf{d}\circ\mathbf{i_M}\circ\mathbf{m}\circ\mathbf{i_C}$.

This proves that $\mathbf{c}$, $\mathbf{m}$ and $\mathbf{d}$ are right adjoints and therefore they are right exact.

□

5. The functors $\mathrm{Hom}_A(-,-)$ and $-\otimes_A-$

PROPOSITION 3.18. *Let M be a module in* CMod-R. *Then the functor* $\mathrm{Hom}_A(-,M):$ CMod-$R\to \mathcal{A}b$ *is left exact.*

PROOF. Suppose

$$X\xrightarrow{f}Y\xrightarrow{g}Z\longrightarrow 0$$

is an exact sequence in CMod-R. If we consider this sequence in Mod-A, it satisfies

1. $\mathrm{Im}(f)\subseteq\mathrm{Ker}(g)$ and $\mathrm{Ker}(g)/\mathrm{Im}(f)\in\mathcal{T}$.
2. $Z/\mathrm{Im}(g)\in\mathcal{T}$.

If we apply the functor $\mathrm{Hom}_A(-,M)$ we get the sequence

$$0 \longrightarrow \mathrm{Hom}_A(Z,M) \xrightarrow{\mathrm{Hom}_A(g,M)} \mathrm{Hom}_A(Y,M) \xrightarrow{\mathrm{Hom}_A(f,M)} \mathrm{Hom}_A(X,M)$$

We have to show that this sequence is exact in $\mathcal{A}b$.

Suppose $h : Z \to M$ belongs to $\mathrm{Ker}(\mathrm{Hom}_A(g,M))$, i.e. $h \circ g = 0$. We have to prove that $h = 0$, but this is true because g is a monomorphism in CMod-R.

Suppose $h : Y \to M$ belongs to $\mathrm{Ker}(\mathrm{Hom}_A(f,M))$, i.e. $h \circ f = 0$. We have to find a homomorphism $\tilde{h} : Z \to M$ such that $h = \tilde{h} \circ g$.

Let $k \in \mathrm{Ker}(g)$ such that $h(k) \notin 0 = \mathbf{t}\,(M)$. Then we can find an element $r \in R$ such that $h(k)r \neq 0$. But $kr \in \mathrm{Ker}(g)R \subseteq \mathrm{Im}(f)$ and then $h(kr) = 0$ because $h \circ f = 0$. This proves that $h(\mathrm{Ker}(g)) = 0$.

Then consider the exact sequence in Mod-A

$$0 \longrightarrow \mathrm{Ker}(g) \xrightarrow{g^k} Y \xrightarrow{g^{ck}} \mathrm{Im}(g) \longrightarrow 0$$

The composition $h \circ g^k = 0$, so we can find $h' : \mathrm{Im}(g) \to M$ such that $h' \circ g^{ck} = h$.

Then if we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(g) & \xrightarrow{g^{kc}} & Z & \longrightarrow & Z/\mathrm{Im}(g) & \longrightarrow & 0 \\ & & {\scriptstyle h'}\downarrow & \swarrow {\scriptstyle \tilde{h}} & & & & & \\ & & M & & & & & & \end{array}$$

using that M is $\mathbf{t}$-injective and $(Z/\mathrm{Im}(g))R = 0$, we can find a morphism $\tilde{h}$ such that $\tilde{h} \circ g^{kc} = h'$. Then

$$\tilde{h} \circ g = \tilde{h} \circ g^{kc} \circ g^{ck} = h' \circ g^{ck} = h.$$

This proves that the sequence

$$0 \longrightarrow \mathrm{Hom}_A(Z,M) \xrightarrow{\mathrm{Hom}_A(g,M)} \mathrm{Hom}_A(Y,M) \xrightarrow{\mathrm{Hom}_A(f,M)} \mathrm{Hom}_A(X,M)$$

is exact in $\mathcal{A}b$. □

We are going to prove that the functor

$$\mathrm{Hom}_A(-,M) : \text{CMod-}R \to \mathcal{A}b$$

is left exact when $M \in$ CMod-R. We would like to prove that this functor is also exact when it maps from the categories DMod-R and Mod-R to $\mathcal{A}b$. In order to prove this we need the following lemma.

Lemma 3.19. *Let* $M \in$ CMod-R, $X, Y \in$ Mod-A *and* $f : X \to Y$ *with* $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{T}$. *Then*

$$\mathrm{Hom}_A(f,M) : \mathrm{Hom}_A(Y,M) \to \mathrm{Hom}_A(X,M)$$

is an isomorphism.

Proof. Let $h : Y \to M$ belong to $\mathrm{Ker}(\mathrm{Hom}_A(f, M))$, i.e. $h \circ f = 0$. Suppose that for some $y \in Y$, $h(y) \notin 0 = \mathbf{t}\,(M)$. Then we can find $r \in R$ such that $h(y)r \neq 0$. Now

$$Y/\mathrm{Im}(f) \in \mathcal{T} \Rightarrow yr \in \mathrm{Im}(f) \Rightarrow yr = f(x) \quad \text{for some } x \in X$$

$$\text{And hence} \quad 0 = (h \circ f)(x) = h(f(x)) = h(yr) = h(y)r \neq 0,$$

the contradiction we were looking for.

On the other hand let $g : X \to M$. We have to find $h : Y \to M$ such that $h \circ f = g$. As $\mathrm{Ker}(f) \in \mathcal{T}$, $\mathrm{Hom}_A(\mathrm{Ker}(f), M) = 0$ and we can find the induced map $\bar{g} : X/\mathrm{Ker}(f) = \mathrm{Im}(f) \to M$, i.e. $g = \bar{g} \circ f^{ck}$. Then consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(f) & \longrightarrow & Y & \longrightarrow & Y/\mathrm{Im}(f) & \longrightarrow & 0 \\ & & {\scriptstyle \bar{g}}\big\downarrow & \swarrow & & & & & \\ & & M & & & & & & \end{array}$$

The condition $Y/\mathrm{Im}(f) \in \mathcal{T}$ and the $\mathbf{t}$-injectivity of M, let us find a homomorphism $h : Y \to M$ such that $h \circ f^{kc} = \bar{g}$. Then

$$g = \bar{g} \circ f^{ck} = h \circ f^{kc} \circ f^{ck} = h \circ f.$$

This proves the surjectivity of $\mathrm{Hom}_A(f, M)$. □

Proposition 3.20. *Let* $M \in \mathrm{CMod}\text{-}R$. *Then* $\mathrm{Hom}_A(-, M)$ *is a left exact functor from the categories* $\mathrm{CMod}\text{-}R$, $\mathrm{Mod}\text{-}R$ *and* $\mathrm{DMod}\text{-}R$ *to* $\mathcal{A}b$.

Proof. Consider an exact sequence in any of the categories. If we apply the equivalence functors, we can find short exact sequences in the other categories. The diagram is as follows

$$\begin{array}{ccccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\ {\scriptstyle \iota_1}\big\uparrow & & {\scriptstyle \iota_2}\big\uparrow & & {\scriptstyle \iota_3}\big\uparrow & & \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 \\ {\scriptstyle \mu_1}\big\uparrow & & {\scriptstyle \mu_2}\big\uparrow & & {\scriptstyle \mu_3}\big\uparrow & & \\ Z_1 & \longrightarrow & Z_2 & \longrightarrow & Z_3 & \longrightarrow & 0 \end{array}$$

The objects of the first row are in $\mathrm{CMod}\text{-}R$, the objects of the second are in $\mathrm{Mod}\text{-}R$ and the objects of the third are in $\mathrm{DMod}\text{-}R$.

If we have any exact sequence $X_1 \to X_2 \to X_3 \to 0$ in CMod-R we define $Y_i = \mathbf{m}(X_i)$ and $Z_i = \mathbf{d}(X_i)$ with the canonical morphisms. The same happens if we start with an exact sequence $Y_1 \to Y_2 \to Y_3 \to 0$ in Mod-R, we define $X_i = \mathbf{c}(Y_i)$ and $Z_i = \mathbf{d}(Y_i)$. Therefore if we start with a exact sequence in any of the categories we can always build a diagram like the previous one in which the morphisms μ_i and ι_i satisfy $\mathrm{Ker}(\iota_i), \mathrm{Coker}(\iota_i), \mathrm{Ker}(\mu_i), \mathrm{Coker}(\mu_i) \in \mathcal{T}$ for all $i = 1, 2, 3$ and the sequences are exact in the corresponding category.

If we apply the functor $\mathrm{Hom}_A(-, M)$ we obtain the following diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_A(X_3, M) & \longrightarrow & \mathrm{Hom}_A(X_2, M) & \longrightarrow & \mathrm{Hom}_A(X_1, M) \\
 & & \downarrow{\scriptstyle \mathrm{Hom}_A(\iota_1, M)} & & \downarrow{\scriptstyle \mathrm{Hom}_A(\iota_2, M)} & & \downarrow{\scriptstyle \mathrm{Hom}_A(\iota_3, M)} \\
0 & \longrightarrow & \mathrm{Hom}_A(Y_3, M) & \longrightarrow & \mathrm{Hom}_A(Y_2, M) & \longrightarrow & \mathrm{Hom}_A(Y_1, M) \\
 & & \downarrow{\scriptstyle \mathrm{Hom}_A(\mu_1, M)} & & \downarrow{\scriptstyle \mathrm{Hom}_A(\mu_2, M)} & & \downarrow{\scriptstyle \mathrm{Hom}_A(\mu_3, M)} \\
0 & \longrightarrow & \mathrm{Hom}_A(Z_3, M) & \longrightarrow & \mathrm{Hom}_A(Z_2, M) & \longrightarrow & \mathrm{Hom}_A(Z_1, M)
\end{array}$$

The first row is exact because of Proposition 3.18 and the morphisms on the columns are isomorphisms because of Lemma 3.19, from that we deduce that all the rows are exact. □

LEMMA 3.21. *Let* $M \in R$-DMod *and* $f : X \to Y \in \mathrm{Hom}_A(X, Y)$ *for some* $X, Y \in$ Mod-A *with* $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{T}$. *Then*

$$f \otimes_A M : X \otimes_A M \to Y \otimes_A M$$

is an isomorphism.

PROOF. We shall use several times Lemma 2.40.

To prove the surjectivity, let $\sum_i y_i \otimes m_i \in Y \otimes_A M$. We can write $m_i = \sum_j r_{ij} m_{ij}$ with $r_{ij} \in R$ and $m_{ij} \in M$. The elements $y_i r_{ij} \in YR \subseteq \mathrm{Im}(f)$, therefore we can find elements $x_{ij} \in X$ such that $y_i r_{ij} = f(x_{ij})$ and then

$$\sum_i y_i \otimes m_i = \sum_{i,j} y_i r_{ij} \otimes m_{i,j} = \sum_{i,j} f(x_{ij}) \otimes m_{ij} = (f \otimes_A M)(\sum_{i,j} x_{ij} \otimes m_{ij})$$

To prove the injectivity, suppose $\sum_i x_i \otimes m_i \in \mathrm{Ker}(f \otimes_A M)$. We can suppose that the set $\{m_i : i \in I\}$ is a generator set of M over A if we add elements $x_i = 0$ whenever necessary.

Because $\sum_i x_i \otimes m_i \in \mathrm{Ker}(f \otimes_A M)$, then $\sum_i f(x_i) \otimes m_i = 0$ in $Y \otimes_A M$ and therefore we can find elements $y_k \in Y$ and $a_{ik} \in A$ such that

$$\sum_i a_{ik} m_i = 0 \ \ \forall k$$

$$\sum_k y_k a_{ik} = f(x_i) \ \ \forall i$$

For the elements $m_i \in M = RM$ we can find elements $r_{ij} \in R$ such that $m_i = \sum_j r_{ij} m_j$, and then $\sum_{i,j} a_{ik} r_{ij} m_j = 0$ for all k and using the fact that $R \otimes_A M \simeq M$ we deduce that $\sum_j (\sum_i a_{ik} r_{ij}) \otimes m_j = 0$ in $R \otimes_A M$ for all k. Therefore we can find elements $\hat{a}_{jkl} \in A$ and $\hat{r}_{kl} \in R$ such that

$$\sum_i a_{ik} r_{ij} = \sum_l \hat{r}_{kl} \hat{a}_{jkl} \ \forall i, j$$

$$\sum_j \hat{a}_{jkl} m_j = 0 \ \ \forall j, l$$

The elements $y_k \hat{r}_{kl} \in YR \subseteq \mathrm{Im}(f)$ and we can find elements $\hat{x}_{kl} \in X$ such that $y_k r_{kl} = f(x_{kl}) \ \forall k, l$. Then

$$\sum_i f(x_i) r_{ij} = \sum_{i,k} y_k a_{ik} r_{ij} = \sum_{k,l} y_k \hat{r}_{kl} \hat{a}_{jkl} = \sum_{k,l} f(x_{kl}) \hat{a}_{jkl}$$

and therefore $\sum_i x_i r_{ij} - \sum_{k,l} x_{kl} \hat{a}_{jkl} \in \mathrm{Ker}(f) \in \mathfrak{T}$.

The element $\sum\limits_{j,k,l} \hat{x}_{kl} \hat{a}_{jkl} \otimes m_j = \sum\limits_{k,l} x_{kl} \otimes \sum_j a_{jkl} m_j = 0$. Therefore, if we prove that $\sum\limits_{k,l} \hat{x}_{kl} \hat{a}_{jkl} \otimes m_j = \sum_i x_i r_{ij} \otimes m_j$ for all j we have finished because we would obtain $\sum_i x_i \otimes m_i = \sum\limits_{i,j} x_i r_{ij} \otimes m_j = 0$. Hence

As $m_j \in M = RM$ we know that $m_j = \sum_t r_{jt} m_t$.

$$\sum_i x_i r_{ij} - \sum_{k,l} x_{kl} \hat{a}_{jkl} \in \mathrm{Ker}(f) \in \mathfrak{T} \Rightarrow$$

$$(\sum_i x_i r_{ij} - \sum_{k,l} x_{kl} \hat{a}_{jkl}) r_{jt} = 0 \ \forall j, t$$

so that

$$\sum_{k,l} \hat{x}_{kl} \hat{a}_{jkl} \otimes m_j = \sum_{k,l,t} \hat{x}_{kl} \hat{a}_{jkl} r_{jt} \otimes m_t =$$

$$= \sum_{i,t} x_i r_{ij} r_{jt} \otimes m_t = \sum_i x_i r_{ij} \otimes m_j$$

This proves our claim. $\square$

LEMMA 3.22. *Let* $M \in \text{DMod-}R$ *and* $f \in \mathrm{Hom}_A(X, Y)$ *for some* $X, Y \in \text{Mod-}A$ *with* $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathfrak{T}$. *Then*

$$\mathrm{Hom}_A(M, f) : \mathrm{Hom}_A(M, X) \to \mathrm{Hom}_A(M, Y)$$

is an isomorphism.

Proof. Let $h : M \to X$ belong to $\mathrm{Ker}(\mathrm{Hom}_A(M, f))$, i.e. $f \circ h = 0$. Then $\mathrm{Im}(h) \subseteq \mathrm{Ker}(f)$. The module $\mathrm{Im}(h) = M/\mathrm{Ker}(h)$ is unitary, and therefore $\mathrm{Im}(h) = \mathrm{Im}(h)R \subseteq \mathrm{Ker}(f)R = 0$ so that $\mathrm{Im}(h) = 0$ and $h = 0$.

On the other hand, let $g : M \to Y$. We have to find a homomorphism $h : M \to X$ such that $f \circ h = g$. The module $Y/\mathrm{Im}(f) \in \mathcal{T}$ and $MR = M$, therefore $\mathrm{Hom}_A(M, Y/\mathrm{Im}(f)) = 0$ and we can find $\bar{g} : M \to \mathrm{Im}(f)$ such that $f^{ck} \circ \bar{g} = g$.

Consider the sequence

$$0 \to \mathrm{Ker}(f) \to X \to \mathrm{Im}(f) \to 0$$

and the morphism $\bar{g} : M \to \mathrm{Im}(f)$. Using the fact that $\mathrm{Ker}(f)R = 0$ and M is $\mathbf{u}$-codivisible, we can find a homomorphism $h : M \to X$ such that $f^{kc} \circ h = \bar{g}$. Then $g = f^{ck} \circ \bar{g} = f^{ck} \circ f^{kc} \circ h = f \circ h$. This proves the surjectivity of $\mathrm{Hom}_A(M, f)$. □

We want to prove the exactness of the functors $\mathrm{Hom}_A(M, -)$ and $M \otimes_A -$ for a module M_A. This could be done in a direct way, as we have done for $\mathrm{Hom}_A(-, M)$, but we are not going to do it like that now. Rather we are going to use the adjoint properties.

Proposition 3.23. *Let R be an idempotent ring, A and A' rings with identity such that R is a two-sided ideal of A. Let ${}_{A'}M_A$ be a bimodule. Then the functor $\mathrm{Hom}_A(M, -) : \mathrm{CMod}\text{-}R \to \mathrm{Mod}\text{-}A'$ has a left adjoint, and therefore it is left exact.*

Proof. Consider the following diagram of categories and functors:

$$\begin{array}{ccccc}
\mathrm{CMod}\text{-}R & \xrightarrow{\ \mathbf{i_C}\ } & \mathrm{Mod}\text{-}A & \xrightarrow{\ \mathrm{Hom}_A(M,-)\ } & \mathrm{Mod}\text{-}A' \\
 & & & & \\
\mathrm{CMod}\text{-}R & \xleftarrow[\ \mathbf{c}\ ]{} & \mathrm{Mod}\text{-}A & \xleftarrow[\ -\otimes_{A'} M\ ]{} & \mathrm{Mod}\text{-}A'
\end{array}$$

with $\mathbf{i_C} : \mathrm{CMod}\text{-}R \to \mathrm{Mod}\text{-}A$ the canonical inclusion. The functor $\mathbf{c}$ is a left adjoint of $\mathbf{i_C}$ and $- \otimes_{A'} P$ is a left adjoint of $\mathrm{Hom}_A(P, -)$. Then $\mathbf{c} \circ - \otimes_{A'} P$ is a left adjoint of $\mathrm{Hom}_A(P, -) \circ \mathbf{i}$. □

Proposition 3.24. *Let R' be an idempotent ring, A and A' rings with identity such that R' is a two-sided ideal on A'. Let ${}_{A'}M_A$ be a bimodule. Then the functor $- \otimes_{A'} M : \mathrm{DMod}\text{-}R' \to \mathrm{Mod}\text{-}A$ has a right adjoint, and therefore it is right exact.*

Proof. Consider the following diagram of categories and functors:

$$\mathrm{DMod}\text{-}R' \xrightarrow{\ \mathbf{i_D}\ } \mathrm{Mod}\text{-}A' \xrightarrow{\ -\otimes_{A'} M\ } \mathrm{Mod}\text{-}A$$

$$\mathrm{DMod}\text{-}R' \xleftarrow[\ \mathbf{d}\ ]{} \mathrm{Mod}\text{-}A' \xleftarrow[\ \mathrm{Hom}_A(M,-)\ ]{} \mathrm{Mod}\text{-}A$$

The functor $\mathbf{i_D}$ is a left adjoint of $\mathbf{d}$ and $-\otimes_{A'} M$ is left adjoint of $\mathrm{Hom}_A(M,-)$, therefore $-\otimes_{A'} M : \mathrm{DMod}\text{-}R' \to \mathrm{Mod}\text{-}A$ has a right adjoint and is right exact. □

Remark 3.25. *Let R be an idempotent ideal of a ring with identity A and A' another ring with identity, then using Lemma 3.22 we can deduce that if $M \in$ CMod-R, the functor $\mathrm{Hom}_A(M,-)$ is left exact from* CMod-R, Mod-R *and* DMod-R *to* Mod-A'.

On the other hand, let R' be an idempotent ideal of a ring A' and A other ring with identity, then using Lemma 3.21 we can deduce that if $M \in R'$-DMod, the functor $-\otimes_{A'} M$ is right exact from CMod-R', Mod-R' *and* DMod-R' *to* Mod-A.

6. Generators

The module $\mathbf{c}(R)$ is a generator of CMod-R. In order to find generators in the other categories, we only have to apply the equivalence functors. Therefore the module $\mathbf{u}(\mathbf{c}(R))$ is a generator of Mod-R. This module is $R/\mathbf{t}(R)$. The generator of DMod-R is $\mathbf{d}(\mathbf{c}(R)) \simeq R \otimes_A R$.

In this section we want to study objects that are finitely generated. The best category to study this kind of property is the category Mod-R, because for a module $M \in$ Mod-R, the submodules are the A-submodules N of M such that $NR = N$, and the sum of a family of submodules is the same if we calculate it in Mod-R or in Mod-A.

The definition of finitely generated object is as follows

Definition 3.26. An object M of a Grothendieck category $\mathcal{G}$ is called *finitely generated* if for every directed family of subobjects of M, $\{M_i : i \in I\}$, if $\sum_{i\in I} M_i = M$ then there exists an $i_0 \in I$ such that $M_{i_0} = M$.

Proposition 3.27. *Let M be a module in* Mod-R. *M is finitely generated if and only if we can find elements $m_1, \cdots, m_k \in M$ such that $M = m_1R + \cdots + m_kR$.*

Proof. Let F be a finite subset of M, we denote $M_F = \sum_{m\in F} mR$. The modules mR are unitary because $R^2 = R$ and torsion free because they are submodules of M and therefore $mR \in$ Mod-R and $M_F \in$

Mod-R. It is clear that $\sum_{F\in\mathcal{P}_0(M)} M_F = M$ where $\mathcal{P}_0(M)$ denote the class of finite subsets of M. If M is finitely generated, we can find $F_0 \in \mathcal{P}_0(M)$ such that $M_{F_0} = M$, and then $M = \sum_{m\in F_0} mR$ as we claimed.

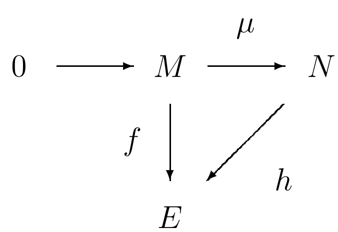
On the other hand suppose $M = m_1R + \cdots + m_kR$ for some $m_1, \cdots, m_k \in M$. Let $\{M_i : i \in I\}$ be a directed family of submodules of M such that $M = \sum_{i\in I} M_i$. The elements $m_t \in M = \sum_{i\in I} M_i$, therefore for every $t \in \{1, \cdots, k\}$, there exists $i_t \in I$ such that $m_t \in M_{i_t}$, if we have an i_0 greater than the $\{i_1, \cdots, i_k\}$ then $m_t \in M_{i_t} \subseteq M_{i_0}$ for all t and then

$$M = m_1R + \cdots + m_kR \subseteq M_{i_0} \subseteq M.$$

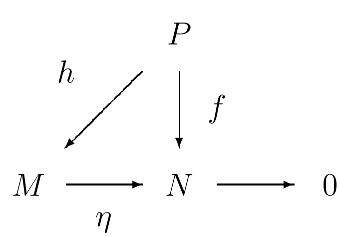
Then $M_{i_0} = M$ and M is finitely generated. □

7. Projective and Injective Modules

Definition 3.28. Let E be a module in CMod-R. We shall say that E is *injective* if for every monomorphism $\mu : M \to N$ in CMod-R and every morphism $f : M \to E$, there exists a morphism $h : N \to E$ such that $h \circ \mu = f$.



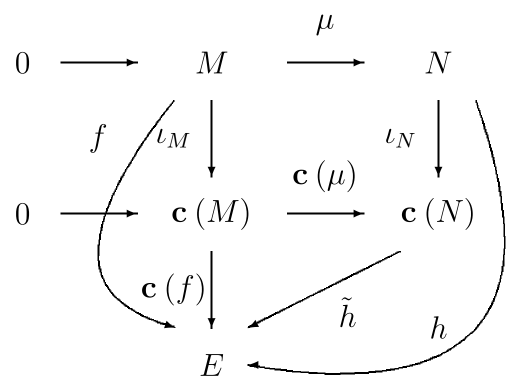
Definition 3.29. Let P be a module in DMod-R. We shall say that P is *projective* if for every epimorphism $\eta : N \to M$ in DMod-R and every morphism $f : P \to N$, there exists a morphism $h : P \to M$ such that $\eta \circ h = f$



Proposition 3.30. *Let* $E \in$ CMod-R. *Then* E *is injective in* CMod-R *if and only if* E *is injective in* Mod-A.

PROOF. If E is injective in Mod-A, then it is injective in CMod-R because every monomorphism in CMod-R is also a monomorphism in Mod-A.

On the other hand, suppose E is injective in CMod-R and $\mu : M \to N$ is a monomorphism in Mod-A. The functor $\mathbf{c}$ is left exact and then $\mathbf{c}(\mu)$ is a monomorphism and we have the following diagram:



Using the fact that E is injective in CMod-R we can find a morphism $\tilde{h} : \mathbf{c}(N) \to E$ such that $\tilde{h} \circ \mathbf{c}(\mu) = \mathbf{c}(f)$. If we define $h := \tilde{h} \circ \iota_N$, then

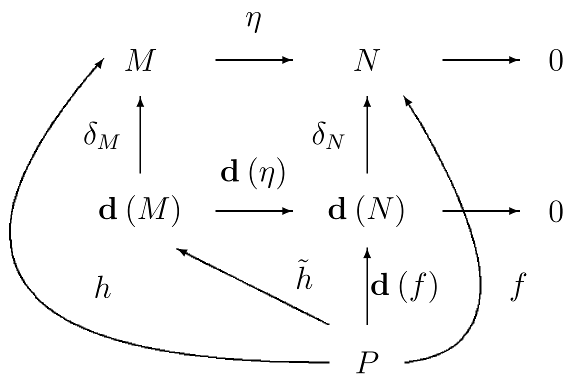
$$h \circ \mu = \tilde{h} \circ \iota_N \circ \mu = \tilde{h} \circ \mathbf{c}(\mu) \circ \iota_M = \mathbf{c}(f) \circ \iota_M = f.$$

□

PROPOSITION 3.31. *Let P be a module in* DMod-R. *Then P is projective in* DMod-R *if and only if P is projective in* Mod-A.

PROOF. ($\Leftarrow$). This is clear because each epimorphism in DMod-R is an epimorphism in Mod-A.

($\Rightarrow$). Let $\eta : M \to N$ be an epimorphism in Mod-A, and $f : P \to N$ a homomorphism. If we apply the functor $\mathbf{d}(N) = NR \otimes_A R$ we get that, if $\sum n_i r_i \otimes s_i \in \mathbf{d}(N)$ and $n_i = \eta(m_i)$ so that $\sum n_i r_i \otimes s_i = \mathbf{d}(\eta)\left(\sum m_i r_i \otimes s_i\right)$. Therefore $\mathbf{d}(\eta)$ is an epimorphism. We have the following diagram:



($\delta_M : MR \otimes_A R \to M$ and $\delta_N : NR \otimes_A R \to N$ are the canonical ones)

Using the fact that P is projective in DMod-R we can find a morphism $\tilde{h} : P \to \mathbf{d}(M)$ such that $\mathbf{d}(f) = \mathbf{d}(\eta) \circ \tilde{h}$. If we define $h = \delta_M \circ \tilde{h}$ we get

$$\eta \circ h = \eta \circ \delta_M \circ \tilde{h} = \delta_N \circ \mathbf{d}(f) = f.$$

□

Proposition 3.32. *Every module in* CMod-R *is a submodule of an injective module in* CMod-R.

Proof. Suppose $M \in$ CMod-R. Let $E(M)$ denote the injective envelope of M in Mod-A. This module is clearly $\mathbf{t}$-injective. We only have to prove that it is torsion free in order to prove that $E(M) \in$ CMod-R. The module $\mathbf{t}(E(M))$ is torsion and $\mathbf{t}(E(M)) \cap M$ is torsion. But M is torsion free, then $\mathbf{t}(E(M)) \cap M = 0$ and then $\mathbf{t}(E(M)) = 0$. □

It is not possible to dualize this result for projective modules. It is even possible to find an example in which the categories CMod-R, Mod-R and DMod-R have no nonzero projectives, see [**8**, Example 3.4.i].

8. Noncommutative Localization

Most of the results about noncommutative localization given for the category of unitary modules Mod-A, for a ring with identity A, are true in a Grothendieck category, and therefore they are true in our categories. We shall recall some of them, but we shall also give something more, because in our case we can generalize the concept of Gabriel filter for the ring R, and this is in general impossible for Grothendieck categories.

In this section we are going to fix the ring A as the Dorroh's extension of R. We could use another ring, but in this case, the filter would have right A-submodules of R and not right ideals (i.e. right $R \times \mathbb{Z}$-submodules).

DEFINITION 3.33. Let $\mathbb{G}$ be a Grothendieck category. A *preradical* r of $\mathbb{G}$ assigns to each object X a subobject $\mu_X : r(X) \to X$ in such a way that every morphism $f : X \to Y$ induces $r(f) : r(X) \to r(Y)$ by restriction.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \mu_X}\uparrow & & \uparrow{\scriptstyle \mu_Y} \\ r(X) & \xrightarrow[r(f)]{} & r(Y) \end{array}$$

(i.e. $\mu : r \to \mathrm{id}$ is a natural transformation such that μ_X is a monomorphism for all $X \in \mathbb{G}$).

DEFINITION 3.34. A preradical r is *idempotent* if $r \circ r = r$ and it is called a *radical* if $r(X/r(X)) = 0$ for all X.

To a preradical r one can associate two classes of objects of $\mathbb{G}$, namely

$$\mathbb{T}_r = \{X \in \mathbb{G} : \mu_X \text{ is an isomorphism}\}$$

$$\mathbb{F}_r = \{X \in \mathbb{G} : \mu_X \text{ is the morphism } 0\}$$

DEFINITION 3.35. A class $\mathbb{C}$ is called a *pretorsion class* if it is closed under quotient objects and coproducts, and it is a *pretorsion-free class* if it is closed under subobjects and products.

PROPOSITION 3.36. *Let r be a preradical in a Grothendieck category $\mathbb{G}$. Then the class $\mathbb{T}_r$ is a pretorsion class and $\mathbb{F}_r$ is a pretorsion-free class.*

PROOF. See [**14**, Section VI.1]. □

PROPOSITION 3.37. *Let $\mathbb{G}$ be a Grothendieck category. Then there is a bijective correspondence between idempotent preradicals of $\mathbb{G}$ and pretorsion classes of objects of $\mathbb{G}$. Dually, there is a bijective correspondence between radicals of $\mathbb{G}$ and pretorsion-free classes of objects of $\mathbb{G}$.*

PROOF. See [**14**, Proposition VI.1.4]. We are going to give here merely the definition of the bijection. If r is an idempotent preradical of $\mathbb{G}$, the pretorsion class that is assigned is $\mathbb{T}_r$ and the pretorsion-free class is $\mathbb{F}_r$. If $\mathbb{T}$ is a pretorsion class, then the preradical that is assigned is defined as follows: if $M \in \mathbb{G}$, $r(M)$ is the largest subobject of M that belongs to $\mathbb{T}$. If $\mathbb{F}$ is a pretorsion-free class, then the corresponding preradical is defined for an object $M \in \mathbb{G}$ as the largest subobject N

of M such that $M/N \in \mathbb{F}$. (This object is the sum of all the the subobjects C of M with the property of $M/C \in \mathbb{F}$. Using the fact that the category is locally small, i.e. the class of subobjects of M is a set, we deduce that this maximal subobject exists). □

PROPOSITION 3.38. *The following assertions are equivalent for a preradical r:*

1. *r is a left exact functor.*
2. *If N is a subobject of M, $r(N) = r(M) \cap N$.*
3. *r is idempotent and $\mathbb{T}_r$ is closed under subobjects.*

PROOF. See [**14**, Proposition VI.1.7]. □

DEFINITION 3.39. A pretorsion class is called *hereditary* if it is closed under subobjects.

COROLLARY 3.40. *There is a bijective correspondence between left exact preradicals and hereditary pretorsion classes.*

PROOF. See [**14**, Section VI.1]. □

DEFINITION 3.41. A *torsion theory* for $\mathbb{G}$ is a pair $(\mathbb{T}, \mathbb{F})$ of classes of objects of $\mathbb{G}$ such that

1. $\mathrm{Hom}(T, F) = 0$ for all $T \in \mathbb{T}$ and $F \in \mathbb{F}$.
2. If $\mathrm{Hom}(M, F) = 0$ for all $F \in \mathbb{F}$, then $M \in \mathbb{T}$.
3. If $\mathrm{Hom}(T, M) = 0$ for all $T \in \mathbb{T}$, then $M \in \mathbb{F}$.

$\mathbb{T}$ is called a *torsion class* and its objects are *torsion objects*, while $\mathbb{F}$ is a *torsion-free class* consisting of *torsion-free objects*.

Any given class $\mathbb{C}$ of objects generates a torsion theory in the following way

$$\mathbb{F} := \{F \in \mathbb{G} : \mathrm{Hom}(C, F) = 0 \text{ for all } C \in \mathbb{C}\}$$

$$\mathbb{T} := \{T \in \mathbb{G} : \mathrm{Hom}(T, F) = 0 \text{ for all } F \in \mathbb{F}\}$$

This pair $(\mathbb{T}, \mathbb{F})$ is a torsion theory and $\mathbb{T}$ is the smallest torsion class containing $\mathbb{C}$.

Dually, any given class $\mathbb{C}$ of objects cogenerates a torsion theory in the following way

$$\mathbb{T} := \{T \in \mathbb{G} : \mathrm{Hom}(T, C) = 0 \text{ for all } C \in \mathbb{C}\}$$

$$\mathbb{F} := \{F \in \mathbb{G} : \mathrm{Hom}(T, F) = 0 \text{ for all } T \in \mathbb{T}\}$$

This pair $(\mathbb{T}, \mathbb{F})$ is a torsion theory and $\mathbb{F}$ is the smallest torsion-free class containing $\mathbb{C}$.

PROPOSITION 3.42. *The following properties of a class $\mathbb{T}$ of objects of $\mathbb{G}$ are equivalent:*

1. $\mathbb{T}$ *is a torsion class for some torsion theory.*
2. $\mathbb{T}$ *is closed under quotient objects, coproducts and extensions.*

Proof. See [**14**, Proposition VI.2.1]. □

Proposition 3.43. *The following properties of a class* $\mathbb{F}$ *of objects of* $\mathbb{G}$ *are equivalent:*

1. $\mathbb{F}$ *is a torsion-free class for some torsion theory.*
2. $\mathbb{F}$ *is closed under subobjects, products and extensions.*

Proof. See [**14**, Proposition VI.2.2]. □

Proposition 3.44. *There is a bijective correspondence between torsion theories and idempotent radicals.*

Proof. See [**14**, Proposition VI.2.3]. The idempotent radical is defined as in Proposition 3.37. □

We are going to start working with the categories CMod-R, DMod-R and Mod-R for an idempotent ring R.

Remark 3.45. *There is a bijective correspondence between torsion theories in* CMod-R*,* DMod-R *and* Mod-R*.*

Proof. The definitions we have given are categorical and therefore, the category equivalences give the bijections between the torsion and torsion-free classes. □

A torsion theory $(\mathbb{T}, \mathbb{F})$ is called hereditary if $\mathbb{T}$ is closed under subobjects. If we combine Corollary 3.40 and Proposition 3.44 we obtain

Proposition 3.46. *There is a bijective correspondence between hereditary torsion theories and left exact radicals.*

Definition 3.47. Let R be an idempotent ring. A *right Gabriel topology* on R is a non-empty set $\mathfrak{G}$ of right ideals on R such that

T1. If $\mathfrak{a} \in \mathfrak{G}$ and $\mathfrak{b} \leq R_R$ with $\mathfrak{a} \leq \mathfrak{b}$, then $\mathfrak{b} \in \mathfrak{G}$.
T2. If $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathfrak{G}$, then $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{G}$.
T3. If $\mathfrak{a} \in \mathfrak{G}$ and $r \in R$, then $(\mathfrak{a} : r) := \{s \in R : rs \in \mathfrak{a}\} \in \mathfrak{G}$.
T4. If $\mathfrak{a}$ is a right ideal on R and there exists $\mathfrak{b} \in \mathfrak{G}$ such that $(\mathfrak{a} : b) \in \mathfrak{G}$ for all $b \in \mathfrak{b}$, then $\mathfrak{a} \in \mathfrak{G}$.

Definition 3.48. Let $\mathfrak{G}$ be a right Gabriel topology on R. We shall say that a module M is $\mathfrak{G}$-*discrete* if for all $m \in M$, $\text{r.ann}(m) \in \mathfrak{G}$.

Lemma 3.49. *Let* $\mathfrak{G}$ *be a right Gabriel topology on* R*. Then*

1. $R \in \mathfrak{G}$.
2. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are in* $\mathfrak{G}$*, then* $\mathfrak{a}\mathfrak{b} \in \mathfrak{G}$.
3. *If* $\mathfrak{a} \in \mathfrak{G}$ *then* $\mathfrak{a}R \in \mathfrak{G}$.

PROOF. The statement (1) is trivial because of T1. The statement (3) is a consequence of (1) and (2). In order to prove (2), let $a \in \mathfrak{a}$. Then $(\mathfrak{a}\mathfrak{b} : a) \supseteq \mathfrak{b}$, therefore $(\mathfrak{a}\mathfrak{b} : a) \in \mathfrak{G}$ because of T1, and then if we apply T4, we obtain our claim. □

PROPOSITION 3.50. *Let $\mathfrak{G}$ be a right Gabriel topology on R. Let $M \in$ Mod-A. Then the following conditions are equivalent:*

1. $\mathbf{c}(M)$ *is $\mathfrak{G}$-discrete in* CMod-R.
2. $\mathbf{u}(M/\mathbf{t}(M)) = \mathbf{u}(M)/\mathbf{t}(\mathbf{u}(M))$ *is $\mathfrak{G}$-discrete in* Mod-R.
3. $\mathbf{d}(M)$ *is $\mathfrak{G}$-discrete in* DMod-R.

PROOF. $(1 \Rightarrow 2)$. Suppose $\mathbf{c}(M)$ is $\mathfrak{G}$-discrete in CMod-R. Then for all $f : R \to M/\mathbf{t}(M)$, r.ann$(f) \in \mathfrak{G}$, i.e.

$$\mathrm{Ker}(f) = \{r \in R : f(r) = 0\} = \{r \in R : fr = 0\} = \mathrm{r.ann}(f) \in \mathfrak{G}$$

Let $\alpha := \sum_i (m_i + \mathbf{t}(M))r_i \in \mathbf{u}(M/\mathbf{t}(M))$. For all i, let $f_i : R \to M/\mathbf{t}(M)$ be the morphism defined by $f_i(r) = m_i r + \mathbf{t}(M)$. Then $\alpha = \sum_i f_i(r_i)$. The element $\sum_i f_i r_i \in \mathbf{c}(M)$, and hence $\mathrm{Ker}(\sum_i f_i r_i) \in \mathfrak{G}$. But

$$\mathrm{Ker}(\sum_i f_i r_i) = \{r \in R : \sum_i f_i r_i(r) = 0\} = \{r \in R : \sum_i f_i(r_i) r = 0\}$$
$$= \mathrm{r.ann}(\sum_i m_i r_i + \mathbf{t}(M)) = \mathrm{r.ann}(\alpha)$$

Hence r.ann$(\alpha) \in \mathfrak{G}$.

$(2 \Rightarrow 3)$. Suppose $\mathbf{u}(M)/\mathbf{t}(\mathbf{u}(M))$ is $\mathfrak{G}$-discrete, and let $\sum_i m_i r_i \otimes s_i \in \mathbf{d}(M)$. Let $\mathfrak{a} = \mathrm{r.ann}(\sum m_i r_i s_i + \mathbf{t}(\mathbf{u}(M))) \in \mathfrak{G}$, and let $\mathfrak{b} = \mathrm{r.ann}(\sum_i m_i r_i \otimes s_i)$. We are going to prove that $\mathfrak{a}R \subseteq \mathfrak{b}$ and applying the previous lemma and T1, we would obtain $\mathfrak{b} \in \mathfrak{G}$.

Let $a \in \mathfrak{a}$ and $\bar{r}, \bar{s} \in R$. Then $\sum_i m_i r_i s_i a \in \mathbf{t}(\mathbf{u}(M))$, so that $\sum_i m_i r_i s_i a \bar{r} = 0$. Now

$$(\sum_i m_i r_i \otimes s_i) a \bar{r} \bar{s} = \sum_i m_i r_i s_i a \bar{r} \otimes \bar{s} = 0,$$

and $a\bar{r}\bar{s} \in \mathfrak{b}$. This proves that $\mathfrak{b} \supseteq \mathfrak{a}R^2 = \mathfrak{a}R$ and that $\mathfrak{b} \in \mathfrak{G}$.

$(3 \Rightarrow 1)$. Suppose $\mathbf{d}(M)$ is $\mathfrak{G}$-discrete, and let $f : R \to M/\mathbf{t}(M)$. We have to prove that $\mathrm{Ker}(f) \in \mathfrak{G}$. Let $r \in R$, $r = \sum_i r_i s_i t_i$ where $r_i, s_i, t_i \in R$. Let $m_i \in M$ be elements such that $f(r_i) = m_i + \mathbf{t}(M)$. Then

$$(\mathrm{Ker}(f) : r) = \{a \in R : ra \in \mathrm{Ker}(f)\} = \{a \in R : f(r)a = 0\}$$
$$= \{a \in R : \sum_i m_i r_i s_i a \in \mathbf{t}(M)\}$$

$$\supseteq \{a \in R : \sum_i m_i r_i s_i a = 0\}$$

$$\supseteq \{a \in R : (\sum_i m_i r_i \otimes s_i) a = 0\}$$

$$= \text{r.ann}(\sum_i m_i r_i \otimes s_i) \in \mathfrak{G}.$$

This proves that $(\mathrm{Ker}(f) : r) \in \mathfrak{G}$ for all $r \in R$, and then using T4 we deduce that $\mathrm{Ker}(f) \in \mathfrak{G}$. □

Proposition 3.51. *There is a bijective correspondence between:*

(1) *Right Gabriel topologies on R.*
(2) *Hereditary torsion theories for* CMod-R.
(*2) *Hereditary torsion theories for* Mod-R.
(**2) *Hereditary torsion theories for* DMod-R.
(3) *Left exact radicals for* CMod-R.
(*3) *Left exact radicals for* Mod-R.
(**3) *Left exact radicals for* DMod-R.

Proof. If we find the bijective correspondence between (1.), (2.) and (3.), then we can find all the other ones. If $\mathfrak{G}$ is a Gabriel topology on R, the corresponding torsion class is the class of $\mathfrak{G}$-discrete modules, and this class of modules is well behaved with respect to the equivalences between the categories because of the previous proposition. The equivalence between (2.) and (3.) has already been proved in Proposition 3.46.

Now we only have to find the bijective correspondence between (1.) and ($*$2.). Suppose $\mathfrak{G}$ is a Gabriel topology on R, and let $\mathbb{T}$ be the class of $\mathfrak{G}$-discrete modules on Mod-R.

Let $M_i \in \mathbb{T}$. We have to prove that $\coprod M_i \in \mathbb{T}$. Consider the module $\coprod M_i$. This module is the same if we calculate it on Mod-R or in Mod-A because if all $M_i \in$ Mod-R, then $\coprod M_i$ calculated in Mod-A is in Mod-R. Let $(m_i)_{i\in I} \in \coprod_{i\in I} M_i$ and define $I_0 = \{i \in I : m_i \neq 0\}$. This set is finite, $\text{r.ann}((m_i)_{i\in I}) = \cap_{i\in I_0}\text{r.ann}(m_i)$. The ideal $\text{r.ann}(m_i) \in \mathfrak{G}$ because $M_i \in \mathbb{T}$, so that $\cap_{i\in I_0}\text{r.ann}(m_i) \in \mathfrak{G}$ using condition T2 several times. Thus $\coprod_{i\in I} M_i \in \mathbb{T}$.

Let $M \in \mathbb{T}$ and $\eta : M \to N$ be an epimorphism, i.e. $\mathrm{Coker}(\eta) \in \mathfrak{T}$. Let $n \in N$ and $r \in R$ and note that $(\text{r.ann}n : r) = \{s \in R : rs \in \text{r.ann}n\} = \{s \in R : nrs = 0\}$. The element $nr \in NR \subseteq \mathrm{Im}(\eta)$ and therefore we can find $m \in M$ such that $\eta(m) = nr$. If $ms = 0$ for some $s \in R$, then $\eta(m)s = nrs = 0$, therefore $\text{r.ann}(m) \subseteq (\text{r.ann}n : r)$, and this is true for all $r \in R$. Then using T4 and that $R \in \mathfrak{G}$ we deduce that $\text{r.ann}n \in \mathfrak{G}$. Thus $N \in \mathbb{T}$.

Let $M \in \mathbb{T}$ and N be a submodule of M. Then N is a submodule in the category Mod-A and then N is a subset of M. If $n \in N \subseteq M$, then $\text{r.ann}(n) \in \mathfrak{G}$ because $M \in \mathbb{T}$.

Consider the following short exact sequence in Mod-R:

$$0 \longrightarrow L \xrightarrow{\mu} M \xrightarrow{\eta} N \longrightarrow 0$$

with $L, N \in \mathbb{T}$. Then $\mathrm{Ker}(\eta)/\mathrm{Im}(\mu) \in \mathfrak{T}$ and $N/\mathrm{Im}(\eta) \in \mathfrak{T}$. Let $m \in M$, we have to prove that $\mathrm{r.ann}(m) \in \mathfrak{G}$. The right ideal $\mathrm{r.ann}(\eta(m)) \in \mathfrak{G}$, then $\mathfrak{b} := \mathrm{r.ann}(\eta(m))R \in \mathfrak{G}$. For all $b \in \mathfrak{b}$, $b = \sum a_i r_i$ with $a_i \in \mathrm{r.ann}(\eta(m))$ and $r_i \in R$, hence $mb = \sum m a_i r_i$. The element $m a_i \in \mathrm{Ker}(\eta)$ and therefore $m a_i r_i \in \mathrm{Im}(\mu)$ and we can find an element $l \in L$ such that $\sum m a_i r_i = \mu(l)$. The ideal $\mathrm{r.ann} mb = \mathrm{r.ann}(\mu(l)) = \mathrm{r.ann}(l)$ because μ is a monomorphism. Then $(\mathrm{r.ann}(m) : b) = \mathrm{r.ann}(mb) = \mathrm{r.ann}(l) \in \mathfrak{G}$. We have proved that for all $b \in \mathfrak{b}$, $(\mathrm{r.ann}(m) : b) \in \mathfrak{G}$ and using T4, that $\mathrm{r.ann}(m) \in \mathfrak{G}$. Thus $\mathbb{T}$ is a torsion class for a hereditary torsion theory.

Conversely, let $(\mathcal{T}, \mathcal{F})$ be a torsion theory in Mod-R. We define

$$\mathfrak{G} = \{\mathfrak{a} \leq R : \mathrm{Hom}_R(R/\mathfrak{a}, F) = 0 \quad \forall F \in \mathbb{F}\}.$$

We are going to see first that $\mathfrak{a} \in \mathfrak{G}$ if and only if $\frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \in \mathbb{T}$. We shall use both characterizations whenever necessary.

If $\mathfrak{a} \in \mathfrak{G}$, suppose $\frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \notin \mathbb{T}$. Then there exists $h : \frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \to F$ ($F \in \mathbb{F}$ and $h \neq 0$). If we consider $\iota : R/\mathfrak{a} \to \frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})}$, then $h \circ \iota = 0$ and this means that we have a morphism $\bar{h} : \mathrm{Coker}(\iota) \to F$. But $\mathrm{Coker}(\iota) = 0$ and $F \in$ Mod-R. This proves that $\bar{h} = 0$ and therefore $h = 0$.

Conversely, if $\frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \in \mathbb{T}$, let $h : R/\mathfrak{a} \to F$ be a R-homomorphism with $F \in \mathbb{F}$. $h(\mathbf{t}\,(R/\mathfrak{a})) = 0$ because $h(\mathbf{t}\,(R/\mathfrak{a}))R = 0$ and $F \in$ Mod-R, and therefore h induces $\bar{h} : \mathrm{Im}(\iota) \to F$. But $\bar{h} : \frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \to F$ has to be the 0 morphism, then $\bar{h} = 0$ and therefore $h = 0$ as we claimed.

We have to check that $\mathfrak{G}$ is a right Gabriel Topology for R.

T1 Suppose $\mathfrak{a} \in \mathfrak{G}$ and $\mathfrak{a} \leq \mathfrak{b}$, then there exists an epimorphism $p : R/\mathfrak{a} \to R/\mathfrak{b}$. If we have $h : R/\mathfrak{b} \to F$ with $h \neq 0$, $F \in \mathbb{F}$, then $h \circ p : R/\mathfrak{a} \to F$ is not 0 because p is an epimorphism, a contradiction. Thus $\mathfrak{b} \in \mathfrak{G}$.

T2 Suppose $\mathfrak{a}$ and $\mathfrak{b}$ belong to $\mathfrak{G}$, and consider the canonical monomorphism $j : R/\mathfrak{a} \cap \mathfrak{b} \to R/\mathfrak{a} \oplus R/\mathfrak{b}$,

$$\begin{array}{ccccc}
\mathbf{t}\left(\frac{R}{\mathfrak{a}\cap\mathfrak{b}}\right) & \longrightarrow & \frac{R}{\mathfrak{a}\cap\mathfrak{b}} & \longrightarrow & \frac{\frac{R}{\mathfrak{a}\cap\mathfrak{b}}}{\mathbf{t}\left(\frac{R}{\mathfrak{a}\cap\mathfrak{b}}\right)} \\
\Big\downarrow \mathbf{t}(j) & & \Big\downarrow j & & \Big\downarrow \mathbf{t}^{-1}(j) \\
\mathbf{t}\left(\frac{R}{\mathfrak{a}}\right) \oplus \mathbf{t}\left(\frac{R}{\mathfrak{b}}\right) & \longrightarrow & \frac{R}{\mathfrak{a}} \oplus \frac{R}{\mathfrak{b}} & \longrightarrow & \frac{\frac{R}{\mathfrak{a}}}{\mathbf{t}\left(\frac{R}{\mathfrak{a}}\right)} \oplus \frac{\frac{R}{\mathfrak{b}}}{\mathbf{t}\left(\frac{R}{\mathfrak{b}}\right)}
\end{array}$$

The morphism $\mathbf{t}^{-1}(j)$ is a monomorphism, and $\frac{\frac{R}{\mathfrak{a}}}{\mathbf{t}\left(\frac{R}{\mathfrak{a}}\right)} \oplus \frac{\frac{R}{\mathfrak{b}}}{\mathbf{t}\left(\frac{R}{\mathfrak{b}}\right)} \in \mathbb{T}$. Then $\frac{\frac{R}{\mathfrak{a}\cap\mathfrak{b}}}{\mathbf{t}\left(\frac{R}{\mathfrak{a}\cap\mathfrak{b}}\right)} \in \mathbb{T}$ and $\mathfrak{a} \cap \mathfrak{b} \in \mathfrak{G}$.

T3 Suppose $\mathfrak{a} \in \mathfrak{G}$ and that $r \in R$. The left multiplication by r induces an exact sequence in Mod-A = MOD-R,

$$0 \to (\mathfrak{a} : r) \to R \to R/\mathfrak{a},$$

and then $R/(\mathfrak{a} : r) \le R/\mathfrak{a}$, this means that $\mathbf{t}^{-1}(R/(\mathfrak{a} : r)) \le \frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \in \mathbb{T}$ and using that $\mathbb{T}$ is closed under subobjects, $\mathbf{t}^{-1}(R/(\mathfrak{a} : r)) \in \mathbb{T}$ and $(\mathfrak{a} : r) \in \mathfrak{G}$ as we claimed.

T4 Suppose $\mathfrak{a}$ is a right ideal such that $(\mathfrak{a} : b) \in \mathfrak{G}$ for all $b \in \mathfrak{b}$ for some $\mathfrak{b} \in \mathfrak{G}$. We consider the exact sequence

$$0 \to \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b} \to R/\mathfrak{a} \to R/(\mathfrak{a} + \mathfrak{b}) \to 0$$

We are going to check that $\mathbf{m}\,(\mathfrak{b}/\mathfrak{a} \cap \mathfrak{b})$ and $\mathbf{m}\,(R/(\mathfrak{a} + \mathfrak{b})) = \mathbf{t}^{-1}(R/(\mathfrak{a} + \mathfrak{b}))$ are in $\mathbb{T}$. If this holds, applying that $\mathbf{m}$ is an exact functor[3], then the sequence

$$0 \to \mathbf{m}\,(\mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}) \to \mathbf{t}^{-1}(R/\mathfrak{a}) \to \mathbf{t}^{-1}(R/(\mathfrak{a} + \mathfrak{b})) \to 0$$

is exact (in Mod-R) and as $\mathbb{T}$ is closed under extensions, $\frac{R/\mathfrak{a}}{\mathbf{t}(R/\mathfrak{a})} \in \mathbb{T}$ and $\mathfrak{a} \in \mathfrak{G}$. Then let us prove the claim.

$\mathbf{t}^{-1}(R/\mathfrak{a} + \mathfrak{b})$ is in $\mathbb{T}$ because $\mathfrak{b} \le \mathfrak{a} + \mathfrak{b}$ and T1.

Suppose $h : \mathbf{m}\,(\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})) \to F$, $F \in \mathbb{F}$, and $\iota : \frac{\mathfrak{b}}{\mathfrak{a}\cap\mathfrak{b}} \to \frac{\frac{\mathfrak{b}}{\mathfrak{a}\cap\mathfrak{b}}}{\mathbf{t}\left(\frac{\mathfrak{b}}{\mathfrak{a}\cap\mathfrak{b}}\right)}$ is the projection. We are going to prove that for all $r \in R$ and $b \in \mathfrak{b}$, $h(\iota(br + \mathfrak{a} \cap \mathfrak{b})) = 0$. Suppose that this does not happen for some $b \in \mathfrak{b}$ and $r \in R$. Then consider the morphism induced by left multiplication by b, $R \to \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$. The kernel of this morphism is $(\mathfrak{a} \cap \mathfrak{b} : b) = (\mathfrak{a} : b) \in \mathfrak{G}$, and then we have a R-monomorphism $\mu : R/(\mathfrak{a} : b) \to \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$. The composition $h \circ \iota \circ \mu : R/(\mathfrak{a} : b) \to F$ carries r to $h(\iota(\mu(r))) = h(\iota(rb + \mathfrak{a} \cap \mathfrak{b})) \neq 0$. But this is a contradiction because $(\mathfrak{a} : b) \in \mathfrak{G}$. Then $h(\iota((\mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b}))R)) = 0$ and therefore $h(\iota(\mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}))R \subseteq \mathbf{t}\,(F) = 0$. This proves that $h \circ \iota = 0$. Let $x \in \mathbf{t}^{-1}(\mathfrak{b}/\mathfrak{a} \cap \mathfrak{b})$ and $r \in R$. Then $xr \in \mathfrak{b}/\mathfrak{a} \cap \mathfrak{b}$ and $h(xr) = 0$. As this can be done for all $r \in R$, $h(x) \in \mathbf{t}\,(F) = 0$ for every x. This proves $h = 0$, as we claimed.

□

[3]In this proof we use the functor $\mathbf{t}^{-1}$ instead of $\mathbf{m}$ everytime that the module considered is unitary. For this kind of modules both functors coincide.

CHAPTER 4

Morita Theory

We shall fix the following notation for the whole chapter. R and R' will be idempotent rings, A and A' rings with identity such that R is a two-sided ideal of A and R' is a two sided ideal of A'.

1. Functors Between the Categories

PROPOSITION 4.1. *Let ${}_{A'}P_A, {}_{A'}\bar{P}_A$ be bimodules and $\varphi : P \to \bar{P}$ a bimodule homomorphism. Then the following conditions are equivalent:*

1. $\mathrm{Hom}_A(\varphi, -)$ *is a natural equivalence between the functors* $\mathrm{Hom}_A(P, -)$ *and* $\mathrm{Hom}_A(\bar{P}, -)$ *from* CMod-R *to* $\mathcal{A}b$.
2. $\varphi \otimes_A -$ *is a natural equivalence between the functors* $P \otimes_A -$ *and* $\bar{P} \otimes_A -$ *from* R-DMod *to* $\mathcal{A}b$.
3. $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Coker}(\varphi)$ *are in* $\mathcal{T}$.

PROOF. The condition (3) implies the conditions (1) because of Lemma 3.19 and the condition (2) because of Lemma 3.21.

Suppose condition (1) holds. Then for all $M \in$ CMod-R,

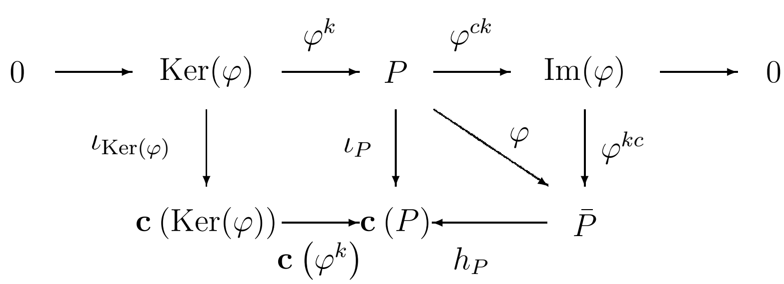
$$\mathrm{Hom}_A(\varphi, M) : \mathrm{Hom}_A(\bar{P}, M) \to \mathrm{Hom}_A(P, M)$$

is an isomorphism. Now

$$\mathrm{Ker}(\mathrm{Hom}_A(\varphi, M)) = \{f : P' \to M : f \circ \varphi = 0\} = \mathrm{Hom}_A(\mathrm{Coker}(\varphi), M)$$

If $\mathrm{Hom}_A(\mathrm{Coker}(\varphi), M) = 0$ for all $M \in$ CMod-R, then $\mathrm{Hom}_A(\mathrm{Coker}(\varphi), \mathbf{c}\,(\mathrm{Coker}(\varphi)) = 0$ and then $\mathbf{c}\,(\mathrm{Coker}(\varphi)) = 0$ and $\mathrm{Coker}(\varphi) \in \mathcal{T}$.

Consider the following diagram:



Using the surjectivity of $\mathrm{Hom}_A(\varphi, \mathbf{c}\,(P))$ we can find a morphism $h_P : \bar{P} \to \mathbf{c}\,(P)$ such that $h_P \circ \varphi = \iota_P$, and then $0 = h_P \circ \varphi \circ \varphi^k = \iota_P \circ \varphi^k$

and then $\mathbf{c}\left(\varphi^k\right) \circ \iota_{\mathrm{Ker}(\varphi)} = 0$. But $\mathbf{c}\left(\varphi^k\right)$ is a monomorphism. Then $\iota_{\mathrm{Ker}(\varphi)} = 0$. But this is true if and only if $\mathrm{Ker}(\varphi) \in \mathcal{T}$.

Suppose now that (2) holds, then for all $M \in R$-DMod, the morphism

$$\varphi \otimes_A M : P \otimes_A M \to \bar{P} \otimes_A M$$

is an isomorphism. The condition $\varphi \otimes_A M$ epimorphism implies $\bar{P}/\mathrm{Im}(\varphi) \otimes_A M = 0$. If we apply this to $R \otimes_A R$, then $P/\mathrm{Im}(\varphi) \otimes_A R \otimes_A R = 0$ and then $(P/\mathrm{Im}(\varphi))R = 0$ and $\mathrm{Coker}(\varphi) = P/\mathrm{Im}(\varphi) \in \mathcal{T}$. Suppose $\mathrm{Ker}(\varphi) \notin \mathcal{T}$. Then $\mathbf{d}(\mathrm{Ker}(\varphi)) \neq 0$. But we know that $\mathbf{d}(\mathrm{Ker}(\varphi)) = \mathrm{Ker}(\varphi) \otimes_A R \otimes_A R$ and

$$\mathrm{Ker}(\varphi \otimes_A R \otimes_A R) = \{\sum_i p_i \otimes r_i \otimes s_i \in P \otimes_A R \otimes_A R : \sum_i \varphi(p_i) \otimes r_i \otimes s_i = 0\}$$

If $\sum_i p_i \otimes r_i \otimes s_i \in \mathbf{d}(\mathrm{Ker}(\varphi)) \setminus \{0\}$ then $\sum_i p_i \otimes r_i \otimes s_i \in \mathrm{Ker}(\varphi \otimes_A R \otimes_A R) \setminus \{0\}$ and this is not possible, because $R \otimes_A R \in R$-DMod and $\phi \otimes_A R \otimes_A R$ is an isomorphism. □

In the next two corollaries we shall deduce that if we chose the bimodules inside the categories CMod-R and DMod-R, then they have a certain uniqueness property.

COROLLARY 4.2. *Let P and $\bar{P}$ be (A', A)-bimodules such that $P_A, \bar{P}_A \in$ DMod-R, and let $\varphi : P \to \bar{P}$ be a bimodule homomorphism. Then the following conditions are equivalent:*

1. $\mathrm{Hom}_A(\varphi, -)$ *is a natural equivalence between the functors* $\mathrm{Hom}_A(P, -)$ *and* $\mathrm{Hom}_A(\bar{P}, -)$ *from* CMod-R *to* $\mathcal{A}b$.
2. $\varphi \otimes_A -$ *is a natural equivalence between the functors* $P \otimes_A -$ *and* $\bar{P} \otimes_A -$ *from* R-DMod *to* $\mathcal{A}b$.
3. φ *is an isomorphism.*

PROOF. These conditions are equivalent to $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ in $\mathcal{T}$. But both are unitary modules using Proposition 2.41 and $\mathrm{Ker}(\varphi) = 0$ and $\mathrm{Coker}(\varphi) = 0$. □

COROLLARY 4.3. *Let P and $\bar{P}$ be (A', A)-bimodules such that $P_A, \bar{P}_A \in$ CMod-R, and let $\varphi : P \to \bar{P}$ be a bimodule homomorphism. Then the following conditions are equivalent:*

1. $\mathrm{Hom}_A(\varphi, -)$ *is a natural equivalence between the functors* $\mathrm{Hom}_A(P, -)$ *and* $\mathrm{Hom}_A(\bar{P}, -)$ *from* CMod-R *to* $\mathcal{A}b$.
2. $\varphi \otimes_A -$ *is a natural equivalence between the functors* $P \otimes_A -$ *and* $\bar{P} \otimes_A -$ *from* R-DMod *to* $\mathcal{A}b$.
3. φ *is an isomorphism.*

PROOF. These conditions are equivalent to $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ in $\mathcal{T}$. But both are torsion-free modules using Proposition 2.30 and $\mathrm{Ker}(\varphi) = 0$ and $\mathrm{Coker}(\varphi) = 0$. □

Corollary 4.4. *Let P be a (A', A)-bimodule. Then the following functors are equivalent:*

$$\mathrm{Hom}_A(\mathbf{c}\,(P)\,, -) \simeq \mathrm{Hom}_A(P, -) \simeq \mathrm{Hom}_A(\mathbf{d}\,(P)\,, -)$$

$$\mathbf{c}\,(P) \otimes_A - \simeq P \otimes_A \simeq \mathbf{d}\,(P) \otimes_A -$$

Proof. We only have to apply the fact that the canonical homomorphisms $\mathbf{d}\,(P) \to P$ and $P \to \mathbf{c}\,(P)$ given in Proposition 2.29 and Proposition 2.38 are bimodule homomorphisms and have torsion kernel and cokernel. □

Proposition 4.5. *Let ${}_{A'}P_A$ be a bimodule and let*

$$\begin{array}{rccc} \mu: & R' \otimes_{A'} P & \to & P \\ & r' \otimes p & \mapsto & r'p \end{array}$$

Then the following conditions are equivalent:

1. $\mathrm{Hom}_A(P, -)$ *is a functor between* CMod-R *and* CMod-R'.
2. $P \otimes_A -$ *is a functor between* R-DMod *and* R'-DMod.
3. $\mathrm{Ker}(\mu)$ *and* $\mathrm{Coker}(\mu)$ *are in* $\mathcal{T}$.

Proof. Condition (1) is equivalent to the following condition,

$$\forall M \in \text{CMod-}R, \mathrm{Hom}_A(P, M) \in \text{CMod-}R' \iff$$

$$\forall M \in \text{CMod-}R, \mathrm{Hom}_{A'}(R', \mathrm{Hom}_A(P, M)) \simeq \mathrm{Hom}_A(P, M) \iff$$

$$\forall M \in \text{CMod-}R, \mathrm{Hom}_A(R' \otimes_{A'} P, M) \simeq \mathrm{Hom}_A(P, M) \iff$$

$$\forall M \in \text{CMod-}R, \mathrm{Hom}_A(\mu, M) : \mathrm{Hom}_A(P, M) \to \mathrm{Hom}_A(R' \otimes_{A'} P, M)$$

is an isomorphism

and using Proposition 4.1, this is equivalent to $\mathrm{Ker}(\mu)$ and $\mathrm{Coker}(\mu)$ in $\mathcal{T}$.

The condition (2) is equivalent to the following condition,

$$\forall M \in R\text{-DMod}, P \otimes_A M \in R'\text{-DMod} \iff$$

$$\forall M \in R\text{-DMod}, R' \otimes_{A'} P \otimes_A M \simeq P \otimes_A M \iff$$

$\forall M \in R\text{-DMod}, \mu \otimes_A M : R' \otimes_{A'} P \otimes_A M \to P \otimes_A M$ is an isomorphism

and using Proposition 4.1, this is equivalent to $\mathrm{Ker}(\mu)$ and $\mathrm{Coker}(\mu)$ in $\mathcal{T}$. □

Corollary 4.6. *Let ${}_{A'}P_A$ be a bimodule such that $\mu : R' \otimes_{A'} P \to P$ has* $\mathrm{Ker}(\mu)$ *and* $\mathrm{Coker}(\mu)$ *in* $\mathcal{T}$. *Then*

1. $\mathrm{Hom}_A(\mu, -)$ *is a natural equivalence between the functors* $\mathrm{Hom}_A(P, -)$: CMod-R $\to$ CMod-R' *and* $\mathrm{Hom}_A(R' \otimes_{A'} P, -)$: CMod-R $\to$ CMod-R'.
2. $\mu \otimes_A -$ *is a natural equivalence between the functors* $P \otimes_A -$: R-DMod $\to$ R'-DMod *and* $R' \otimes_{A'} P \otimes_A -$: R-DMod $\to$ R'-DMod.

2. Morita Contexts and Equivalences

In this chapter we want to study the equivalences between the categories CMod-R and CMod-R' (or equivalently Mod-$R \simeq$ Mod-R' or DMod-$R \simeq$ DMod-R') for two idempotent rings R and R'. For these category equivalences we are going to find bimodules ${}_{A'}P_A$ and ${}_AQ_{A'}$ such that the following functors are equivalences:

$$\begin{array}{cc} \mathrm{Hom}_A(P,-) : \text{CMod-}R \to \text{CMod-}R' & \mathrm{Hom}_{A'}(Q,-) : \text{CMod-}R' \to \text{CMod-}R \\ P \otimes_A - : R\text{-DMod} \to R'\text{-DMod} & Q \otimes_{A'} - : R'\text{-DMod} \to R\text{-DMod} \\ \mathrm{Hom}_{A'}(P,-) : R'\text{-CMod} \to R\text{-CMod} & \mathrm{Hom}_A(Q,-) : R\text{-CMod} \to R'\text{-CMod} \\ - \otimes_{A'} P : \text{DMod-}R' \to \text{DMod-}R & - \otimes_A Q : \text{DMod-}R \to \text{DMod-}R' \end{array}$$

If the bimodules have to satisfy all these properties, we can choose the bimodule that $P \in$ DMod-R and R'-DMod, and the bimodule $Q \in$ DMod-R' and R-DMod. This is the reason we are going to add this conditions to the definition of a Morita context.

Proposition 4.7. *Let ${}_AQ_{A'}, {}_{A'}P_A$ bimodules such that $Q_{A'} \in$ DMod-R', ${}_AQ \in$ R-DMod, $P_A \in$ DMod-R and ${}_{A'}P \in R'$-DMod. Let $(-,-) : Q \times P \to R$, $[-,-] : P \times Q \to R'$ be mappings. Then the following conditions are equivalent:*

1. $\begin{pmatrix} R & Q \\ P & R' \end{pmatrix}$ *is a ring with the sum defined componentwise and the product*

$$\begin{pmatrix} r_1 & q_1 \\ p_1 & {r'}_1 \end{pmatrix}\begin{pmatrix} r_2 & q_2 \\ p_2 & {r'}_2 \end{pmatrix} = \begin{pmatrix} r_1r_2 + (q_1,p_2) & r_1q_2 + q_1{r'}_2 \\ p_1r_2 + {r'}_1p_2 & [p_1,q_2] + {r'}_1{r'}_2 \end{pmatrix}$$

2. $[-,-]$ *is A'-bilinear A-balanced, $(-,-)$ is A-bilinear A'-balanced and the following associativity conditions hold:*

$$(q,p)\bar{q} = q[p,\bar{q}] \qquad [p,q]\bar{p} = p(q,\bar{p})$$

for all $p, \bar{p}$ in P and $q, \bar{q}$ in Q.

Proof. Note first that

$$\left(\begin{pmatrix} r_1 & q_1 \\ p_1 & {r'}_1 \end{pmatrix} + \begin{pmatrix} r_2 & q_2 \\ p_2 & {r'}_2 \end{pmatrix}\right)\begin{pmatrix} \bar{r} & \bar{q} \\ \bar{p} & \bar{r}' \end{pmatrix} =$$

$$\begin{pmatrix} (r_1+r_2)\bar{r} + (q_1+q_2,\bar{p}) & (r_1+r_2)\bar{q} + (q_1+q_2)\bar{r}' \\ (p_1+p_2)\bar{r} + ({r'}_1+{r'}_2)\bar{r}' & [p_1+p_2,\bar{q}] + ({r'}_1+{r'}_2)\bar{p} \end{pmatrix}$$

On the other hand

$$\begin{pmatrix} r_1 & q_1 \\ p_1 & {r'}_1 \end{pmatrix}\begin{pmatrix} \bar{r} & \bar{q} \\ \bar{p} & \bar{r}' \end{pmatrix} + \begin{pmatrix} r_2 & q_2 \\ p_2 & {r'}_2 \end{pmatrix}\begin{pmatrix} \bar{r} & \bar{q} \\ \bar{p} & \bar{r}' \end{pmatrix} =$$

$$\begin{pmatrix} (r_1+r_2)\bar{r} + (q_1,\bar{p}) + (q_2,\bar{p}) & (r_1+r_2)\bar{q} + (q_1+q_2)\bar{r}' \\ (p_1+p_2)\bar{r} + ({r'}_1+{r'}_2)\bar{p} & [p_1,\bar{q}] + [p_2,\bar{q}] + ({r'}_1+{r'}_2)\bar{r}' \end{pmatrix}$$

Then, the distributivity of multiplication over addition on the right is equivalent to the additivity of $(-,-)$ and $[-,-]$ in their first variables. The other distributivity law is equivalent to the additivity in the second variables.

If we apply the definition of the multiplication to the equalities given by the associativity of the multiplication

$$\left(\begin{pmatrix} r_1 & q_1 \\ p_1 & {r'}_1 \end{pmatrix}\begin{pmatrix} r_2 & q_2 \\ p_2 & {r'}_2 \end{pmatrix}\right)\begin{pmatrix} r_3 & q_3 \\ p_3 & {r'}_3 \end{pmatrix} =$$
$$\begin{pmatrix} r_1 & q_1 \\ p_1 & {r'}_1 \end{pmatrix}\left(\begin{pmatrix} r_2 & q_2 \\ p_2 & {r'}_2 \end{pmatrix}\begin{pmatrix} r_3 & q_3 \\ p_3 & {r'}_3 \end{pmatrix}\right)$$

we obtain the following relations

$$(q_1,p_2)r_3 + (r_1q_2,p_3) + (q_1{r'}_2,p_3) = r_1(q_2,p_3) + (q_1,p_2r_3) + (q_1,{r'}_2p_3)$$
$$(q_1,p_2)q_3 = q_1[p_2,q_3]$$
$$[p_1,q_2]p_3 = p_1(q_2,p_3)$$
$$[p_1r_2,q_3] + [{r'}_1p_2,q_3] + [p_1,q_2]{r'}_3 = [p_1,r_2p_3] + [p_1,q_2{r'}_3] + {r'}_1[p_2,q_3]$$

These conditions are satisfied if (2) holds. Now if we suppose that the previous conditions hold and we apply the additivity of $(-,-)$ and $[-,-]$ we can make for example in the first relation $p_2 = 0$ and $q_2 = 0$ and we obtain $(q_1{r'}_2,p_3) = (q_1,{r'}_2p_3)$. In this way we can find all the following relations:

$$\begin{aligned} (q,pr) &= (q,p)r & (rq,p) &= r(q,p) \\ [p,qr'] &= [p,q]r' & [r'p,q] &= r'[p,q] \\ (qr',p) &= (q,r'p) & [pr,q] &= [p,rq] \\ (q,p)\bar{p} &= q[p,\bar{p}] & [p,q]\bar{q} &= p(q,\bar{q}) \end{aligned}$$

This relations prove that $[-,-]$ is R'-bilinear R-balanced and $(-,-)$ is R-bilinear R'-balanced. In order to prove that they satisfy this properties for A and A', but this is clear using that P and Q are unitary on both sides. □

If the equivalent conditions of Proposition 4.7 hold, then the mappings $(-,-)$ and $[-,-]$ (called the pairings) define bimodule homomorphisms $\varphi : Q \otimes_{A'} P \to R$ and $\psi : P \otimes_A Q \to R'$.

Definition 4.8. A *Morita context* is a six-tuple $(R, R', P, Q, \varphi, \psi)$ satisfying the conditions given in proposition 4.7. The associated ring $\begin{pmatrix} R & Q \\ P & R' \end{pmatrix}$ is called the *Morita ring* of the context. The ideals $\varphi(Q\otimes P)$ and $\psi(P \otimes Q)$ are called the *trace ideals* of the context.

REMARK 4.9. *Let $(R, R', P, Q, \varphi, \psi)$ be a Morita context. Then the trace ideals are two sided ideals of R and R'.*

PROOF. The trace ideals are two sided ideals because $Q \otimes P$ and $P \otimes Q$ are bimodules and φ, ψ are bimodule homomorphisms. □

Associated with any Morita context $(R, R', P, Q, \varphi, \psi)$ are eight natural maps:

$$
\begin{aligned}
[*,-] :& P \to \mathrm{Hom}_{A'}(Q, R') & \quad (*,-) :& Q \to \mathrm{Hom}_{A}(P, R) \\
& p \mapsto [p,-] & & q \mapsto (q,-) \\
[-,*] :& Q \to \mathrm{Hom}_{A'}(P, R') & \quad (-,*) :& P \to \mathrm{Hom}_{A}(Q, R) \\
& q \mapsto [-,q] & & p \mapsto (-,p) \\
& R \to \mathrm{End}_{A'}(Q) & & R \to \mathrm{End}_{A'}(P) \\
& r \mapsto (q \mapsto rq) & & r \mapsto (p \mapsto pr) \\
& R' \to \mathrm{End}_{A}(P) & & R' \to \mathrm{End}_{A}(Q) \\
& s \mapsto (p \mapsto sp) & & s \mapsto (q \mapsto qs)
\end{aligned}
$$

We are specially interested in the contexts for which φ and ψ are epimorphisms. This contexts have several "beautiful" properties, and we are going to give one of them.

PROPOSITION 4.10. *Let $(R, R', P, Q, \varphi, \psi)$ be a Morita context. Then the bimodules $P \otimes_A Q$ and $Q \otimes_{A'} P$ satisfy $P \otimes_A Q \otimes_{A'} R' \simeq P \otimes_A Q$ and $Q \otimes_{A'} P \otimes_A R \simeq Q \otimes_{A'} P$.*

If φ and ψ are epimorphisms, then the morphisms $\varphi \otimes_A R : Q \otimes_{A'} P \to R \otimes_A R$ and $\psi \otimes_{A'} R' : P \otimes_A Q \to R' \otimes_{A'} R'$ are isomorphisms.

PROOF. The first part is clear because $P \simeq P \otimes_A R$ and $Q \simeq Q \otimes_{A'} R'$.

In the second part, the proof is symmetric, and we only have to prove it for φ.

As R is unitary, the functor $- \otimes_A R$ is the same as the functor **d**. This functor preserves epimorphisms, and therefore $\varphi \otimes_A R$ is an epimorphism. Note that $Q \otimes_{A'} P \in$ DMod-R and $R \otimes_A R \in$ DMod-R, so if we apply Proposition 2.41 we obtain that $\mathrm{Ker}(\varphi \otimes_A R)$ is unitary. Let $\sum k_i r_i \in \mathrm{Ker}(\varphi \otimes_A R)$ with $k_i \in \mathrm{Ker}(\varphi \otimes_A R)$ for all i. The elements $r_i \in R = \mathrm{Im}(\varphi)$, and therefore we can find elements $p_{ij} \in P$ and $q_{ij} \in Q$ such that $r_i = \sum_j \varphi(q_{ij} \otimes p_{ij})$. For the elements $k_i \mathrm{Ker}(\varphi \otimes_A R) = \mathrm{Ker}(\varphi \otimes_A R)R$ we can find also $\bar{q}_{it} \in Q$, $\bar{p}_{it} \in P$ and $\bar{r}_{it} \in R$ such that

$$k_i = \sum_t \bar{q}_{it} \otimes \bar{p}_{it} r_{it} \quad \text{and}$$

$$\sum_t \varphi(q_{it} \otimes p_{it}) \otimes r_{it} = (\varphi \otimes_A R)(\sum_t \varphi(q_{it} \otimes p_{it}) \otimes r_{it}) = 0.$$

Therefore, if we apply the canonical epimorphism $R \otimes_A R \to R$ we deduce $\sum_t \varphi(q_{it} \otimes p_{it}) r_{it} = 0$. Then

$$\begin{aligned}\sum k_i r_i &= \sum_{i,j,t} \bar{q}_{it} \otimes \bar{p}_{it} r_{it} \varphi(q_{ij} \otimes p_{ij}) \\ &= \sum_{i,j,t} \bar{q}_{it} \otimes \psi(\bar{p}_{it} r_{it} \otimes q_{ij}) p_{ij} \\ &= \sum_{i,j,t} \bar{q}_{it} \psi(\bar{p}_{it} r_{it} \otimes q_{ij}) \otimes p_{ij} \\ &= \sum_{i,j,t} \varphi(\bar{q}_{it} \otimes \bar{p}_{it} r_{it}) q_{ij} \otimes p_{ij} = 0\end{aligned}$$

This proves that $\operatorname{Ker}(\varphi \otimes_A R) = 0$ as we claimed. □

We are going to define the composition of contexts.

PROPOSITION 4.11. *Let R,R' and R'' be idempotent rings, A,A' and A'' rings with identity such that R is a two sided ideal of A, R' of A' and R'' of A''.*

Given two Morita contexts[1]

$$(-,-) : Q \times P \to R \quad [-,-] : P \times Q \to R'$$

and

$$(-,-) : V \times U \to R' \quad [-,-] : U \times V \to R''$$

the new pairings

$$(-,-) : (Q \otimes_{A'} V) \times (U \otimes_{A'} P) \to R \quad (q \otimes v, u \otimes p) := (q, (v,u)p)$$

and

$$[-,-] : (U \otimes_{A'} P) \times (Q \otimes_{A'} V) \to R'' \quad [u \otimes p, q \otimes v] := [u, [p,q]v]$$

define a new context between the rings R and R''. If

$$\begin{array}{cc} \varphi : Q \otimes_{A'} P \to R & \psi : P \otimes_A Q \to R' \\ \xi : V \otimes_{A''} U \to R' & \zeta : U \otimes_{A'} V \to R'' \\ \delta : (U \otimes_{A'} P) \otimes_A (Q \otimes_{A'} V) \to R'' & \epsilon : (Q \otimes_{A'} V) \otimes_{A''} (U \otimes_{A'} P) \to R \end{array}$$

are the induced pairings, then the trace ideals are

$$\operatorname{Im}(\delta) = \zeta(U \otimes_{A'} \psi(P \otimes_A Q) V) \quad \operatorname{Im}(\epsilon) = \varphi(Q \otimes_{A'} \xi(V \otimes_{A''} U) P).$$

[1] With the additional properties that we are assuming for all Morita contexts given in Proposition 4.7, i.e.

$$\begin{array}{cc} Q \otimes_{A'} R' \simeq Q & R \otimes_A Q \simeq Q \\ P \otimes_A R \simeq P & R' \otimes_{A'} P \simeq P \\ U \otimes_{A'} R' \simeq U & R'' \otimes_{A''} U \simeq U \\ V \otimes_{A''} R'' \simeq V & R' \otimes_{A'} V \simeq V \end{array}$$

Furthermore, if the first two contexts satisfy that φ,ψ, ξ *and* ζ *are epimorphisms, then* δ *and* ϵ *are epimorphisms.*

PROOF. The bimodules $Q \otimes_{A'} V$ and $U \otimes_{A'} P$ satisfy

$$\begin{array}{rr} Q \otimes_{A'} V \otimes_{A''} R'' \simeq Q \otimes_{A'} V & R \otimes_A Q \otimes_{A'} V \simeq Q \otimes_{A'} V \\ U \otimes_{A'} P \otimes_A R \simeq U \otimes_{A'} P & R'' \otimes_{A''} U \otimes_{A'} P \simeq U \otimes_{A'} P \end{array}$$

because of the associativity of tensor products.

To prove that the new context satisfy the other properties of Proposition 4.7 it can be checked directly as can the computation of $\mathrm{Im}(\epsilon)$ and $\mathrm{Im}(\delta)$.

The last claim is also clear because all the modules are unitary. □

PROPOSITION 4.12. *The property "R is related with R' if and only if there exists a Morita context between R and R' with epimorphisms", is an equivalence relation.*

PROOF. Let R be an idempotent ring. Thus taking $P = Q = R \otimes_A R$ and the pairings

$$\begin{array}{rrcl} \varphi = \psi : & (R \otimes_A R) \otimes_A (R \otimes_A R) & \to & R \\ & (r \otimes s) \otimes (t \otimes u) & \mapsto & rstu \end{array}$$

define a Morita context between R and R.

If $(R, R', P, Q, \varphi, \psi)$ is a Morita context with epimorphisms, then $(R', R, Q, P, \psi, \varphi)$ is a Morita context with epimorphisms.

If $(R, R', P, Q, \varphi, \psi)$ and $(R', R'', U, V, \zeta, \xi)$ are Morita contexts with epimorphisms, then the composition defined in the previous proposition $(R, R'', U \otimes_{A'} P, Q \otimes_{A'} V, \delta, \epsilon)$ is a Morita context with epimorphisms between R and R''. □

PROPOSITION 4.13. *Let $(R, R', P, Q, \varphi, \psi)$ be a Morita context. Then the following conditions are equivalent:*

1. *The functors*

$$\begin{array}{rrcl} \mathrm{Hom}_A(P, -) : & \text{CMod-}R & \to & \text{CMod-}R' \\ \mathrm{Hom}_{A'}(Q, -) : & \text{CMod-}R' & \to & \text{CMod-}R \end{array}$$

are inverse category equivalences.

2. *The functors*

$$\begin{array}{rrcl} \mathrm{Hom}_{A'}(P, -) : & R'\text{-CMod} & \to & R\text{-CMod} \\ \mathrm{Hom}_A(Q, -) : & R\text{-CMod} & \to & R'\text{-CMod} \end{array}$$

are inverse category equivalences.

3. *The functors*

$$\begin{array}{rrcl} P \otimes_A - : & R\text{-DMod} & \to & R'\text{-DMod} \\ Q \otimes_{A'} - : & R'\text{-DMod} & \to & R\text{-DMod} \end{array}$$

are inverse category equivalences.

4. *The functors*

$$\begin{array}{rcl} -\otimes_{A'} P: & \text{DMod-}R' & \to & \text{DMod-}R \\ -\otimes_{A} Q: & \text{DMod-}R & \to & \text{DMod-}R' \end{array}$$

are inverse category equivalences.

5. *φ and ψ are epimorphisms.*

PROOF. First we are going to prove that (5) implies all the other conditions. Suppose (5) holds and let $\sum_i q_i \otimes p_i \in \mathrm{Ker}(\varphi)$, $r \in R$. For this element r we can find elements $\bar{p}_j \in P$ and $\bar{q}_j \in Q$ such that $r = \varphi(\sum_j \bar{q}_j \otimes \bar{p}_j)$, and hence

$$(\sum_i q_i \otimes p_i)r = \sum_{i,j} q_i \otimes p_i\varphi(\bar{q}_j \otimes \bar{p}_j)$$

$$= \sum_{i,j} q_i \otimes \psi(p_i \otimes \bar{q}_j)\bar{p}_j = \sum_{i,j} q_i\psi(p_i \otimes \bar{q}_j) \otimes \bar{p}_j$$

$$= \sum_{i,j} \varphi(q_i \otimes p_i)\bar{q}_j \otimes \bar{p}_j = \varphi(\sum_i q_i \otimes p_i) \sum_j \bar{q}_j \otimes \bar{p}_j = 0$$

On the other hand we also have $r(\sum_i q_i \otimes p_i) = 0$. This proves that $\mathrm{Ker}(\varphi)R = 0 = R\mathrm{Ker}(\varphi)$. For ψ the proof is similar, and we obtain $\mathrm{Ker}(\psi)R' = R'\mathrm{Ker}(\psi) = 0$.

Using the fact that $\mathrm{Ker}(\varphi)$, $\mathrm{Coker}(\varphi)$, $\mathrm{Ker}(\psi)$ and $\mathrm{Coker}(\psi)$ are torsion on both sides, we can apply Proposition 4.1 and its dual to deduce

1.1

$$\mathrm{Hom}_A(P,-)\circ\mathrm{Hom}_{A'}(Q,-) = \mathrm{Hom}_{A'}(P\otimes_A Q,-) : \text{CMod-}R' \to \text{CMod-}R'$$

is equivalent to the functor $\mathrm{Hom}_{A'}(R',-) = \mathrm{id}_{\text{CMod-}R'}$ by the natural equivalence $\mathrm{Hom}_{A'}(\psi,-)$.

1.2

$$\mathrm{Hom}_{A'}(Q,-)\circ\mathrm{Hom}_A(P,-) = \mathrm{Hom}_A(Q\otimes_{A'} P,-) : \text{CMod-}R \to \text{CMod-}R$$

is equivalent to the functor $\mathrm{Hom}_A(R,-) = \mathrm{id}_{\text{CMod-}R}$ by the natural equivalence $\mathrm{Hom}_A(\varphi,-)$.

2.1

$$\mathrm{Hom}_{A'}(P,-)\circ\mathrm{Hom}_A(Q,-) = \mathrm{Hom}_A(Q\otimes_{A'} P,-) : R\text{-CMod} \to R\text{-CMod}$$

is equivalent to the functor $\mathrm{Hom}_A(R,-) = \mathrm{id}_{R\text{-CMod}}$ by the natural equivalence $\mathrm{Hom}_A(\varphi,-)$.

2.2

$$\mathrm{Hom}_A(Q,-)\circ\mathrm{Hom}_{A'}(P,-) = \mathrm{Hom}_{A'}(P\otimes_A Q,-) : R'\text{-CMod} \to R'\text{-CMod}$$

is equivalent to the functor $\mathrm{Hom}_{A'}(R',-) = \mathrm{id}_{R'\text{-CMod}}$ by the natural equivalence $\mathrm{Hom}_A(\varphi,-)$.

3.1

$$(P \otimes_A -) \circ (Q \otimes_{A'} -) = P \otimes_A Q \otimes_{A'} - : R'\text{-DMod} \to R'\text{-DMod}$$

is equivalent to the functor $R' \otimes_{A'} - = \mathrm{id}_{R'\text{-DMod}}$ by the natural equivalence $\psi \otimes_{A'} -$.

3.2

$$(Q \otimes_{A'} -) \circ (P \otimes_A -) = Q \otimes_{A'} P \otimes_A - : R\text{-DMod} \to R\text{-DMod}$$

is equivalent to the functor $R \otimes_A - = \mathrm{id}_{R\text{-DMod}}$ by the natural equivalence $\varphi \otimes_{A'} -$.

4.1

$$(- \otimes_{A'} P) \circ (- \otimes_A Q) = - \otimes_{A'} P \otimes_A Q : \text{DMod-}R' \to \text{DMod-}R'$$

is equivalent to the functor $- \otimes_{A'} R' = \mathrm{id}_{\text{DMod-}R'}$ by the natural equivalence $- \otimes_{A'} \psi$.

4.2

$$(- \otimes_A Q) \circ (- \otimes_{A'} P) = - \otimes_A Q \otimes_{A'} P : \text{DMod-}R \to \text{DMod-}R$$

is equivalent to the functor $- \otimes_A R = \mathrm{id}_{\text{DMod-}R}$ by the natural equivalence $- \otimes_A \varphi$.

On the other hand, suppose (1) holds, then we have (1.1) and (1.2) and this is equivalent to $\mathrm{Ker}(\varphi)$, $\mathrm{Coker}(\varphi)$, $\mathrm{Ker}(\psi)$ and $\mathrm{Coker}(\psi)$ torsion. But $\mathrm{Coker}(\varphi)$ and $\mathrm{Coker}(\psi)$ are unitary, and if they are torsion, they have to be 0 and this is the condition (5).

With the others, the proof is similar. We have to use Proposition 4.1 and its dual and deduce that $\mathrm{Coker}(\varphi)$ and $\mathrm{Coker}(\psi)$ are torsion on one side or the other, and use that both are left and right unitary to conclude that they have to be 0. □

This proposition and Proposition 2.45 say that the categories Mod-R and Mod-R' are also equivalent, but there are several posibilities for finding this equivalence. We can go through CMod-$R \to$ CMod-R' or through DMod-$R \to$ DMod-R'. We have the following commutative diagrams:

$$\begin{array}{ccc}
\text{Mod-}R & \xrightarrow{\mathbf{m}' \circ \mathbf{i}'_{\mathbf{C}} \circ \mathrm{Hom}_A(P,-) \circ \mathbf{c} \circ \mathbf{i}_{\mathbf{M}}} & \text{Mod-}R' \\
\Big\downarrow {\scriptstyle \mathbf{c} \circ \mathbf{i}_{\mathbf{M}}} & & \Big\downarrow {\scriptstyle \mathbf{c}' \circ \mathbf{i}'_{\mathbf{M}}} \\
\text{CMod-}R & \xrightarrow[\mathrm{Hom}_A(P,-)]{} & \text{CMod-}R'
\end{array}$$

$$\begin{array}{ccc}
\text{Mod-}R & \xrightarrow{\mathbf{m}'\circ\mathbf{i}'_{\mathbf{C}}\circ\mathrm{Hom}_A(P,-)\circ\mathbf{c}\circ\mathbf{i}_{\mathbf{M}}} & \text{Mod-}R' \\
\mathbf{m}\circ\mathbf{i}_{\mathbf{C}}\uparrow & & \uparrow\mathbf{m}'\circ\mathbf{i}'_{\mathbf{C}} \\
\text{CMod-}R & \xrightarrow[\mathrm{Hom}_A(P,-)]{} & \text{CMod-}R'
\end{array}$$

$$\begin{array}{ccc}
\text{Mod-}R & \xrightarrow{\mathbf{m}'\circ\mathbf{i}'_{\mathbf{D}}\circ(-\otimes_A Q)\circ\mathbf{d}\circ\mathbf{i}_{\mathbf{M}}} & \text{Mod-}R' \\
\mathbf{d}\circ\mathbf{i}_{\mathbf{M}}\downarrow & & \downarrow\mathbf{d}'\circ\mathbf{i}'_{\mathbf{M}} \\
\text{DMod-}R & \xrightarrow[-\otimes_A Q]{} & \text{DMod-}R'
\end{array}$$

$$\begin{array}{ccc}
\text{Mod-}R & \xrightarrow{\mathbf{m}'\circ\mathbf{i}'_{\mathbf{D}}\circ(-\otimes_A Q)\circ\mathbf{d}\circ\mathbf{i}_{\mathbf{M}}} & \text{Mod-}R' \\
\mathbf{m}\circ\mathbf{i}_{\mathbf{D}}\uparrow & & \uparrow\mathbf{m}'\circ\mathbf{i}'_{\mathbf{D}} \\
\text{DMod-}R & \xrightarrow[-\otimes_A Q]{} & \text{DMod-}R'
\end{array}$$

And something similar on the left. What we are going to prove is that it is the same if we go through CMod-$R \to$ CMod-R' or DMod-$R \to$ DMod-R'.

PROPOSITION 4.14. *Let R,R' be idempotent rings, and let $(R, R', P, Q, \varphi, \psi)$ be a Morita context with φ and ψ epimorphisms. Then the functors $\mathbf{m}'\circ\mathbf{i}'_{\mathbf{D}}\circ(-\otimes_A Q)\circ\mathbf{d}\circ\mathbf{i}_{\mathbf{M}}$ and $\mathbf{m}'\circ\mathbf{i}'_{\mathbf{C}}\circ\mathrm{Hom}_A(P,-)\circ\mathbf{c}\circ\mathbf{i}_{\mathbf{M}}$ are equivalent.*

PROOF. We have to find for all $M \in$ Mod-R an isomorphism

$$\eta_M : \mathbf{m}'\circ\mathbf{i}'_{\mathbf{D}}\circ(-\otimes_A Q)\circ\mathbf{d}\circ\mathbf{i}_{\mathbf{M}}(M) \to \mathbf{m}'\circ\mathbf{i}'_{\mathbf{C}}\circ\mathrm{Hom}_A(P,-)\circ\mathbf{c}\circ\mathbf{i}_{\mathbf{M}}(M)$$

natural in M. First we are going to calculate these modules.

$$\mathbf{m}' \circ \mathbf{i}'_{\mathbf{D}} \circ (- \otimes_A Q) \circ \mathbf{d} \circ \mathbf{i}_{\mathbf{M}}(M) = \mathbf{m}'\left(\mathbf{d}\left(M\right) \otimes_A Q\right).$$

Using Lemma 3.21 and that $Q \in R$-DMod, we deduce that $\mathbf{d}\left(M\right) \otimes_A Q = M \otimes_A Q$, and using that $Q \in$ DMod-R', $M \otimes_A Q$ is in $\mathcal{U}'$, we have

$$\mathbf{m}'\left(\mathbf{d}\left(M\right) \otimes_A Q\right) = \mathbf{m}'\left(M \otimes_A Q\right) = (M \otimes_A Q)/\mathbf{t}'\left(M \otimes_A Q\right)$$

Using similar arguments we deduce that

$$\mathbf{m}' \circ \mathbf{i}'_{\mathbf{C}} \circ \mathrm{Hom}_A(P, -) \circ \mathbf{c} \circ \mathbf{i}_{\mathbf{M}}(M) = \mathbf{u}'\left(\mathrm{Hom}_A(P, M)\right).$$

Let us define $\beta : M \times Q \to \mathrm{Hom}_A(P, M)$ by

$$\begin{array}{rccc} \beta(m, q) : & P & \to & M \\ & p & \to & m\varphi(q \otimes p) \end{array}$$

It is staightforward to check that $\beta(m, q) \in \mathrm{Hom}_A(P, M)$ and that β is A'-bilinear and A-balanced. Then we have a homomorphism

$$\epsilon_M : M \otimes_A Q \to \mathrm{Hom}_A(P, M)$$

We are going to prove that $\mathrm{Ker}(\epsilon_M) = \mathbf{t}'\left(M \otimes_A Q\right)$ and that $\mathrm{Im}(\epsilon_M) = \mathbf{u}'\left(\mathrm{Hom}_A(P, M)\right)$.

$\boxed{\mathrm{Im}(\epsilon_M) \subseteq \mathbf{u}'\left(\mathrm{Hom}_A(P, M)\right)}$ Let $mr \in M = MR$ and $q \in Q$. As φ is an epimorphism we can find elements $q_i \in Q$ and $p_i \in P$ such that $r = \sum_i \varphi(q_i \otimes p_i)$ and then

$$\epsilon_M(mr \otimes q) = \epsilon_M(m \otimes rq) = \sum_i \epsilon_M(m \otimes \varphi(q_i \otimes p_i)q)$$

$$= \sum_i \epsilon_M(m \otimes q_i \psi(p_i \otimes q)) = \sum_i \epsilon_M(m \otimes q_i)\psi(p_i \otimes q) \in \mathrm{Hom}_R(P, M)R'$$

$\boxed{\mathrm{Im}(\epsilon_M) \supseteq \mathbf{u}'\left(\mathrm{Hom}_A(P, M)\right)}$ Let $f : P \to M$ and $r' \in R'$. As ψ is an epimorphism we can find elements $p_i \in P$ and $q_i \in Q$ such that $r' = \sum_i \psi(p_i \otimes q_i)$. We are going to prove that $fr' = \epsilon_M(\sum_i f(p_i) \otimes q_i)$:

$$fr'(p) = f(r'p) = f(\sum_i \psi(p_i \otimes q_i)p) = f(\sum_i p_i \varphi(q_i \otimes p))$$

$$= \sum_i f(p_i)\varphi(q_i \otimes p) = \epsilon_M(\sum_i f(p_i) \otimes q_i)(p)$$

for all $p \in P$.

$\boxed{\mathrm{Ker}(\epsilon_M) \supseteq \mathbf{t}'\left(M \otimes_A Q\right)}$ Let $\sum_i m_i \otimes q_i \in \mathbf{t}'\left(M \otimes_A Q\right)$. We have to prove that $\epsilon_M(\sum_i m_i \otimes q_i) = 0$ and for that let $r' \in R'$. Now $\epsilon_M(\sum_i m_i \otimes q_i)r' = \epsilon_M(\sum_i m_i \otimes q_i r') = 0$. As $\mathbf{t}'\left(\mathrm{Hom}_A(P, M)\right) = 0$ (because $P \in R'$-DMod), then $\epsilon_M(\sum_i m_i \otimes q_i) = 0$ as we claimed.

$\boxed{\mathrm{Ker}(\epsilon_M) \subseteq \mathbf{t}'(M \otimes_A Q)}$ Let $\sum_i m_i \otimes q_i \in \mathrm{Ker}(\epsilon_M)$ and $r' \in R'$ such that $\sum_i m_i \otimes q_i r' \neq 0$. As ψ is an epimorphism we can find elements $\bar{p}_j \in P$ and $\bar{q}_j \in Q$ such that $r' = \sum_j \psi(\bar{p}_j \otimes \bar{q}_j)$. Then $\sum_{ij} m_i \varphi(q_i \otimes \bar{p}_j) \otimes \bar{q}_j \neq 0$ and then for at least one j we have $\eta_M(\sum_i m_i \otimes q_i)(\bar{p}_j) = \sum_i m_i \varphi(q_i \otimes \bar{p}_j) \neq 0$ and this is a contradiction.

All these facts let us define

$$\eta_M : \frac{M \otimes_A Q}{\mathbf{t}'(M \otimes_A Q)} \to \mathbf{u}'(\mathrm{Hom}_A(P, M))$$

and prove that it is an isomorphism. The naturality of η_M can be easily verified.

□

3. Building Morita Contexts from Equivalences

Let R and R' be idempotent rings. In this section we shall try to build Morita contexts with epimorphisms from equivalences between the categories we have already built for R and R'. We know that the categories

$$\text{CMod-}R \leftrightarrow \text{Mod-}R \leftrightarrow \text{DMod-}R$$

are equivalent, and the same happens for R'. Therefore, if we want to study the equivalence between, for instance, DMod-R and DMod-R', we can study the equivalence between the categories CMod-R and CMod-R' or between Mod-R and Mod-R'.

In the case of idempotent rings it is better to consider the equivalence between the categories CMod-R and CMod-R' because localization techniques and the fact that these categories are quotient categories will be very helpful.

In this section we are going to use the following notation:

1. R and R' are idempotent rings.
2. A and A' are rings with identity such that R is a two-sided ideal of A and R' of A'.
3. $\bar{R} = R/\mathbf{t}(R)$ and $\bar{R}' = R'/\mathbf{t}'(R')$.
4. $B = \mathbf{c}(R)$ and $B' = \mathbf{c}'(R')$.
5. F : CMod-$R \to$ CMod-R' and G : CMod-$R' \to$ CMod-R are inverse category equivalences.

Lemma 4.15. *$B = \mathrm{End}_{\bar{R}}(\bar{R})$.*

Proof. We know that $B = \mathrm{Hom}_A(R, \bar{R})$. But $\bar{R}$ is torsion-free so that $\mathrm{Hom}_A(\mathbf{t}(R), \bar{R}) = 0$. Then $\mathrm{Hom}_A(R, \bar{R}) = \mathrm{Hom}_A(R/\mathbf{t}(R), \bar{R})$. □

The ring $\bar{R}$ can be considered to be inside B via the canonical monomorphism $\bar{R} = R/\mathbf{t}(R) \to \mathbf{c}(R) = B$, and with this it is true that $b\bar{r} = b(\bar{r})$. We shall use many times this inclusion. All modules in

CMod-R are R-modules, $\bar{R}$-modules and B-modules, and we could have some problems when we talk about morphisms, because they could be R-homomorphisms, $\bar{R}$-homomorphisms or B-homomorphisms. What we are going to do in the next lemma is to prove that they are the same in the cases we are interested in.

LEMMA 4.16. *Let $X, Y \in$ Mod-B with $Y \in \mathfrak{F}$. Then*

$$\mathrm{Hom}_A(X,Y) = \mathrm{Hom}_R(X,Y) = \mathrm{Hom}_{\bar{R}}(X,Y) = \mathrm{Hom}_B(X,Y)$$

PROOF. The proof of the fact that the first three sets are equal and $\mathrm{Hom}_{\bar{R}}(X,Y) \supseteq \mathrm{Hom}_B(X,Y)$ can be verified directly. The only problem is with the inclusion $\mathrm{Hom}_{\bar{R}}(X,Y) \subseteq \mathrm{Hom}_B(X,Y)$. In order to prove that, let $f : X \to Y$ be an $\bar{R}$-homomorphism and let $b \in B$. Then for all $x \in X$,

$$(f(xb) - f(x)b)\bar{r} = f(xb\bar{r}) - f(x)b\bar{r} =$$
$$f(xb(\bar{r})) - f(x)b(\bar{r}) = f(xb(\bar{r})) - f(xb(\bar{r})) = 0$$

and this is true for all $\bar{r} \in \bar{R}$. Using the fact that Y is torsion-free we deduce that $f(xb) = f(x)b$. □

We know that the category CMod-R is a quotient category of Mod-A. We are going to see now that this category can also be seen as a quotient category of Mod-B.

PROPOSITION 4.17. *$\bar{R}B$ is a two sided ideal of B, which is torsion free and idempotent as a ring that contains $\bar{R}$. If we denote $\mathcal{G} = \{I \leq B_B : \bar{R}B \subseteq I\}$, then this is a Gabriel topology on B and* CMod-R = Mod-$(B, \mathcal{G})$.

PROOF. As we have seen previously, $B\bar{R} \subseteq \bar{R}$ ($b\bar{r} = b(\bar{r})$) and then $B(\bar{R}B) \subseteq \bar{R}B$. This proves that $\bar{R}B$ is a two-sided ideal of B.

To see that $\bar{R}B$ contains $\bar{R}$ we only have to notice that $\mathrm{id}_{\bar{R}} = 1_B \in B$ and then $\bar{R} = \bar{R}1_B \subseteq \bar{R}B$.

To see that $\bar{R}B$ is idempotent, we know that $\bar{R} \subseteq \bar{R}B$ and then $B\bar{R} \subseteq B\bar{R}B$ and $\bar{R}B\bar{R} \subseteq (\bar{R}B)^2$. This proves that $\bar{R} = \bar{R}^2 \subseteq (\bar{R}B)^2$ and then $\bar{R}B \subseteq (\bar{R}B)^2$.

The ring B is torsion-free because $\bar{R}$ is.

To see that $\mathcal{G}$ is a Gabriel topology, conditions T1 and T2 are immediate. If $I \in \mathcal{G}$ and $b \in B$, then $\bar{R} \subseteq (I : b)$ because $b\bar{R} \subseteq \bar{R} \subseteq I$ and using the fact that I is a right ideal of B we deduce that $\bar{R}B \subseteq I$. This proves T3. To prove T4 suppose I is a right ideal of B such that for some $J \in \mathcal{G}$, it is true that $\forall x \in J$, $(I : x) \in \mathcal{G}$. Then using the fact that $\bar{R} \subseteq J$ we deduce that $\bar{R} \subseteq (I : \bar{r})$ for all $\bar{r} \in \bar{R}$ and then $\bar{R} = (\bar{R})^2 \subseteq I$.

It is clear that CMod-$R \subseteq$ Mod-B. To see that CMod-R = Mod-$(B, \mathcal{G})$ we only have to see that for a module $M \in$ Mod-B, $\mathrm{Hom}_A(R, M) = M$ if and only if $\forall I \in \mathcal{G}$, $\mathrm{Hom}_B(I, M) = M$.

For that we shall adopt the following notations. For all $m \in M$, $\lambda_m : R \to M$ in $\mathrm{Hom}_A(R, M)$ is defined as $\lambda_m(r) = mr$. On the other hand $\lambda'_m : I \to M$ in $\mathrm{Hom}_B(I, M)$ is defined also as $\lambda'_m(b) = mb$. Define $\lambda : M \to \mathrm{Hom}_A(R, M)$ and $\lambda' : M \to \mathrm{Hom}_B(I, M)$ by $\lambda(m) = \lambda_m$ and $\lambda'(m) = \lambda'_m$ for all $m \in M$. We have to prove that λ is an isomorphism if and only if λ' is an isomorphism for all $I \in \mathcal{G}$.

($\Rightarrow$). Suppose that for some $m \in M$ we have $\lambda'_m = 0$. Then $m\bar{R} = 0$ because $\bar{R} \subseteq I$ and then $mR = 0$. But this implies that $\lambda_m = 0$ and $m = 0$ because λ is an isomorphism. If $f : I \to M$ belongs to $\mathrm{Hom}_B(I, M)$, then f is an A-homomorphism (M is torsion-free) and we can compose with the inclusion $j : \bar{R} \to I$ and with the projection $p : R \to \bar{R}$ to obtain $f \circ j \circ p : R \to M$ which belongs to $\mathrm{Hom}_A(R, M)$. We deduce that there exists an $m \in M$ such that $(f \circ j \circ p)(r) = mr$ for all $r \in R$ and then $f(\bar{r}) = m\bar{r}$ for all $\bar{r} \in \bar{R}$. Suppose that for some $x \in I$ we have $f(x) \neq mx$. Then $f(x) - mx \in \mathbf{t}\,(M)$ because for all $r \in R$, $(f(x) - mx)r = f(xr) - mxr = 0$ ($xr \in \bar{R}$ for all $r \in R$ and $b \in B$), then $f(x) = \lambda'_m(x)$ for all $x \in I$. This proves that λ' is an isomorphism.

($\Leftarrow$). Suppose $\lambda_m = 0$ for some $m \in M$. Then $\bar{R} \leq \mathrm{r.ann}_B(m) \in \mathcal{G}$ and this is not possible unless $m = 0$ because $\mathrm{Hom}_B(\mathrm{r.ann}_B(m), M) = M$. To see that λ is surjective let $f : R \to M$ belong to $\mathrm{Hom}_A(R, M)$. For all $b : \bar{R} \to \bar{R}$ in B we define $\bar{f} : \bar{R}B \to M$ in $\mathrm{Hom}_B(\bar{R}B, M)$ by $\bar{f}((r + \mathbf{t}\,(R))b) := f(r)b$. The morphism $\bar{f}$ is well defined because if $r \in \mathbf{t}\,(R)$ then $f(r)b \in \mathbf{t}\,(M) = 0$.

For this $\bar{f}$ we can find $m \in M$ such that $\bar{f}(\bar{r}b) = m\bar{r}b$ for all $\bar{r}b \in \bar{R}B$ and then $f(r) = mr$ for all $r \in R$. □

Lemma 4.18. *$\bar{P} = G(B')$ and $\bar{Q} = F(B)$ are bimodules such that $\bar{P}_A$ and $\bar{Q}_{A'}$ are generators of the categories* CMod-R *and* CMod-R' *and the functors F and G are, up to natural isomorphisms, $F \simeq \mathrm{Hom}_A(\bar{P}, -)$ and $G \simeq \mathrm{Hom}_{A'}(\bar{Q}, -)$.*

Proof. Using the category equivalence we see that $B' = \mathrm{End}_B(\bar{P}_B)$ and $B = \mathrm{End}_{B'}(\bar{Q}_{B'})$. This gives the bimodule structure for P and Q. They are generators because B and B' are.

Let $M \in$ CMod-R. Then $F(M) = \mathrm{Hom}_{B'}(B', F(M)) = \mathrm{Hom}_B(G(B'), G(F(M))) = \mathrm{Hom}_B(\bar{P}, M)$, and the same holds for G. □

Lemma 4.19. *For all $X \in$ Mod-B, if $\mathrm{Hom}_B(\bar{P}, X) = 0$ then $X\bar{R}B = 0$.*

Proof. Suppose $X\bar{R}B \neq 0$ and $\mathrm{Hom}_B(\bar{P}, X) = 0$. As $B \in$ CMod-R and $\bar{P}_B$ is a generator of CMod-R we can find an epimorphism (in CMod-R) $q : \bar{P}^{(I)} \to B$, i.e, $\mathrm{Coker}(q)\bar{R}B = 0$. Let $x \in X$ and $\bar{r} \in \bar{R}$ such that $x\bar{r} \neq 0$ and consider $\lambda_x : B \to X$ ($\lambda_x(b) = xb$). As $\mathrm{Coker}(q)\bar{R}B = 0$, $\bar{r} = 1_B\bar{r} \in \mathrm{Im}(q)$ and we can find elements $(y_i)_{i\in I} \in \bar{P}^{(I)}$ such that $q((y_i)_{i\in I}) = \bar{r}$. Then $(\lambda_x \circ q)((y_i)_{i\in I}) = x\bar{r} \neq 0$

and we have found a nonzero morphism $\lambda_x \circ q : \bar{P}^{(I)} \to X$, but this is a contradiction to $\mathrm{Hom}_B(\bar{P}, X) = 0$. □

LEMMA 4.20. *$\mathcal{G} = \{I \leq B_B : \bar{Q}R' \subseteq I\bar{Q}\}$ and $\mathcal{G}' = \{J \leq B'_{B'} : \bar{P}R \subseteq J\bar{P}\}$.*

PROOF. Let us denote $\mathbf{i} : \mathrm{CMod}\text{-}R \to \mathrm{Mod}\text{-}B$ be the canonical inclusion.

The functor

$$\mathrm{Hom}_{A'}(\bar{Q}, -) \circ \mathbf{i} : \mathrm{CMod}\text{-}R' \to \mathrm{Mod}\text{-}B$$

has two left adjoints

$$\mathbf{c} \circ (- \otimes_A \bar{Q}) : \mathrm{Mod}\text{-}B \to \mathrm{CMod}\text{-}R'$$

and

$$\mathrm{Hom}_A(\bar{P}, -) \circ \mathbf{c} : \mathrm{Mod}\text{-}B \to \mathrm{CMod}\text{-}R'$$

The first is a left adjoint because $\mathbf{c}$ is a left adjoint of $\mathbf{i}$ and $- \otimes_A \bar{Q}$ is a left adjoint of $\mathrm{Hom}_{A'}(\bar{Q}, -)$. The second is a left adjoint because of the equivalence.

Using the uniqueness of the adjunction, we deduce that for all $X \in \mathrm{Mod}\text{-}B$, $\mathrm{Hom}_A(\bar{P}, \mathbf{c}\,(X)) \simeq \mathbf{c}\left(X \otimes_A \bar{Q}\right)$ and then we claim that $X\bar{R}B = 0$ if and only if $(X \otimes_A Q)\bar{R}'B' = 0$.

To see why this is the case, suppose $(X \otimes_A Q)\bar{R}'B' = 0$. Then $\mathrm{Hom}_A(\bar{P}, \mathbf{c}\,(X)) = 0$ and using the previous lemma we deduce that $\mathbf{c}\,(X)\,\bar{R}B = 0$ and then $\mathbf{c}\,(X) = 0$ and $X\bar{R}B = 0$.

On the other hand suppose $X\bar{R}B = 0$. Then $\mathbf{c}\,(X) = 0$ and $\mathrm{Hom}_A(\bar{P}, \mathbf{c}\,(X)) = 0$. Hence $\mathbf{c}\left(X \otimes_A \bar{Q}\right) = 0$ and hence $(X \otimes_A Q)\bar{R}'B' = 0$.

If we apply this fact to compute the Gabriel topology, we obtain

$$\mathcal{G} = \{I_B \leq B_B : (B/I)\bar{R}B = 0\} = \{I : ((B/I) \otimes \bar{Q})\bar{R}'B' = 0\}$$
$$= \{I : (\bar{Q}/I\bar{Q})\bar{R}'B' = 0\} = \{I : \bar{Q}\bar{R}' \subseteq I\bar{Q}\}$$

as we have claimed. The result for $\mathcal{G}'$ is obtained because of the symmetry. □

COROLLARY 4.21. *$\bar{P}R \subseteq R'\bar{P}$ and $\bar{Q}R' \subseteq R\bar{Q}$.*

PROOF. By Lemma 4.20. □

The bimodules $\bar{P}$ and $\bar{Q}$ are going to be used to build the Morita context, but they are not exactly the modules that appear. We are going to build a context between the rings B and B' with identity and from this we shall find one for R and R'.

PROPOSITION 4.22. *The bimodules* $\bar{P}$ *and* $\bar{Q}$ *establish a Morita context between the rings* B *and* B'*, namely* $(B, B', \bar{P}, \bar{Q}, \bar{\varphi}, \bar{\psi})$*, such that* $\bar{R} \subseteq \mathrm{Im}(\bar{\varphi})$ *and* $\bar{R}' \subseteq \mathrm{Im}(\bar{\psi})$*.*

PROOF. Using Lemma 4.18 we know that $F \simeq \mathrm{Hom}_B(\bar{P}, -)$, and then $\bar{Q} = F(B) \simeq \mathrm{Hom}_B(\bar{P}, B)$. We also have the same fact in the case of G, $\bar{Q} = \mathrm{Hom}_B(\bar{P}, B)$. With this we can define $\bar{\varphi}(\bar{q} \otimes \bar{p}) = \bar{q}(\bar{p})$ and $\bar{\psi}(\bar{p} \otimes \bar{q}) = \bar{p}(\bar{q})$. Using this definition it is straightforward to prove that $(B, B', \bar{P}, \bar{Q}, \bar{\varphi}, \bar{\psi})$ is a Morita context. Therefore we only have to prove the last conditions.

As $\bar{P}$ is a generator in CMod-R, we can find an epimorphism $h : \bar{P}^{(I)} \to B$ in CMod-R (i.e. $(B/\mathrm{Im}(h))R = \mathrm{Coker}(h)R = 0$). Then for all $\bar{r} \in \bar{R}$ there exists $(\bar{p}_i)_{i \in I} \in \bar{P}^{(I)}$ such that $\bar{r} = h((\bar{p}_i)_{i \in I})$. If we denote by $j_i : \bar{P} \to \bar{P}^{(I)}$ the canonical inclusions, then $h \circ j_i : \bar{P} \to B$ are elements in $\bar{Q}$ and $\bar{r} = \sum_i \bar{\varphi}(h \circ j_i \otimes \bar{p}_i)$. This proves that $\bar{R} \subseteq \mathrm{Im}(\bar{\varphi})$. The proof for $\bar{\psi}$ is similar. □

PROPOSITION 4.23. *Consider the canonical homomorphisms*

$$\begin{array}{rrcl} \eta: & R' \otimes_{A'} R' \otimes_{A'} \bar{P} \otimes_A R \otimes_A R & \to & \bar{P} \\ & r' \otimes s' \otimes \bar{p} \otimes s \otimes r & \mapsto & r's'\bar{p}sr \\ \epsilon: & R \otimes_A R \otimes_A \bar{P} \otimes_{A'} R' \otimes_{A'} R' & \to & \bar{P} \\ & r \otimes s \otimes \bar{p} \otimes s' \otimes r' & \mapsto & rs\bar{p}s'r' \end{array}$$

Then $\mathrm{Ker}(\eta), \mathrm{Coker}(\eta) \in \mathcal{T}$ *and* $\mathrm{Ker}(\epsilon), \mathrm{Coker}(\epsilon) \in \mathcal{T}'$*.*

PROOF. Let $\bar{p} \in \bar{P}$, $t, r, s \in R$. The element $\bar{p}t \in \bar{P}R \subseteq R'\bar{P} = R'^2\bar{P}$ and we can find elements $r'_j, s'_j \in R'$ and $\bar{p}_j \in \bar{P}$ such that $\bar{p}t = \sum_j s'_j r'_j \bar{p}_j$. Then

$$\bar{p}trs = \sum_j s'_j r'_j \bar{p}_j rs = \sum_j \eta(s'_j \otimes r'_j \otimes \bar{p} \otimes r \otimes s) \in \mathrm{Im}(\eta)$$

This proves that $\mathrm{Coker}(\eta)R = \mathrm{Coker}(\eta)R^3 = 0$.

In order to prove that $\mathrm{Ker}(\eta) \in \mathcal{T}$, we shall make some abuse of the language in the following sense: if we consider an element of the form $\bar{p}(\bar{q})r's'$, this element is in $\bar{R}'$, but we would like to consider it in R', and for that we would have to assign a unique element of R'. The element $\bar{q}(\bar{p})r' = w' + \mathbf{t}(R')$ for some $w' \in R'$. The element that we assign to $\bar{p}(\bar{q})r's'$ is $w's'$. We have to prove that this element is not dependent on the element w'. Suppose $w' + \mathbf{t}(R') = v' + \mathbf{t}(R')$. Then $(w' - v')s' = 0$ and $w's' = v's'$. The same holds for R.

Let $\sum_i s'_i \otimes r'_i \otimes \bar{p}_i \otimes r_i \otimes s_i \in \mathrm{Ker}(\eta)$. Then $\sum_i s'_i r'_i \bar{p}_i r_i s_i = 0$. We have to prove that for all $r \in R$, $\sum_i s'_i \otimes r'_i \otimes \bar{p}_i \otimes r_i \otimes s_i r = 0$. If we prove this for certain elements in R such that any element in R is a finite sum of elements of this type, it is clear that we would have obtained what we need. The special type of elements are the elements of the form $tuvwx$ with $t, u, v, w, x \in R$ and $t + \mathbf{t}(R) = \bar{q}^*(u'v'w'\bar{p}^*)$

with $\bar{q}^* \in \bar{Q}$, $\bar{p}^* \in \bar{P}$, $u', v', w' \in R'$. We have to prove first that all the elements in R are a finite sum of elements of this type. But this is clear, first because R is idempotent, and then R is sum of products $t_1 t_2 uvwx$ with $t_1, t_2, u, v, w \in R$, and second because the condition $t_1 + \mathbf{t}\,(R) \in \bar{R} \subseteq \mathrm{Im}(\varphi)$ gives that t_1 is a finite sum of elements of the form $\bar{q}(\bar{p})$, and the element $\bar{p}t_2 \in \bar{P}R \subseteq R'\bar{P} = {R'}^3\bar{P}$ so that $t_1t_2 + \mathbf{t}\,(R)$ is a sum of elements of the form $\bar{q}(u'v'w'\bar{p})$.

Let $t, u, v, w, x \in R$ with $t + \mathbf{t}\,(R) = \bar{q}^*(u'v'w'\bar{p}^*)$. Then

$$(\sum_i s_i' \otimes r_i' \otimes \bar{p}_i \otimes r_i \otimes s_i)tuvwx = \sum_i s_i' \otimes r_i' \otimes \bar{p}_i r_i s_i tuv \otimes w \otimes x.$$

The module $\bar{P}$ is a B-module, and therefore its R-module structure comes from the identification of $\bar{R} \subseteq B$ and it is the same to multiply by t or by $t + \mathbf{t}\,(R)$. Thus we have

$$\sum_i s_i' \otimes r_i' \otimes \bar{p}_i r_i s_i tuv \otimes w \otimes x = \sum_i s_i' \otimes r_i' \otimes \bar{p}_i (r_i s_i t + \mathbf{t}\,(R))uv \otimes w \otimes x$$

$$= \sum_i s_i' \otimes r_i' \otimes \bar{p}_i r_i s_i \bar{q}^*(u'v'w'\bar{p}^*)uv \otimes w \otimes x$$

$$= \sum_i s_i' \otimes r_i' \otimes \bar{p}_i (r_i s_i \bar{q}^*)u'v'w'\bar{p}^* uv \otimes w \otimes x.$$

The element $\bar{p}_i(r_i s_i \bar{q}^*)u' \in \bar{R}'$,then we can find an element $x_i' \in R'$ such that $\bar{p}_i(r_i s_i \bar{q}^*)u' = x_i' + \mathbf{t}\,(R')$, and then

$$\sum_i s_i' \otimes r_i' \otimes \bar{p}_i (r_i s_i \bar{q}^*)u'v'w'\bar{p}^* uv \otimes w \otimes x =$$

$$\sum_i s_i' \otimes r_i' \otimes x_i'v'w'\bar{p}^* uv \otimes w \otimes x = \sum_i s_i' r_i' x_i' v' \otimes w' \otimes \bar{p}^* uv \otimes w \otimes x$$

$$= \sum_i (s_i' r_i' \bar{p}_i r_i s_i)(\bar{q}^*)u'v' \otimes w' \otimes \bar{p}^* uv \otimes w \otimes x = 0.$$

The proof for $\bar{\psi}$ is similar. □

THEOREM 4.24. *Let R and R' be idempotent rings and F : CMod-R $\to$ CMod-R', G : CMod-R' $\to$ CMod-R inverse category equivalences. Then there exists a Morita context $(R, R', P, Q, \varphi, \psi)$ with φ, ψ epimorphisms such that $F \simeq \mathrm{Hom}_A(P, -)$ and $G \simeq \mathrm{Hom}_{A'}(Q, -)$. This contexs induce the following equivalences*

$$\begin{array}{rlrl} \text{CMod-}R & \simeq \text{CMod-}R' & R\text{-CMod} & \simeq R'\text{-CMod} \\ \text{Mod-}R & \simeq \text{Mod-}R' & R\text{-Mod} & \simeq R'\text{-Mod} \\ \text{DMod-}R & \simeq \text{DMod-}R' & R\text{-DMod} & \simeq R'\text{-DMod} \end{array}$$

PROOF. The modules that we have to use are $P = R' \otimes_{A'} R' \otimes_{A'} \bar{P} \otimes_A R \otimes_A R$ and $Q = R \otimes_A R \otimes_A \bar{Q} \otimes_{A'} R' \otimes_{A'} R'$. These modules define the same functors as P and Q because of the previous proposition, Proposition 4.1 and its dual. The pairings are defined in the natural way with the composition

$$(R' \otimes R' \otimes \bar{P} \otimes R \otimes R) \otimes (R \otimes R \otimes \bar{Q} \otimes R' \otimes R') \xrightarrow{can} \bar{P} \otimes \bar{Q} \xrightarrow{\bar{\varphi}} R'$$

and similarly for ψ.

Then using Propositions 4.13 and 4.14 we deduce that the pairings are epimorphisms and all the categories are equivalent. □

4. Some Consequences of the Morita Theorems

PROPOSITION 4.25. *Let R and R' be idempotent rings and $(R, R', P, Q, \varphi, \psi)$ a Morita context with epimorphisms. Then* $\mathrm{Cen}(\mathbf{c}(R)) = \mathrm{Cen}(\mathbf{c}'(R'))$ *and also with the functors on the left.*

PROOF. See [**7**, Proposition 3.1]. □

LEMMA 4.26. *Let R be an idempotent commutative ring. Then $\mathbf{c}(R)$ is commutative.*

PROOF. Let $f : R \to R/\mathbf{t}(R)$ in $\mathbf{c}(R)$ and let $r, s \in R$. Then

$$(rf)s = rf(s) = f(s)r = f(sr) = f(r)s = (fr)s.$$

Using the fact that $\mathbf{c}(R)$ is torsion-free, we deduce that $rf = fr$. This proves that $R/\mathbf{t}(R)$ is a two-sided ideal of $\mathbf{c}(R)$ that is inside its center. Let $f, g \in \mathbf{c}(R)$ and $s \in R$. Then

$$(fg)s = f(gs) = f(g(s)) = g(s)f = (gs)f = g(sf) = (gf)s.$$

Using again the fact that $\mathbf{c}(R)$ is torsion-free we deduce that $gf = fg$ for all $f, g \in \mathbf{c}(R)$. □

PROPOSITION 4.27. *Let R and R' be commutative and idempotent rings such that there exists a Morita context $(R, R', P, Q, \varphi, \psi)$ with φ and ψ epimorphisms. Then*

1. *$\mathbf{c}(R)$ and $\mathbf{c}'(R')$ are isomorphic commutative rings with identity.*
2. *$\mathbf{m}(R)$ and $\mathbf{m}'(R')$ are isomorphic commutative and idempotent rings.*
3. *$\mathbf{d}(R)$ and $\mathbf{d}'(R')$ are isomorphic commutative and idempotent rings.*

The same holds for the functors on the other side.

PROOF. Because of the previous lemma, $\mathbf{c}(R)$ and $\mathbf{c}'(R')$ are commutative rings with identity. The rings $\mathbf{m}(R) = R/\mathbf{t}(R)$ and $\mathbf{m}'(R') = R'/\mathbf{t}'(R')$ are clearly commutative idempotent rings and $\mathbf{d}(R) = R \otimes_A$

R and $\mathbf{d}'(R') = R' \otimes_{A'} R'$ are also commutative and idempotent rings with the multiplication $(r \otimes s)(x \otimes y) = rsx \otimes y$.

Because of Proposition 4.25 we obtain $\mathbf{c}(R) = \mathbf{c}'(R')$. The second isomorphism is proved in [**7**, Proposition 3.2]. The third one comes from the previous one because

$$R \otimes_A R \simeq R/\mathbf{t}(R) \otimes_A R/\mathbf{t}(R) \simeq R'/\mathbf{t}'(R') \otimes_{A'} R'/\mathbf{t}'(R') \simeq R' \otimes_{A'} R'$$

□

PROPOSITION 4.28. *Let R and R' be Morita equivalent idempotent rings and let $\mathfrak{L}_R$ denote the lattice $\{I \subseteq R : I \text{ is an ideal of } R, RIR = I\}$ and similarly for $\mathfrak{L}_{R'}$. Then there is an isomorphism between $\mathfrak{L}_R$ and $\mathfrak{L}_{R'}$ which may be given by $I \mapsto \varphi(QI \otimes P)$ and $J \mapsto \psi(PJ \otimes Q)$.*

PROOF. See [**7**, Proposition 3.5] □

5. The Picard Group of an Idempotent Ring

In this section we are going to generalize the concept of the Picard group of a ring. This group could be defined as the group of equivalences CMod-$R \to$ CMod-R (or with any other of the categories on the right or on the left). The problem with this definition is that we cannot deduce directly that it is a group because this might not be a set. We are going to prove that this is a group from the results given in this chapter.

We would deduce that the equivalences CMod-$R \to$ CMod-R contitute a group if we prove that, up to natural isomorphisms, they lie inside a set. Every equivalence CMod-$R \to$ CMod-R is given by a Morita context $(R, R, P, Q, \varphi, \psi)$ with φ and ψ epimorphisms because of Theorem 4.24.

Let $\{x_\lambda \in R : \lambda \in \Lambda\}$ be a generator set of R. Using the fact that φ is an isomorphism we can find elements $p_i^\lambda \in P$ and $q_i^\lambda \in Q$ such that $x_\lambda = \varphi(\sum_{i=1}^{n_\lambda} q_i^\lambda \otimes p_i^\lambda)$. We define $p_i^\lambda = 0$ and $q_i^\lambda = 0$ for all $i > n_\lambda$. Using these notations we obtain

LEMMA 4.29. *The following morphism, is an epimorphism*

$$\begin{array}{rccl} \eta: & R^{(\Lambda\times\mathbb{N})} & \to & P \\ & (r_i^\lambda) & \mapsto & \sum_{\lambda,i} r_i^\lambda p_i^\lambda \end{array}$$

PROOF. Let $p \in P$. As $P = PR$, there exists an element $(\tilde{p}_\lambda) \in P^{(\Lambda)}$ such that $p = \sum_{\lambda\in\Lambda} \tilde{p}_\lambda x_\lambda$. Then

$$\begin{aligned} p = \sum_{\lambda\in\Lambda} \tilde{p}_\lambda x_\lambda &= \sum_{(\lambda,i)\in\Lambda\times\mathbb{N}} \tilde{p}_\lambda \varphi(q_i^\lambda \otimes p_i^\lambda) \\ &= \sum_{(\lambda,i)\in\Lambda\times\mathbb{N}} \psi(\tilde{p}_\lambda \otimes q_i^\lambda) p_i^\lambda \in \operatorname{Im}(\eta). \end{aligned}$$

□

The set $R^{(\Lambda\times\mathbb{N})}$ is not dependent on the equivalence. Thus we have proved that for every equivalence $F : \text{CMod-}R \to \text{CMod-}R$, there exists P such that $F \simeq \text{Hom}_A(P, -)$ and P is a quotient of $R^{(\Lambda\times\mathbb{N})}$. As the quotient modules of $R^{(\Lambda\times\mathbb{N})}$ constitute a set, we have found an injection between the equivalences $\text{CMod-}R \to \text{CMod-}R$ and a set. Therefore, the equivalences $\text{CMod-}R \to \text{CMod-}R$ constitute a set.

Once we have proved that this is a set, it is clear that $\text{Pic}(R) = \{F : \text{CMod-}R \to \text{CMod-}R | F \text{ is an equivalence}\}$ is a group.

CHAPTER 5

Special Properties for Special Rings

1. Coclosed Rings

DEFINITION 5.1. Let R be an idempotent ring and A be a ring with identity such that R is a two sided ideal of it. We shall say that R is *coclosed* if the canonical morphism $R \otimes_A R \to R$ is an isomorphism.

PROPOSITION 5.2. *This definition does not depend on the ring A.*

PROOF. This condition is equivalent to the condition $R \in$ DMod-R, and we proved that the objects that are in this category are not dependent on the ring A (see Proposition 2.46). □

PROPOSITION 5.3. *Let R be an idempotent two sided ideal of a ring A with identity. Then*

1. *$S := R \otimes_A R$ is a coclosed ring.*
2. *The definition of S does not depend on the choice of A.*
3. *The following categories for the ring R are the same as for S, namely*

$$\begin{array}{cc} \text{CMod-}R \simeq \text{CMod-}S & R\text{-CMod} \simeq S\text{-CMod} \\ \text{Mod-}R \simeq \text{Mod-}S & R\text{-Mod} \simeq S\text{-Mod} \\ \text{DMod-}R \simeq \text{DMod-}S & R\text{-DMod} \simeq S\text{-DMod} \end{array}$$

PROOF. Using Proposition 2.46 we deduce that $S = \mathbf{d}(R)$, and therefore is independent of the ring A. If A and A' are rings with identity such that R is a two sided ideal on each and for all $r, s \in R$, the multiplication in A is the same as in A', then there exists an isomorphism $\sigma : R \otimes_A R \to R \otimes_{A'} R$. This isomorphism is an A-isomorphism and A'-isomorphism, and what we have to prove is that it is a ring isomorphism. But this is true because the definition is $\sigma(r \otimes s) = r \otimes s$ and this definitions preserves the multiplication. The structure of S comes with the sum defined as a module and the multiplication given by $(r \otimes r')(t \otimes t') = rr' \otimes tt'$.

Consider the epimorphism $\mu : S \to R$ with $K = \text{Ker}(\mu)$, $R \simeq S/K$, and let B be the Dorroh's extension of S. The morphism μ is an A-homomorphisms but also a B-homomorphism. Therefore K is a two-sided ideal of B and R is an ideal of $A' := B/K$. If $\sum_i r_i \otimes r'_i \in K$ and $\sum_j t_j \otimes t'_j \in S$ then

$$(\sum_i r_i \otimes r_i')(\sum_j t_j \otimes t_j') = \sum_i r_i r_i' \otimes \sum_j t_j t_j' = 0.$$

We know that $S \simeq R \otimes_{A'} R$ and $S \in \text{DMod-}R$, and therefore

$$S \simeq R \otimes_{A'} R \otimes_{A'} R \simeq R \otimes_{A'} R \otimes_{A'} R \otimes_{A'} R \simeq S \otimes_{A'} S \simeq S \otimes_B S.$$

The last isomorphism comes because $KS = 0 = SK$ and the B-module structure of S is the same as the B/K-structure. This proves that S is coclosed.

Because of the symmetry of the condition we only have to prove that the categories are the same on the right.

$$\boxed{\text{CMod-}R \simeq \text{CMod-}S}$$

Let $M \in \text{CMod-}R$, then $\text{Hom}_{A'}(R, M) = M$. The A'-module M has a B-module structure by the epimorphism $B \to A'$, then $\text{Hom}_B(S, M) = \text{Hom}_{A'}(S, M)$ and

$$\begin{aligned} \text{Hom}_B(S, M) &= \text{Hom}_{A'}(S, M) = \text{Hom}_{A'}(R \otimes_{A'} R, M) \\ &=^{1} \ \text{Hom}_{A'}(R, \text{Hom}_{A'}(R, M)) = M \end{aligned}$$

This proves that $M \in \text{CMod-}S$. On the other hand suppose $M \in \text{CMod-}S$. If we want to give M an A'-module structure, we have to prove that $MK = 0$. Let $m \in M$ and $k \in K$. We know that $kS = 0$ and therefore $mkS = 0$. But M is torsion-free with respect to S, and then $mk = 0$. With this A'-module structure we have to prove that M is torsion-free and **t**-injective with respect to R.

For every $m \in M$, $m(r \otimes s) = mrs$, and then $mR = 0$ if and only if $m(R \otimes_{A'} R) = 0$. But this happens if and only if $m = 0$. Let

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence in Mod-A' with $ZR = 0$ and let $f : X \to M$. This short exact sequence is also a short exact sequence in Mod-B with the B-module structures that come from the epimorphism $B \to A'$. The A'-homomorphism f is also a B-homomorphism and because $ZR = 0$ then $ZS = 0$. Then we can find an B-homomorphism $g : M \to Y$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & & & & & \\ & & M & & & & & & \end{array}$$

[1]See [**14**, Lemma 19.11] for this identification.

The B-homomorphism g is also an A'-homomorphism. We have proved that M is $\mathbf{t}$-injective and $M \in \text{CMod-}R$.

$$\boxed{\text{Mod-}R \simeq \text{Mod-}S}$$

Let $M \in \text{Mod-}R$. Bacause of the construction, the category Mod-R is a full subcategory of Mod-A' and then M is an A'-module. With the epimorphism $B \to A'$ we can give B-module structure to M (see the comments before [**?**, Proposition 2.11] for this possibility). The multiplication is defined as $m(r \otimes r') = mrr'$ for all $m \in M$, $r, r' \in R$. As $MR = M$, then $MS = M(R \otimes_{A'} R) = MR^2 = MR = M$, then M is unitary with respect to S. Let $m \in M$ such that $mS = 0$. Then $0 = m(R \otimes_{A'} R) = mR^2 = mR$ and $m = 0$. This proves that $M \in \text{Mod-}S$.

On the other hand, suppose $M \in \text{Mod-}S$. We have to prove that $MK = 0$. But this is true since $MK \subseteq \mathbf{t}(M) = 0$ because $KR = 0$. Then M has a B-module structure. As $MS = M$, then $MR = MR^2 = M(R \otimes_{A'} R) = MS = M$. If $mR = 0$, then $mS = 0$ and $m = 0$.

$$\boxed{\text{DMod-}R \simeq \text{DMod-}S}$$

Let $M \in \text{DMod-}R$. The A'-module M has a B-module structure with the epimorphism $B \to A'$ and the multiplication $m(r \otimes r') = mrr'$.

$$M \otimes_B S = M \otimes_{A'} S = M \otimes_{A'} R \otimes_{A'} R = M$$

On the other hand, suppose $M \in \text{DMod-}S$. We have to prove that $MK = 0$. But this is clear because $MK = (MS)K = M\underbrace{(SK)}_{=0} = 0$. The module M satisfies $MR = M(S/K) = MS = M$ and it is $\mathbf{u}$-codivisible with a proof dual to the one we have made for CMod-R.

□

This proves that the study of idempotent rings could be reduced to the study of coclosed rings because with respect to the categories we are studying, the rings R and $R \otimes_A R$ are the same.

The coclosed rings have nice properties with respect to the functors $\mathbf{d}$ and $\mathbf{c}$.

Proposition 5.4. *Let R be a coclosed ring. Then*

1. $\mathbf{c} \simeq \text{Hom}_A(R, -)$.
2. $\mathbf{d} \simeq - \otimes_A R$.

Proof. As $\mu : R \otimes_A R \to R$ is an isomorphism, $\mathbf{c} \simeq \text{Hom}_A(R \otimes_A R, -) \simeq \text{Hom}_A(R, -)$ and $\mathbf{d} \simeq - \otimes_A R \otimes_A R \simeq - \otimes_A R$. □

Proposition 5.5. *Let R and R' be coclosed rings and $(R, R', P, Q, \varphi, \psi)$ be a Morita context with φ and ψ epimorphisms. Then φ and ψ are isomorphisms.*

Proof. This is a consequence of Proposition 4.10 □

PROPOSITION 5.6. *Let R and R' be commutative coclosed and Morita equivalent rings. Then they are isomorphic.*

PROOF. This is a consequence of Proposition 4.27. □

2. Rings With Local Units

DEFINITION 5.7. Let R be a ring. We shall say that R is a ring with *local units* if there exists a set $E \subseteq R$ of commuting idempotents of R such that for every finite subset of elements in R, $\{r_1, \cdots, r_n\}$ there exists $e \in E$ such that $r_i = r_i e = e r_i$ for $i = 1, \cdots, n$.

PROPOSITION 5.8. *Let R be a ring with a set of local units E. Then R is coclosed.*

PROOF. Let A be any ring with identity such that R is a two-sided ideal of it. We have to prove that $R \otimes_A R \simeq R$. Let $r \in R$. We can find an element $e \in E \subseteq R$ such that $re = r$, and then $\mu(r \otimes e) = re = r$. This proves that $\mu : R \otimes_A R \to R$ is an epimorphism and therefore R is idempotent.

Suppose $\sum_{i=1}^n r_i \otimes s_i \in \mathrm{Ker}(\mu)$. Then we can find an $e \in E$ such that $e r_i = r_i$ for all $i = 1, \cdots, n$, and then

$$\sum_{i=1}^n r_i \otimes s_i = \sum_{i=1}^n e r_i \otimes s_i = e \otimes \sum_{i=1}^n r_i s_i$$
$$= e \otimes \mu(\sum_{i=1}^n r_i \otimes s_i) = e \otimes 0 = 0.$$

□

PROPOSITION 5.9. *Let R be a ring with a set of local units E, A a ring with identity such that R is an ideal of it. Let $M \in \mathrm{Mod}\text{-}A$. Then the following conditions are equivalent.*

1. $M \in \mathrm{Mod}\text{-}R$.
2. $M \in \mathrm{DMod}\text{-}R$.
3. $MR = M$.

PROOF. It is clear that conditions (1) or (2) imply (3).

Suppose (3) holds and let $m \in \mathbf{t}(M)$. For m we can find elements m_i and r_i with $i = 1, \cdots, n$ such that $m = \sum_{i=1}^n m_i r_i$. For the elements r_i we can find an $e \in E$ such that $r_i = r_i e$ for $i = 1, \cdots, n$. Then

$$0 = me = \sum_{i=1}^n m_i r_i e = \sum_{i=1}^n m_i r_i = m.$$

Suppose (3) holds and let $\mu : M \otimes_A R \to M$ be the canonical epimorphism with $\sum_{i=1}^n m_i \otimes r_i \in \mathrm{Ker}(\mu)$. Again we can find $e \in E$ such that $r_i = r_i e$ for all i and then

$$\sum_{i=1}^{n} m_i \otimes r_i = \sum_{i=1}^{n} m_i r_i \otimes e = \mu(\sum_{i=1}^{n} m_i \otimes r_i) \otimes e = 0 \otimes e = 0.$$

□

COROLLARY 5.10. *The categories* DMod-R *and* Mod-R *are equal and the functors* $\mathbf{d} \circ \mathbf{i_M}$ *and* $\mathbf{m} \circ \mathbf{i_D}$ *are the identity functors.*

PROOF. Because of the previous proposition, a module $M \in$ Mod-A is in Mod-R if and only if $M \in$ DMod-R. The modules of DMod-R are torsion free and the modules of Mod-R are coclosed, therefore the functors $\mathbf{d} \circ \mathbf{i_M}$ and $\mathbf{m} \circ \mathbf{i_D}$ are the identity functors over the modules and over the morphisms. □

In the study of rings with local units and Morita equivalences, the traditional category that it is used is Mod-$R =$ DMod-R, and therefore we are going to write the Morita theorem for this case.

THEOREM 5.11. *Let R and R' be rings with local units. Let*

$$F : \text{Mod-}R \to \text{Mod-}R' \quad G : \text{Mod-}R' \to \text{Mod-}R$$

be inverse category equivalences. Then, there exists a Morita context $(R, R', P, Q, \varphi, \psi)$ with φ and ψ isomorphisms such that F and G are, up to natural isomorphisms $F \simeq - \otimes_R Q \simeq \mathbf{u}' \circ \mathrm{Hom}_R(P, -)$ and $G \simeq - \otimes_{R'} P \simeq \mathbf{u} \circ \mathrm{Hom}_{R'}(Q, -)$. This context establishes also equivalences for the categories on the left.

PROOF. This is an immediate consequence of Theorem 4.24 and Proposition 4.14. □

Probably the biggest difference between the idempotent or coclosed rings and the rings with local units, is the general existence of projective modules.

PROPOSITION 5.12. *Let R be a ring with a set E of local units. Let $e \in E$. Then the module eR is finitely generated and projective.*

PROOF. Suppose $M_i \leq eR$ with $i \in I$, are submodules such that $\sum_i M_i = eR$. The element $e = e^2 \in eR = \sum M_i$ and therefore we can find a finite subset $I_0 \subseteq I$ such that $e = \sum_{i \in I_0} m_i$ with $m_i \in M_i$. Then $eR = \sum_{i \in I_0} M_i$.

Let $\eta : M \to N$ be an epimorphism and $f : eR \to N$. Because the element $f(e) \in N$ we can find $m \in M$ such that $\eta(m) = f(e)$. If we define $h : eR \to M$ by $h(er) = mr$, we obtain $\eta \circ h = f$. □

PROPOSITION 5.13. *Let R be a ring with a set E of local units and P be a module in* DMod-R. *The module P is projective if and only if it is a direct summand of a module of the form $\bigoplus_{e \in E} (eR)^{(I_e)}$.*

PROOF. Any module $\bigoplus_{e\in E}(eR)^{(I_e)}$ is a direct sum of projectives, therefore projective, and any direct summand is projective. On the other hand suppose P is projective. As $\sum_{e\in E} eR = R$ we can find an epimorphism $\eta : \bigoplus_{e\in E}(eR)^{(I_e)} \to P$ for some sets I_e. This epimorphism is a split epimorphism because P is projective and then P is a direct summand of $\bigoplus_{e\in E}(eR)^{(I_e)}$. $\square$

REMARK 5.14. *The module* $\bigoplus_{e\in E} eR$ *is a projective generator of* DMod-R.

With respect to the Morita theorems, the case of rings with local units introduces the concept of progenerator. To this end let us define the following relation in the set of local units.

DEFINITION 5.15. Let R be a ring with a set of local units E, and let $e, f \in E$. We define

$$e \leq f \text{ if and only if } ef = e (= fe)^2$$

Let R and R' be rings with local units E and E' and $(R, R', P, Q, \varphi, \psi)$ be a Morita context with φ and ψ epimorphisms (and then isomorphisms).

Let us define the following homomorphisms for $e \leq f \in E$ and $e' \leq f' \in E'$

$$\begin{array}{rrcl}
\mu_{f'e'}: & e'P & \to & f'P \\
 & e'p & \mapsto & f'e'p = e'p \\
\mu_{fe}: & Pe & \to & Pf \\
 & pe & \mapsto & pef = pe \\
\mu'_{fe}: & eQ & \to & fQ \\
 & eq & \mapsto & feq = eq \\
\mu'_{f'e'}: & Qe' & \to & Qf' \\
 & qe' & \mapsto & qe'f' = qe'
\end{array}
\qquad
\begin{array}{rrcl}
\epsilon_{e'f'}: & f'P & \to & e'P \\
 & f'p & \mapsto & e'f'p \\
\epsilon_{ef}: & Pf & \to & Pe \\
 & pf & \mapsto & pfe \\
\epsilon'_{ef}: & fQ & \to & eQ \\
 & fq & \mapsto & efq \\
\epsilon'_{e'f'}: & Qf' & \to & Qe' \\
 & qf' & \mapsto & qf'e'
\end{array}$$

PROPOSITION 5.16. *With the previous notations*

1. *For all* $e' \in E'$, $\mu_{e'e'} = \epsilon_{e'e'} = \mathrm{id}_{e'P}$
2. *For all* $e' \leq f' \leq g'$

$$\mu_{g'f'} \circ \mu_{f'e'} = \mu_{g'e'}$$

$$\epsilon_{e'f'} \circ \epsilon_{f'g'} = \epsilon_{e'g'}$$

3. *For all* $e' \leq f'$, $\epsilon_{e'f'} \circ \mu_{f'e'} = \mathrm{id}_{e'P}$
4. *For all* $g' \geq e', f'$,

$$\mu_{g'e'} \circ \epsilon_{e'g'} \circ \mu_{g'f'} \circ \epsilon_{f'g'} = \mu_{g'f'} \circ \epsilon_{f'g'} \circ \mu_{g'e'} \circ \epsilon_{e'g'}$$

5. *The modules* $e'P$ *are finitely generated and projective.*
6. $\varinjlim_{e'\in E'} e'P$ *is a generator of* Mod-R.

[2] E is a set of commuting idempotents

PROOF. Almost all this properties can be checked directly. We shall prove only the last two ones.

□

This fact make us give the following definition

DEFINITION 5.17. Let R be a ring with a set of local units E. Let E' be a partially ordered set, $\{P_{e'} : e' \in E'\}$ be a family of right R-modules in Mod-R such that for all $e', f' \in E'$ there exists $g' \geq e', f'$ in E'. Let $(\mu_{f'e'} : P_{e'} \to P_{f'})_{e' \leq f'}$ and $(\epsilon_{e'f'} : P_{f'} \to P_{e'})_{e' \leq f'}$ be families of R-homomorphisms such that

1. For all $e' \in E'$, $\mu_{e'e'} = \epsilon_{e'e'} = \mathrm{id}_{P_{e'}}$
2. For all $e' \leq f' \leq g'$
$$\mu_{g'f'} \circ \mu_{f'e'} = \mu_{g'e'}$$
$$\epsilon_{e'f'} \circ \epsilon_{f'g'} = \epsilon_{e'g'}$$
3. For all $e' \leq f'$, $\epsilon_{e'f'} \circ \mu_{f'e'} = \mathrm{id}_{P_{e'}}$
4. For all $g' \geq e', f'$,
$$\mu_{g'e'} \circ \epsilon_{e'g'} \circ \mu_{g'f'} \circ \epsilon_{f'g'} = \mu_{g'f'} \circ \epsilon_{f'g'} \circ \mu_{g'e'} \circ \epsilon_{e'g'}$$
5. The modules $P_{e'}$ are finitely generated and projective.
6. $\varinjlim_{e' \in E'} P_{e'}$ is a generator of Mod-R.

We shall call
$$(\{P_{e'} : e' \in E'\}, (\mu_{f'e'} : P_{e'} \to P_{f'})_{e' \leq f'}, (\epsilon_{e'f'} : P_{f'} \to P_{e'})_{e' \leq f'})$$
a progenerator in Mod-R.

PROPOSITION 5.18. *With the previous notations*

1. $(\{e'P : e' \in E'\}, (\mu_{f'e'} : e'P \to f'P)_{e' \leq f'}, (\epsilon_{e'f'} : f'P \to e'P)_{e' \leq f'})$ *is a progenerator in* Mod-R
2. $\big(\{eQ : e \in E\}, (\mu'_{fe} : eQ \to fQ)_{e \leq f}, (\epsilon'_{ef} : fQ \to eQ)_{e \leq f}\big)$ *is a progenerator in* Mod-R'
3. $(\{Pe : e \in E\}, (\mu_{fe} : Pe \to Pe)_{e \leq f}, (\epsilon_{ef} : Pf \to Pe)_{e \leq f})$ *is a progenerator in* R'-Mod
4. $\big(\{Qe' : e' \in E'\}, (\mu'_{f'e'} : Qe' \to Qf')_{e' \leq f'}, (\epsilon'_{e'f'} : Qf' \to Qe')_{e' \leq f'}\big)$ *is a progenerator in* R-Mod

If R is a ring with a set of local units E and
$$(\{P_{e'} : e' \in E'\}, (\mu_{f'e'} : P_{e'} \to P_{f'})_{e' \leq f'}, (\epsilon_{e'f'} : P_{f'} \to P_{e'})_{e' \leq f'})$$
is a progenerator for a certain set E', it is possible to build a ring R' with a set of local units bijective with E' such that R and R' are Morita equivalent. All these results be seen in [**1**], together with some of the previous ones with a direct proof that does not use the idempotent rings.

3. Rings With Enough Idempotents

DEFINITION 5.19. Let R be a ring. We shall say that R has enough idempotents if there exists a set of orthogonal idempotents $\{e_\lambda : \lambda \in \Lambda\}$ in R (that will be called a complete set of idempotents for R) such that $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$.

PROPOSITION 5.20. *Let R be a ring with a complete set of idempotents $\{e_\lambda : \lambda \in \Lambda\}$. Then R is a ring with a set of local units $\{e_I = \sum_{\lambda\in I} e_\lambda | I \subseteq \Lambda\ ,\ I$ finite$\}$.*

PROOF. Let $I, J \subseteq \Lambda$ finite.

$$e_I e_J = \sum_{\lambda\in I}\sum_{\mu\in J} e_\lambda e_\mu = \sum_{\lambda\in I\cap J} e_\lambda e_\lambda = e_J e_I$$

Let $r_1, \cdots, r_t$ be a finite family of elements in $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R$. We can find there a finite set $I \subseteq \Lambda$ and elements $\{s_{k\mu} | k = 1, \cdots, t \ \ \mu \in I\}$ such that $r_k = \sum_{\mu\in I} e_\mu s_{k\mu}$, then

$$e_I r_k = \sum_{\lambda,\mu\in I} e_\lambda e_\mu s_{k\mu} = \sum_{\mu\in I} e_\mu s_{k\mu} = r_k$$

for al $k = 1, \cdots, t$. □

DEFINITION 5.21. Let B be a ring with identity, $\{U_\lambda : \lambda \in \Lambda\}$ a family of finitely generated right B-modules, $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$. Let

$$R = \{r : U_B \to U_B | r(U_\lambda) = 0 \text{ for almost all } \lambda \in \Lambda\}$$

and let $\{e_\lambda : \lambda \in \Lambda\}$ be the set of idempotents in R that satisfy $e_\lambda(U) = U_\lambda$. The ring R is called the functor ring of the finitely generated B-modules $\{U_\lambda : \lambda \in \Lambda\}$.

PROPOSITION 5.22. *The functor ring of $\{U_\lambda : \lambda \in \Lambda\}$, R, is a ring with enough idempotents.*

PROOF. Consider the following elements in R:

$$\begin{array}{rccc} e_\mu : & \bigoplus_{\lambda\in\Lambda} U_\lambda & \to & \bigoplus_{\lambda\in\Lambda} U_\lambda \\ & (u_\lambda)_{\lambda\in\Lambda} & \mapsto & (u'_\lambda)_{\lambda\in\Lambda} \end{array}$$

with $u'_\lambda = u_\mu$ if $\lambda = \mu$ and $u'_\lambda = 0$ if $\lambda \neq \mu$.

If $r \in R$, let $I = \{\lambda \in \Lambda : r(U_\lambda) \neq 0\}$. Because of the definition we have given, I is finite and clearly $r = \sum_{\lambda\in I} re_\lambda$. All the U_λ are finitely generated, then $r(U_\mu) \subseteq \sum_{\lambda\in J_\mu} U_\lambda$ with J_μ finite for all μ. If we define $J = \bigcup_{\mu\in I} J_\mu$, then $r = \sum_{\lambda\in J} e_\lambda r$ and this sum is finite. We have proved that $R = \sum_{\lambda\in\Lambda} e_\lambda R = \sum_{\lambda\in\Lambda} Re_\lambda$. This sum is direct because the elements $\{e_\lambda | \lambda \in \Lambda\}$ are orthogonal. □

There exists some special examples of this kind of rings, is the following. Suppose Λ is an arbitrary index set, and $U_\lambda = B$ for all $\lambda \in \Lambda$ with B a ring with identity. The functor ring of the modules $(U_\lambda)_{\lambda\in\Lambda}$ as right B-modules is denoted by $\mathrm{FM}_\Lambda(B)$ and consist in the ring of $\Lambda \times \Lambda$-matrices with a finite number of entries.

In the case of rings with enough idempotents it is possible to rebuild the ring as a functor ring, it is as follows

Proposition 5.23. *Let R and R' be rings with complete sets of idempotents $\{e_\lambda : \lambda \in \Lambda\}$ and $\{e'_{\lambda'} : \lambda' \in \Lambda'\}$, and let $(R, R', P, Q, \varphi, \psi)$ be a Morita context with φ and ψ epimorphisms. Then*

1. *R' is the functor ring of $\{e'_{\lambda'}P : \lambda' \in \Lambda'\}$*
2. *R is the functor ring of $\{Pe_\lambda : \lambda \in \Lambda\}$*
3. *R is the functor ring of $\{e_\lambda Q : \lambda \in \Lambda\}$*
4. *R' is the functor ring of $\{Qe'_{\lambda'} : \lambda' \in \Lambda'\}$*

Proof. We are going to prove only one of them because all the others are proved by symmetry. First of all, we have to notice that

$$P = R'P = \sum_{\lambda'\in\Lambda'} e'_{\lambda'}P = \bigoplus_{\lambda'\in\Lambda'} e'_{\lambda'}P$$

As φ and ψ are isomorphisms, we can find elements $p_{j\lambda'} \in P$ and $q_{j\lambda'} \in Q$ such that $e'_{\lambda'} = \psi(\sum_{j=1}^{n_{\lambda'}} \psi(p_{j\lambda'} \otimes q_{j\lambda'})$.

The functor ring of the family $\{e'_{\lambda'}P : \lambda' \in \Lambda'\}$ consist in the $\sigma; P_R \to P_R$ such that $\sigma(e'_{\lambda'}P) = 0$ for almost all $\lambda' \in \Lambda'$. If $r' \in R'$, the left multiplication by r' has this property because $r'e'_{\lambda'} = 0$ for almost all $\lambda' \in \Lambda'$. Conversely, let σ be in the functor ring of $\{e'_{\lambda'}P : \lambda' \in \Lambda'\}$, we are going to prove that σ is the left multiplication by $s := \sum_{\lambda'\in\Lambda'} \sum_{j=1}^{n_{\lambda'}} \psi(\sigma(e'_{\lambda'}p_{j\lambda'} \otimes q_{j\lambda'})$ (notice that this sum is finite because $\sigma(e'_{\lambda'}p_{j\lambda}) = 0$ for almost all λ'). For that let $p \in P$,

$$sp = \sum_{\lambda'\in\Lambda'} \sum_{j=1}^{n_{\lambda'}} \psi(\sigma(e'_{\lambda'}p_{j\lambda'} \otimes q_{j\lambda'})p = \sum_{\lambda'\in\Lambda'} \sum_{j=1}^{n_{\lambda'}} \sigma(e'_{\lambda'}p_{j\lambda'})\varphi(q_{j\lambda'} \otimes p) =$$

$$= \sum_{\lambda'\in\Lambda'} \sum_{j=1}^{n_{\lambda'}} \sigma(e'_{\lambda'}p_{j\lambda'}\varphi(q_{j\lambda'} \otimes p)) =$$

$$= \sum_{\lambda'\in\Lambda'} \sigma(e'_{\lambda'} \sum_{j=1}^{n_{\lambda'}} \psi(p_{j\lambda'} \otimes q_{j\lambda'})) = \sum_{\lambda'\in\Lambda'} \sigma(e'_{\lambda'}e'_{\lambda'}p) = \sum_{\lambda'\in\Lambda'} \sigma(e'_{\lambda'}p) = \sigma(p)$$

□

4. Rings With Identity

Every ring with identity R is a ring with a complete set of idempotents $\{1_R\}$, the functors $\mathbf{c}\circ\mathbf{i_M}$ and $\mathbf{m}\circ\mathbf{i_C}$ are the identity functors, therefore, the three categories are equal. This property caracterize the rings with identity because $\mathbf{c}\,(R)$ is always a ring with identity.

The definition of a progenerator in this case, is a module that is finitely generated, projective and generator. This definition generalize the one for rings with local units if we consider $(R, \{1_R\})$ as a ring with local units.

If R and R' are rings with identity, and $(R, R', P, Q, \varphi, \psi)$ is a Morita context with φ and ψ epimorphisms, then all the following maps, are isomorphisms:

$$\begin{aligned}
[*,-]:&P \to \mathrm{Hom}_{R'}(Q,R') &\qquad (*,-):&Q \to \mathrm{Hom}_R(P,R)\\
&p \mapsto [p,-] & &q \mapsto (q,-)\\
[-,*]:&Q \to \mathrm{Hom}_{R'}(P,R') & (-,*):&P \to \mathrm{Hom}_R(Q,R)\\
&q \mapsto [-,q] & &p \mapsto (-,p)\\
&R \to \mathrm{End}_{R'}(Q) & &R \to \mathrm{End}_{R'}(P)\\
&r \mapsto (q \mapsto rq) & &r \mapsto (p \mapsto pr)\\
&R' \to \mathrm{End}_R(P) & &R' \to \mathrm{End}_R(Q)\\
&s \mapsto (p \mapsto sp) & &s \mapsto (q \mapsto qs)
\end{aligned}$$

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