An introduction to Lorentzian Geometry and its applications

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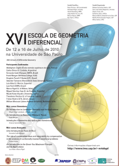

Hendrik Lorentz and Albert Einstein

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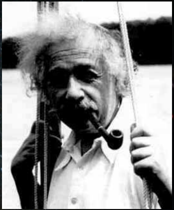

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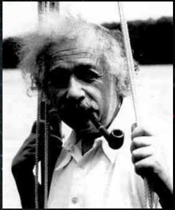

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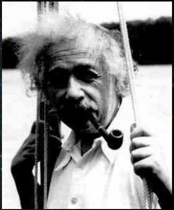

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Omnipresence of Einstein equations - Salar de Uyuni (Bolivia)

Scalar product: basic properties

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Moreover, given $v \in V$, $q_b(v) = b(v, v)$, we say that v is

- *timelike* if $q_b(v) < 0$,
- *lightlike* if $q_b(v) = 0$ and $v \neq 0$,
- *spacelike* if $q_b(v) > 0$,
- *causal* if v is timelike or lightlike.

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A *scalar product* g on V is a nondegenerate sym. bilinear form.

- Given $v, w \in V$, then $v \perp w$ (v and w are *orthogonal*) if $g(v, w) = 0$.

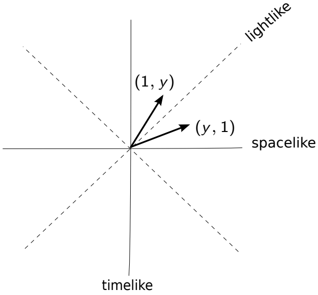

Figure: Two orthogonal vectors in $\mathbb{R}^2$ with $g((x, t), (x', t)) = xx' - tt'$

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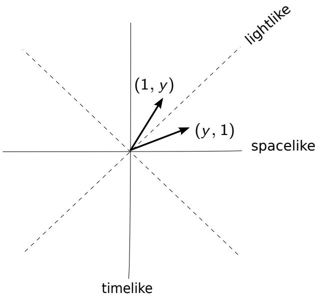

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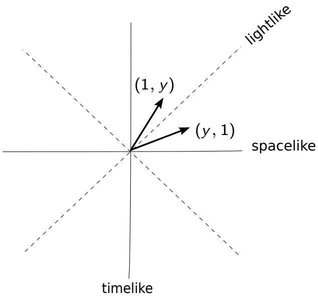
- $A^{\perp} = \{w \in V : g(v, w) = 0, \forall v \in A\}$.



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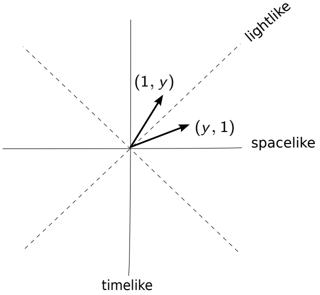
- $A^{\perp} = \{w \in V : g(v, w) = 0, \forall v \in A\}$.
- a basis $e_1, e_2, \ldots, e_n$ of V is said *orthonormal* if $|e_i| = \sqrt{|g(e_i, e_i)|} = 1$, $g(e_i, e_j) = 0$, $i, j = 1, \ldots, n$.



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The number ν of timelike vectors in a basis B of (V, g) does not depend on the basis, but only on (V, g). ν is called the index of (V, g).

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Proof.

- Let $\overbrace{e_1, e_2, \ldots, e_{n-\mu}}^{spacelike}, \overbrace{e_{n-\mu+1}, \ldots, e_n}^{timelike}$ and $\underbrace{e'_1, e'_2, \ldots, e'_{n-\mu'}}_{spacelike}, \underbrace{e'_{n-\mu+1}, \ldots, e'_n}_{timelike}$ be two (ordered) orthonormal bases.

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- If $\mu < \mu'$, $U = \langle e_{n-\mu+1}, \ldots, e_n \rangle_{\mathbb{R}} \cap \langle e'_1, e'_2, \ldots, e'_{n-\mu'} \rangle_{\mathbb{R}} \neq \{0\}$ because of dimensions

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- Contradiction!! if $v \in U$, $g(u, u) < 0$ and $g(u, u) > 0$.

□

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If $W < V$, then

(i) $\dim W + \dim W^{\perp} = \dim V$,

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Proof.

(i) Let $e_1, \ldots, e_n$ be a basis of V such that $e_1, \ldots, e_\rho$ is a basis of W. If $v = \sum_{i=1}^n a^i e_i$, then $v \in W^\perp \Leftrightarrow g(v, e_i) = 0 \quad \forall i = 1, \ldots, \rho$ $\Leftrightarrow \sum_{j=1}^n g_{ij} a^j = 0 \quad \forall i = 1, \ldots, \rho$, where $g_{ij} = g(e_i, e_j)$. □

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- Then

$$e_1, \ldots, e_{n-1}, \frac{u}{|u|}$$

is the orthonormal basis.

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- *Euclidean* if $\nu = 0$,
- *Lorentzian* if $\nu = 1$ and $n \geq 2$.

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Example

In $\mathbb{R}^n$ we will define the usual scalar product of index ν, $\langle \cdot, \cdot \rangle_\nu$, as

$$\langle (a^1, \ldots, a^n), (b^1, \ldots, b^n) \rangle_\nu = \sum_{i=1}^{n-\nu} a^i b^i - \sum_{i=n-\nu+1}^{n} a^i b^i .$$

The origin of the index 1

- In the (Newtonian) 3-dimensional space, the distance does not depend on the inertial frame of reference

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2}$$
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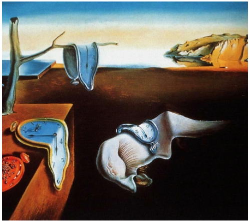

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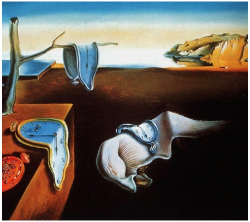
- This leads to consider non positive metrics on index 1



Lorentzian vector spaces: timelike cones

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The subset of the timelike vectors (resp., causal; lightlike if $n > 2$) has two connected parts.
Each one of these parts will be called timelike cone, *(resp.* causal cone*;* lightlike cone*).*

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Proof.

Let $e_1, \ldots, e_n$ be an orthonormal basis of V, and $v \in V$ such that $v = \sum_{i=1}^{n} a^i e_i$. Obviously,

$$v \text{ is lightlike } \Leftrightarrow \begin{cases} |a^n| = \sqrt{(a^1)^2 + \ldots + (a^{n-1})^2} \\ a^n \neq 0 \end{cases}$$

$$v \text{ is timelike } \Leftrightarrow |a^n| > \sqrt{(a^1)^2 + \ldots + (a^{n-1})^2},$$

$$v \text{ is causal } \Leftrightarrow \begin{cases} |a^n| \geq \sqrt{(a^1)^2 + \ldots + (a^{n-1})^2} \\ a^n \neq 0 \end{cases}$$

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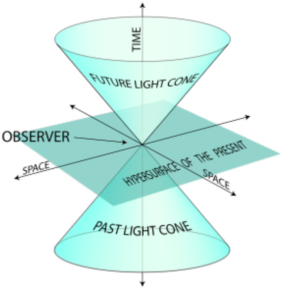
A *time orientation* is a choice of one of the two timelike cones. The chosen cone will be called *future*, and the other one, *past*.



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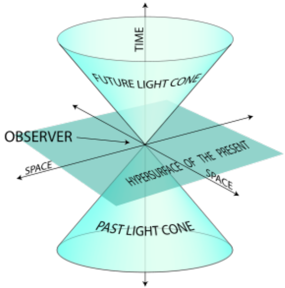

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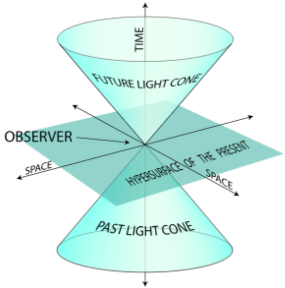
v can be completed to an orthonormal basis $e_1, e_2, \ldots, e_{n-1}, \frac{v}{|v|}$. Observing that

$$w = g(e_1, w)e_1 + \ldots + g(e_{n-1}, w)e_{n-1} - g(v, w)\frac{v}{|v|^2},$$

v and w are in the same cone iff $-g(v, w) > 0$ □



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Proof.

We know that $g(v, w) < 0$, and then

$$g(v, av + bw) = ag(v, v) + bg(v, w) < 0,$$
$$g(av + bw, av + bw) = a^2 g(v, v) + b^2 g(w, w) + 2ab g(v, w) < 0.$$

□

Reverse inequalities

Theorem (Reverse Cauchy-Schwarz inequality)

If $v, w \in V$ are timelike vectors, then

- $|g(v, w)| \geq |v||w|$, *and equality holds iff v, w are colinear.*
- *If v and w lie in the same cone, then $\exists! \ \varphi \geq 0$, called the hyperbolic angle between v and w such that*

$$g(v, w) = -|v||w| \cosh(\varphi).$$

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Proof.

Let $a \in \mathbb{R}$ and $\overline{w} \in \langle v \rangle_{\mathbb{R}}^{\perp}$ such that $w = av + \overline{w}$. Then

$$g(w,w) = a^2 g(v,v) + g(\overline{w}, \overline{w}),$$

and hence $g(v,w)^2 = a^2 g(v,v)^2 = g(v,v)(g(w,w) - g(\overline{w},\overline{w})) \geq g(v,v)g(w,w) = |v|^2|w|^2$.

Reverse inequalities

Theorem (Reverse triangular inequality)

If $v, w \in V$ are timelike vectors in the same cone, then

$$|v| + |w| \leq |v + w|,$$

and the equality holds if and only if v, w are colinear.

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Proof.

As v, w lie in the same cone, $v + w$ is timelike and $g(v, w) < 0$. Therefore

$$|v + w|^2 = -g(v + w, v + w)$$
$$= |v|^2 + |w|^2 + 2|g(v, w)| \geq |v|^2 + |w|^2 + 2|v||w| = (|v| + |w|)^2.$$

Moreover, equality holds iff $|g(v, w)| = |v||w|$, that is, iff v, w are colinear. □

Subspaces

Definition

Let (V, g) be a Lorentzian vector space. We will say that a subspace of V, $W < V$ is

- *spacelike*, if $g_{|W}$ is Euclidean,
- *timelike*, if $g_{|W}$ is nondegenerate with index 1 (that is, Lorentzian whenever $\dim W \geq 2$),
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Proposition

A subspace $W < V$ is timelike if and only if $W^{\perp}$ is spacelike.

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If $W < V$*, with* $\dim(W) \geq 2$*, the following conditions are equivalent:*

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Proof.

- $(i) \Rightarrow (ii)$. As W is timelike, given an orthonormal basis $e_1, e_2, \ldots, e_k$ of W, e_1 is spacelike and e_k timelike. Then $e_1 + e_k$ and $e_1 - e_k$ are two lin. ind. lightlike vectors.

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- *(ii)* $\Rightarrow$ *(iii)*. Let v, w be two lin. ind. lightlike vectors of W, then either $v + w$ or $v - w$ is timelike because $g(v, w) \neq 0$

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(iii) *W contains one timelike vector.*

Proof.

- *(iii)* $\Rightarrow$ *(i)*. Let u be a timelike vector of W. Assume by contradiction that $g|_W$ is degenerate. Then $\exists\, z \neq 0$ in $\mathrm{rad}(g|_W)$. As u, z are lin. ind., we know that

$$\begin{cases} \text{if } u, z \text{ are in the same causal cone} & \Rightarrow g(u,z) < 0 \\ \text{if } u, z \text{ are in different causal cones} & \Rightarrow g(u,z) > 0. \end{cases}$$

Subspaces

Proposition

If $W < V$, the following conditions are equivalent:

(i) *W is lightlike.*

(ii) *W contains a lightlike vector, but* not *a timelike one.*

(iii) *The intersection of W with the subset of null vectors (lightlike or zero) forms a vector subspace of dimension 1.*

The Lorentz group: the four connected components

Denote $\mathbb{L}^n$ the Lorentz-Minkowski spacetime, that is, $\mathbb{R}^n$ endowed with $\langle\cdot,\cdot\rangle_1$ and

$$\eta = \left(\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & -1 \end{array}\right)$$

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Definition

We define the Lorentz transformation group as

$$\mathrm{Iso}(\mathbb{L}^n) = \{f : \mathbb{L}^n \longrightarrow \mathbb{L}^n \mid f \text{ is a vector isometry}\},$$

and the *Lorentz group* as $O_1(n) = \{A \in M_n(\mathbb{R}) \,/\, A^t \eta A = \eta\}$.

The Lorentz group: the four connected components

Definition

Let f be a Lorentz transformation. Then

- f is *proper* if $\det f (= \det A_f) = 1$,
- f is *improper* otherwise.
- $\mathrm{Iso}^+(\mathbb{L}^n) =$ proper Lorentz transformations; $O_1^+(n) := \Phi(\mathrm{Iso}^+(\mathbb{L}^n))$
- $\mathrm{Iso}^-(\mathbb{L}^n) =$ improper Lorentz transformations; $O_1^-(n) := \Phi(\mathrm{Iso}^-(\mathbb{L}^n))$

The Lorentz group: the four connected components

From the usual basis $e_1, \ldots, e_n$ of $\mathbb{L}^n$, we can fix the standard time orientation:

$$\left\{ \begin{array}{l} \textit{Future} \text{ causal cone } C^{\uparrow} : \text{the one that contains } e_n, \\ \textit{Past} \text{ causal cone } C^{\downarrow} : \text{the one that contains } -e_n. \end{array} \right.$$

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Definition

- We will say that f is *orthocronous* if $f(C^{\uparrow}) = C^{\uparrow}$
- $\mathrm{Iso}^{\uparrow}(\mathbb{L}^n) =$ the subgroup of orthocronous transformations
- $O_1^{\uparrow}(n) = \Phi(\mathrm{Iso}^{\uparrow}(n))$.
- $\mathrm{Iso}^{\downarrow}(\mathbb{L}^n) =$ the subset of nonorthocronous transformations,
- $O_1^{\downarrow}(n) = \Phi(\mathrm{Iso}^{\downarrow}(\mathbb{L}^n))$.

The Lorentz group: the four connected components

- We will combine the notation in an obvious way:

$$O_1^{+\downarrow}(n), O_1^{+\uparrow}(n), O_1^{-\downarrow}(n), O_1^{-\uparrow}(n).$$

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- Nevertheless, we will use the special notation $SO_1^{\uparrow}(n)$ for the *restricted Lorentz group*, that is, the subgroup of proper orthocronus transformations.

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- Nevertheless, we will use the special notation $SO_1^{\uparrow}(n)$ for the *restricted Lorentz group*, that is, the subgroup of proper orthocronus transformations.

Proposition

If $f \in \mathrm{Iso}(\mathbb{L}^n)$, then the following conditions are equivalent:

- $f \in \mathrm{Iso}^{\uparrow}(\mathbb{L}^n)$,
- $\exists$ *a causal vector* $v \in \mathbb{L}^n$ *such that* $\langle v, f(v)\rangle < 0$,
- $\forall$ *timelike vector* $v \in \mathbb{L}^n$, $\langle v, f(v)\rangle < 0$,
- *in every orthonormal basis, the element* (n, n) *of the matrix of* f *is* ≥ 0 *(and, actually,* ≥ 1*).*

The Lorentz group: the four connected components

$$\left(\begin{array}{c|cc} I_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right) \in SO_1^{\uparrow}(n); \qquad \left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{array}\right) \in O_1^{-\downarrow}(n)$$

$$\left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{array}\right) \in O_1^{+\downarrow}(n); \qquad \left(\begin{array}{c|c} I_{n-2} & 0 \\ \hline 0 & \begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \end{array}\right) \in O_1^{-\uparrow}(n)$$

The Lorentz group in dimension 2

Isometries with determinant equal to 1: all of them admit a basis of lightlike eigenvectors and

$$SO_1^{\uparrow}(2) = \left\{ \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} : \theta \in \mathbb{R} \right\};\; O_1^{+\downarrow}(2) = \left\{ -A : A \in O_1^{+\uparrow}(2) \right\}.$$

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Isometries with determinant equal to -1: all of them admit an orthonormal basis of eigenvectors and

$$O_1^{-\uparrow}(2) = \left\{ \begin{pmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & -\cosh\theta \end{pmatrix} : \theta \in \mathbb{R} \right\};\; O_1^{-\downarrow}(2) = \left\{ -A : A \in O_1^{-\uparrow}(2) \right\}$$

Observe that, unlike the Euclidean case, all these matrices are diagonalizable.

Properties of Lorentz group in greater dimensions

Proposition

If $A \in O_1(n)$, then:

(i) *non-lightlike eigenvectors of A, if any, have $+1$ or -1 as eigenvalues,*

(ii) *the product of the eigenvalues of two lin. indep. lightlike eigenvectors is 1,*

(iii) *if U is an eigenspace of A that contains a non-lightlike eigenvector, then any other eigenspace is orthogonal to U,*

(iv) *if U is an A-invariant subspace, then $U^\perp$ is also A-invariant.*

Proof.

(i) Let v be a non-lightlike eigenvector of A, $Av = av$. Then

$$\langle v, v\rangle = \langle Av, Av\rangle = a^2\langle v, v\rangle \Rightarrow a = \pm 1.$$

□

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Proof.

(ii) Let v, w be two lin. indep. lightlike eigenvectors, $Av = av$, $Aw = bw$.

$$0 \neq \langle v, w\rangle = \langle Av, Aw\rangle = ab\langle v, w\rangle \Rightarrow ab = 1.$$

□

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Proof.

(iii) Let $z \in U$ be a non-lightlike eigenvector of A. By (i), $Az = \epsilon z$, $\epsilon = \pm 1$. Let w be an eigenvector of λ distinct from ϵ, $\forall\ u \in U$,

$$\langle u, w\rangle = \langle Au, Aw\rangle = \lambda\epsilon\langle u, w\rangle.$$

Thus, $\langle u, w\rangle = 0$ ($\lambda\epsilon \neq 1$). □

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Proof.

(iv) As A is an isometry, $A(U) = U$. Moreover, $A^{-1}(U) = U$. Consider $w \in U^{\perp}$, then

$$\langle Aw, u\rangle = \langle w, A^{-1}u\rangle = 0, \qquad \forall u \in U,$$

and therefore, $Aw \in U^{\perp}$, which concludes. □

Properties of Lorentz group in greater dimensions

Theorem

If $A \in O_1(n)$ and $f_A \in \mathrm{Iso}(\mathbb{L}^n)$, then one of the following three mutually exclusive cases holds:

(i) *A admits a timelike eigenvector. Then $M(f_A, B)$ is*

$$\left(\begin{array}{c|c} R_{n-1} & 0 \\ \hline 0 & \pm 1 \end{array}\right),$$

where B is orthon. and $R_{n-1} \in O(n-1)$.

(ii) *A admits a lightlike eigenvector with eigenvalue $\lambda \neq \pm 1$. Then there exists an o. b. B such that $M(f_A, B)$ is*

$$\left(\begin{array}{c|c} R_{n-2} & 0 \\ \hline 0 & R \end{array}\right),$$

where $R_{n-2} \in O(n-2)$ and $R \in O_1(2)$.

(iii) *A admits a unique indep. lightlike eigenvector of eigenvalue ± 1.*

Properties of Lorentz group in greater dimensions

Proof.

- Reasoning by induction, for $n = 2$ the conclusion follows from the study of $O_1(2)$

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- Reasoning by induction, for $n = 2$ the conclusion follows from the study of $O_1(2)$
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- Let λ be an eigenvalue of A

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    - If $\lambda \in \mathbb{R}$ and it has a causal eigenvector, we conclude.
    - If $\lambda \in \mathbb{R}$ and it has a spacelike eigenvector v, apply induction to $\langle v \rangle_{\mathbb{R}}^{\perp}$
    - If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $Az = \lambda z$, then $P = \langle z, \bar{z} \rangle_{\mathbb{R}}$ is spacelike. Apply induction to $P^{\perp}$.

□

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1) If v is timelike, we obtain (i).
2) If v is lightlike, it can happen that
   a) $\lambda \notin \{\pm 1\}$, in this case we obtain (ii). $\Longrightarrow 1/\lambda$ is eigenvalue, and if v, w eigenvectors of λ, $1/\lambda$ resp., $\langle v, w\rangle_{\mathbb{R}}$ is timelike.

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    b) $\lambda \in \{\pm 1\}$, and (iii) does not hold, then (i) occurs:

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   b) $\lambda \in \{\pm 1\}$, and (iii) does not hold, then (i) occurs:
   c) if w lightlike eigen. lin. indep. of v, then the eigenvalues of w and v are equal (the product is 1), so either $u + w$ or $u - w$ is a timelike eigenvector.

□

The spin covering of the restricted Lorentz group

- Our aim is to show that the universal covering of $SO_1^{\uparrow}(4)$ is the group $Sl(2,\mathbb{C})$, constructing explicitly the universal covering homomorphism $Sl(2,\mathbb{C}) \to SO_1^{\uparrow}(4)$ or *spin map*.

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- A brief study of the topology of $Sl(2,\mathbb{C})$, showing in particular that it is 1-connected.

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- The Hermitian matrices $H(2,\mathbb{C})$ constitute naturally a (real) Lorentz vector space, canonically isomorphic to $\mathbb{L}^4$.
- The natural action $Sl(2,\mathbb{C}) \times H(2,\mathbb{C}) \to H(2,\mathbb{C})$ induces the required spinor map

$$Sl(2,\mathbb{C}) \to \mathrm{Iso}^{+\uparrow}(H(2,\mathbb{C}), g_L) \equiv SO_1^{\uparrow}(4).$$

The spin covering of the restricted Lorentz group

Lemma

If $|a|^2+|b|^2 \neq 0$ $A \in Sl(2,\mathbb{C})$ *iff* $\exists$ *(a unique)* $\lambda \in \mathbb{C}$ *such that*

$$\begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{|a|^2+|b|^2}\begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \end{pmatrix}; \quad \text{where } A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Proof.

Recall that

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \begin{vmatrix} a & -\frac{\bar{b}}{|a|^2+|b|^2} \\ b & \frac{\bar{a}}{|a|^2+|b|^2} \end{vmatrix} + \begin{vmatrix} a & c+\frac{\bar{b}}{|a|^2+|b|^2} \\ b & d-\frac{\bar{a}}{|a|^2+|b|^2} \end{vmatrix},$$

the second determinant equal to 1. So, the first determinant is 1 iff the last one is 0. □

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Proposition

The map

$$F:(\mathbb{C}^2\setminus\{\mathbf{0}\})\times\mathbb{C}\to Sl(2,\mathbb{C}), \quad \left(\begin{pmatrix} a \\ b \end{pmatrix},\lambda\right)\mapsto \begin{vmatrix} a & -\frac{\bar{b}}{|a|^2+|b|^2}+\lambda a \\ b & \frac{\bar{a}}{|a|^2+|b|^2}+\lambda b \end{vmatrix}$$

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Corollary

$Sl(2,\mathbb{C})$ *is diffeomorphic to* $\mathbb{R}^3\times\mathbb{S}^3$.

The spin covering of the restricted Lorentz group

$$H(2,\mathbb{C}) = \{\begin{pmatrix} a & z \\ \bar{z} & d \end{pmatrix} \in M_2(\mathbb{C}) : a, d \in \mathbb{R}, z = x + iy \in \mathbb{C}\}$$

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- the (minus) determinant

$$- \begin{vmatrix} a & z \\ \bar{z} & d \end{vmatrix} = x^2 + y^2 - ad$$

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- the subspace $H(2,\mathbb{C})_*$ of the traceless matrices $(d = -a)$ constitutes a natural spacelike hyperplane, $(H(2,\mathbb{C})_*, g_E)$.

Action of $Sl(2,\mathbb{C})$ on $H(2,\mathbb{C})$ and $\mathbb{L}^4$

Consider the following map:

$$\begin{array}{rcl} Sl(2,\mathbb{C})\times H(2,\mathbb{C}) & \to & H(2,\mathbb{C}) \\ (A,X) & \mapsto & A*X := AXA^{\dagger}. \end{array}$$

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Straightforward relevant properties are:

- It is well-defined: $(AXA^{\dagger})^{\dagger} = AXA^{\dagger}$.
- It is an action: $(A_1 \cdot A_2) * X = A_1 * (A_2 * X)$.
- It gives a linear isometry for $(H(2,\mathbb{C}), g_L)$:

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- It gives a linear isometry for $(H(2,\mathbb{C}), g_L)$:
  - Linearity in the second variable: $A * (aX_1 + bX_2) = a(A * X_1) + b(A * X_2)$,
  - Preserves g_L: $\det(A * X) = \det(X)$,

for all $A, A_1, A_2 \in Sl(2,\mathbb{C}), X, X_1, X_2 \in H(2,\mathbb{C}), a, b \in \mathbb{R}$.

Action of $Sl(2,\mathbb{C})$ on $H(2,\mathbb{C})$ and $\mathbb{L}^4$

The third property implies that the well-defined map

$$A_* : H(2,\mathbb{C}) \to H(2,\mathbb{C}), \qquad X \mapsto A * X,$$

is an isometry of $(H(2,\mathbb{C}), g_L)$.

$$* : Sl(2,\mathbb{C}) \to \mathrm{Iso}(H(2,\mathbb{C}), g_L), \qquad A \mapsto A_*.$$

Moreover, $(A_1 \cdot A_2)_* = (A_1)_* \circ (A_2)_*$. That is, the map $*$ is a Lie group homomorphism.

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Theorem

The spin map defined above is a double covering map, which yields the universal covering group of $SO_1^{\uparrow}(4)$. Thus,

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- So, $AX = XA$ for all X, and $A = \pm I_2$ follows easily.

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Decomposition in rotations and boosts

Our plan of work is:

- recall the polar decomposition $A = PR$ of any $A \in Gl(n, \mathbb{C})$ by means of $P \in H^+(n, \mathbb{C})$ and $R \in U(n)$.

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- As a consequence any matrix on $SO_1^{\uparrow}(4)$ can be written as the composition of a rotation and a boost.

Decomposition in rotations and boosts

Lemma

Let (V, G) be a complex vector space endowed with an inner product and $f \in Aut_{\mathbb{C}} V$. Then:

- $f \circ f^{\dagger}$ *is self-adjoint, and all its eigenvalues are positive. So, there exists a G-orthonormal basis B such that $M(f \circ f^{\dagger}, B)$ is diagonal, real and definite positive.*
- $\exists ! h \in Aut_{\mathbb{C}} V$ *self-adjoint and with all its eigenvalues positive, such that* $h \circ h = f \circ f^{\dagger}$.
- $h^{-1} \circ f \in Iso\,(V, G)$.

Decomposition in rotations and boosts

Theorem

For all $A \in \mathrm{Gl}(n, \mathbb{C})$ *there exist* $!P \in H_+(n, \mathbb{C})$ *and* $!R \in U(n)$ *so that:* $A = P\,R$*.*
Moreover, the map

$$H_+(n, \mathbb{C}) \times U(n) \to \mathrm{Gl}(n, \mathbb{C}), \quad (P, R) \mapsto P\,R,$$

is a homeomorphism.

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Corollary

If $A \in Sl(2, \mathbb{C})$ *and* P, R *are the matrices obtained in its polar decomposition, then* $P \in SH_+(2, \mathbb{C})$ *and* $R \in SU(2)$*.*
Therefore, the restricted map

$$SH_+(2, \mathbb{C}) \times SU(2) \to Sl(2, \mathbb{C}), \quad (P, R) \mapsto P\,R,$$

is a homeomorphism.

Decomposition in rotations and boosts

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Proof.

Clearly, $SH_+(2,\mathbb{C})$ is homeomorphic to $\mathbb{R}^3$, as

$$SH_+(2,\mathbb{C}) = \{\begin{pmatrix} a & x+iy \\ x-iy & d \end{pmatrix} \in M_2(\mathbb{C}) : x,y \in \mathbb{R}, \quad a,d > 0,\ ad - x^2 - y^2 = 1\}$$

and one can remove the last restriction substituting $a = (1+x^2+y^2)/d$. Moreover, $SU(2)$ is homeomorphic to $\mathbb{S}^3$. □

Decomposition in rotations and boosts

Proposition

Let $R \in Sl(2,\mathbb{C})$. $R \in SU(2)$ iff $\Lambda(R) \in \widetilde{SO}(3)$, that is, $\Lambda(R)$ is a rotation, being x^4 an axis of rotation.

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Proof.

The Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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- So, the restriction of R_* to $\sigma_4^\perp = (H(2,\mathbb{C})_*, g_E)$ is an isometry which preserves the orientation.

□

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If $P \in SH_+(2, \mathbb{C})$, there exists a timelike plane π which contains the x^4-axis such that $\Lambda(P)$ is a boost on π.

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For $\alpha > 0, (\alpha \neq 1)$, let $P_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ $(\in SH_+(2, \mathbb{C}))$. Then $\Lambda(P_\alpha)$ is a boost on $\langle x^3, x^4 \rangle_{\mathbb{R}}$ with eigenvalues α^2, α^{-2}.

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Proof of Proposition.

- as P is Hermitian, there exists $R \in SU(2)$ such that $P = R^{-1}P_\alpha R$ for some positive $\alpha(\neq 1)$.
- Let σ_i' so that $R_*\sigma_i' = \sigma_i$ for $i = 1,2,3$. Clearly, $P_*\sigma_i' = \sigma_i'$ for $i = 1,2$.
- So P_* is a boost on the orthogonal plane $\pi = \langle \sigma_3', \sigma_4\rangle_{\mathbb{R}}$.

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Theorem

Let $L \in SO_1^{\uparrow}(4)$. Then $\exists$ a boost B on a timelike plane π_1 which contains the x_4 axis, and a rotation S on a spacelike plane π_2 orthogonal to the x^4 axis (but not necessarily orthogonal to π_1) such that $L = B\,S$.

The Möbius group in the starred nights

Proposition

The smooth map

$$\tilde{j}:\mathbb{C}^2\setminus\{\mathbf{0}\}\to H(2,\mathbb{C}),\quad \begin{pmatrix}\xi\\ \eta\end{pmatrix}\mapsto\begin{pmatrix}\xi\\ \eta\end{pmatrix}(\bar{\xi},\bar{\eta})=\begin{pmatrix}|\xi|^2 & \xi\bar{\eta}\\ \bar{\xi}\eta & |\eta|^2\end{pmatrix}$$

satisfies:

(i) *The image of* $\tilde{j}$ *is the set of all the future-directed lightlike vectors of the Lorentzian vector space* $(H(2,\mathbb{C}),g_L)$.

(ii) $\tilde{j}$ *induces a bijection* j *between* $\mathbb{C}P^1$ *and the set of all the future-pointing lightlike directions*

$$j:\mathbb{C}P^1\ (\equiv S_R)\to\mathbb{S}^2\times\{1\}\ (\equiv\ \textit{future lightlike directions of}\ (H(2,\mathbb{C}),g_L)\equiv\ S_R).$$

(iii) *For any* $A\in Sl(2,\mathbb{C})$ *and the corresponding restricted isometry* A_* *of* $(H(2,\mathbb{C}),g_L)$*:*

$$\tilde{j}(A\begin{pmatrix}\xi\\ \eta\end{pmatrix})=(A_*\tilde{j})\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$
$$j([A\begin{pmatrix}\xi\\ \eta\end{pmatrix}])=[(A_*\tilde{j})\begin{pmatrix}\xi\\ \eta\end{pmatrix}].$$

Self-adjoint endomorphisms with Lorentzian products

Proposition

Let (V, g) be a vector space V endowed with a Lorentzian scalar product g and $A : V \to V$ a self-adjoint endomorphism with respect to g. Then any of the following possibilities happens

- *there exists an orthonormal basis in that the matrix of A is diagonal or of the form*

$$\left(\begin{array}{c|cc} D_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & a & b \\ & -b & a \end{array}\right)$$

- *there exists a basis $e_1, e_2, \ldots, e_{n-2}, u, v$ with $g(e_i, e_i) = 1$ for $i = 1, \ldots, n-2$, $g(u, v) = 1$ and all the other products equal to zero, in that the matrix representation of A is either*

$$\left(\begin{array}{c|cc} D_{n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \lambda & \epsilon \\ & 0 & \lambda \end{array}\right) \quad \text{with } \epsilon = \pm 1, \text{ or } \quad \left(\begin{array}{c|ccc} D_{n-2} & \multicolumn{3}{c}{0} \\ \hline & \lambda & 0 & 1 \\ 0 & 1 & \lambda & 0 \\ & 0 & 0 & \lambda \end{array}\right).$$