Finsler metrics (Flag Curvature)

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Universidad de Granada

Seminario del departamento de Geometría y Topología
16 de diciembre de 2009
Main reference:


DEFINITION: a Finsler metric $F$ in a manifold $M$ is a continuous function $F : TM \to [0, +\infty)$ such that:

1. It is $C^\infty$ in $TM \setminus \{0\}$
2. Positively homogeneous of degree one
   \[ F(x, \lambda y) = \lambda F(x, y) \text{ for all } \lambda > 0 \]
3. Fiberwise strictly convex square:
   \[ g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (F^2) \bigg|_{x, y} \]
   is positively defined.

It can be showed that this implies:

- $F$ is positive in $TM \setminus \{0\}$
- Triangle inequality holds in the fibers
- $F^2$ is $C^1$ on $TM$.
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Non-symmetric “distance”

We can define the length of a curve:
\[ L(\gamma) = \int_{a}^{b} F(\gamma, \dot{\gamma}) \, ds \]

and then the distance between two points:
\[ \text{dist}(p, q) = \inf_{\gamma \in C^\infty(p, q)} L(\gamma) \]

\text{dist} is non-symmetric because \( F \) is non-reversible. The length of a curve \( t \to \gamma(t) \) is different from the length of its reverse \( t \to \gamma(t) \)!! We have to distinguish between forward and backward.
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- geodesical completeness
Closed Geodesics

A geodesic of \((M,F)\) (parameterized by the arclength) is a critical curve of the energy function

\[ E(\gamma) = \int_0^1 F^2(\gamma, \dot{\gamma}) \, ds \]

Existence of closed geodesics in compact manifolds:
At least one: Fet and Lyusternik (51), F. Mercuri (78)

Multiplicity results under topological hypotheses:
Gromoll-Meyer theorem: Betti numbers of \(\Lambda M\) are unbounded
(Matthias 78)
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Katok metrics (73) in \(S^n\) admit a finite number of closed geodesics.

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Chern Connection

\[\pi: \{0\} \to \mathcal{M}\]

is the natural projection.

Now we take the pullback of \(\mathcal{T}\) by \(d\pi = \pi^*\), that is, \(\pi^*\mathcal{T}\).

We have a metric over this vector bundle given by

\[g_{ij}(x, y) = \frac{1}{4} \partial_x^2 (F^2) \partial_y^i \partial_y^j (x, y),\]

where \(g_{ij}(x, y) = 1\).
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Given a connection $\nabla$, the connection 1-forms $\omega^i$:

$$\nabla_v \frac{\partial}{\partial x^j} = \omega^i_j(v) \frac{\partial}{\partial x^i}$$
Chern Connection

- Given a connection $\nabla$, the connection 1-forms $\omega^i_j$: $\nabla_v \frac{\partial}{\partial x^j} = \omega^i_j(v) \frac{\partial}{\partial x^i}$
- The **Chern connection** $\nabla$ is the unique linear connection on $\pi^* TM$ whose connection 1-forms $\omega^i_j$ satisfy:

\[ d\pi^* g_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = 2 F A^i_{jks} \delta^s_y \]

where $\delta^s_y$ are the 1-forms on $\pi^* TM$ given as $\delta^s_y := dy^s + N^s_j dx^j$.

$N^i_j(x, y)$ are the coefficients of the so-called nonlinear connection on $TM \setminus 0$, and $\gamma^i_{jk}(x, y) = \frac{1}{2} g^{rs} y^r y^s$.
Chern Connection

- Given a connection $\nabla$, the connection 1-forms $\omega^i_j$: $\nabla_v \frac{\partial}{\partial x^j} = \omega^i_j(v) \frac{\partial}{\partial x^i}$
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$$dx^j \wedge \omega^i_j = 0 \quad \text{torsion free} \quad (1)$$

$$dg_{ij} - g_{kj} \omega^k_i - g_{ik} \omega^k_j = \frac{2}{F} A_{ijs} \delta y^s \quad \text{almost g-compatibility} \quad (2)$$
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\[
N^i_j(x, y) := \gamma^i_{jk} y^k - \frac{1}{F} A^i_{jk} \gamma^r_{rs} y^r y^s
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are the coefficients of the so called *nonlinear connection* on \( TM \setminus 0 \), and
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\[
\gamma^i_{jk}(x, y) = \frac{1}{2} g^{is} \left( \frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right), A_{ijk}(x, y) = \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{F}{4} \frac{\partial^3 (F^2)}{\partial y^i \partial y^j \partial y^k},
\]
Covariant derivatives

The components of the Chern connection are given by:

\[ \Gamma_{ij}^{k}(x, y) = \gamma_{ij}^{k} - g_{il} F(A_{ljs}^{N} s^{k} - A_{jks}^{N} s^{i} + A_{kls}^{N} s^{j}) \]

that is,

\[ \omega_{ij} = \Gamma_{ij}^{k} dx^{k} \]

The Chern connection gives two different covariant derivatives:

\[ D_{T}W = \left( dW_{i} dt + W_{j} T^{k} \Gamma_{i}^{j} (\gamma, T) \right) \frac{\partial}{\partial x^{i}} \bigg|_{\gamma(t)} \]

with ref. vector \( T \),

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\end{align*}
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Other connections

- Cartan connection: metric compatible but has torsion
- Hashiguchi connection
- Berwald connection: no torsion. Specially good to treat with Finsler spaces of constant flag curvature
- Rund connection: coincides with Chern connection
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E. Cartan (1861-1940)
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Curvature 2-forms of the Chern connection

The curvature 2-forms of the Chern connection are:

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\[ \Omega_{j}^{i} := d\omega_{j}^{i} - \omega_{j}^{k} \wedge \omega_{k}^{i} \]

- It can be expanded as

\[ \Omega_{j}^{i} := \frac{1}{2} R_{j}^{i} {}_{kl} dx^{k} \wedge dx^{l} + P_{j}^{i} {}_{kl} dx^{k} \wedge \frac{\delta y^{l}}{F} + \frac{1}{2} Q_{j}^{i} {}_{kl} \frac{\delta y^{k}}{F} \wedge \frac{\delta y^{l}}{F} \]
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\[ R_j^i_{kl} = \frac{\delta \Gamma^i_j}{\delta x^k} - \frac{\delta \Gamma^i_k}{\delta x^k} + \Gamma^i_{hk} \Gamma^h_{jl} - \Gamma^i_{hl} \Gamma^h_{jk} \left( \frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i_k \frac{\partial}{\partial y^i} \right) \]
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$$P_j^i_{kl} = -F \frac{\partial \Gamma^i}{\partial y^l}$$
Bianchi Identities

First Bianchi Identity for $R$

Other identities:

$$P_{ikjl} = -R_{ijkl} + R_{jikl} = 2B_{ijkl},$$

where $B_{ijkl} := -A_{iju} R_{u kl}$, $R_{u kl} = y_{j} F_{u j kl}$ and $R_{ijkl} = g_{j}^{\mu} R_{i kl} R_{\mu ji} - R_{jikl} = (B_{klji} - B_{jikl}) + (B_{kilj} + B_{ljki}) + (B_{ilji} + B_{jkil})$.

Second Bianchi identities: very complicated, mix terms in $R_{i j kl}$ and $P_{i j kl}$.

Luigi Bianchi (1856-1928)
Bianchi Identities

First Bianchi Identity for $R$

\[ R^i_{\ jkl} + R^i_{\ klj} + R^i_{\ ilj} = 0 \]

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Bianchi Identities

First Bianchi Identity for $R$

- $R_j^i{}_{kl} + R_k^i{}_{lj} + R_l^i{}_{jk} = 0$

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Bianchi Identities

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\[ B_{ijkl} := -A_{iju}R^{u}{}_{kl}, \quad R^{u}{}_{kl} = \frac{v^{j}}{F}R_{j}^{u}{}_{kl} \]

and

\[ R_{ijkl} = g_{j\mu}R_{i}^{\mu}{}_{kl} \]
Bianchi Identities

First Bianchi Identity for $R$

- $R^i_{j\, kl} + R^i_{k\, lj} + R^i_{l\, jk} = 0$

Other identities:

- $P^i_{k\, jl} = P^i_{j\, kl}$
- $R_{ijkl} + R_{jikl} = 2B_{ijkl}$, where
  \[ B_{ijkl} := -A_{iju}R^u_{\ kl}, \ R^u_{\ kl} = \frac{y^j}{F}R^u_{\ j\ kl} \]
  and
- $R_{ijkl} = g_{j\mu}R^\mu_{i\ kl}$
- $R_{klji} - R_{jikl} = (B_{klji} - B_{jikl}) + (B_{kilj} + B_{ljki}) + (B_{ilji} + B_{jkil})$
Bianchi Identities

First Bianchi Identity for $R$

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Second Bianchi identities: very complicated, mix terms in $R_{j}^{i}{}_{kl}$ and $P_{j}^{i}{}_{kl}$
We must fix a flagpole $y$ and then a transverse edge $V$.  

\[ K(y, V) := V_i R_{jikl} y^l V^k g(y, y) g(V, V) - g(y, V)^2 \]

We can change $V$ by $W = \alpha V + \beta y$, that is, $K(y, W) = K(y, V)$. We obtain the same quantity with the other connections (Cartan, Berwald, Hashiguchi...).
We must fix a **flagpole** \( y \) and then a **transverse edge** \( V \)

\[
K(y, V) := \frac{V^i (y^j R^l_{ijk} y^l) V^k}{g(y, y)g(V, V) - g(y, V)^2}
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*M. A. Javaloyes (*)*
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- We obtain the same quantity with the other connections (Cartan, Berwald, Hashiguchi).
Computing Flag curvature

\[ G_i := \gamma_{jk} y_j y_k \] (spray coefficients)

\[ 2 \mathcal{F} R_{ik} = 2(G_i) x_k - \frac{1}{2} (G_i) y_j (G_j) y_k - y_j (G_i) y_k x_j + G_j (G_i) y_k y_j \]

\[ K(y, V) = K(l, V) = V_i (R_{ik} V_k) g(V, V) - g(l, V) \frac{2}{2} , \]

where \( l = y / \mathcal{F} \).

If we consider \( \mathcal{F}(x, y) = \sqrt{\langle y, y \rangle} + df[y] \), with \( \langle \cdot, \cdot \rangle \) the Euclidean metric,

\[ G_i = \frac{1}{F f} x_j x_k y_j y_k \], very simple!!

\[ K(y, V) = K(x, y) = \frac{3}{4} F^4 (f x_i x_j y_i y_j)^2 - \frac{1}{2} F^3 (f x_i x_j x_k y_i y_j y_k) \]

the flag curvature does not depend on the transverse edge!! it is scalar.
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- \( G^i := \gamma^i_{jk} y^j y^k \) (spray coefficients)
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If we consider $F(x, y) = \sqrt{\langle y, y \rangle} + df[y]$, with $\langle \cdot, \cdot \rangle$ the Euclidean metric, then
Computing Flag curvature

- $G^i := \gamma^i_{jk} y^j y^k$ (spray coefficients)

- $2F^2 R^i_k = 2(G^i)_x - \frac{1}{2}(G^i)_{yj}(G^j)_{yk} - y^j(G^i)_{ykxj} + G^j(G^i)_{ykj}$

- $K(y, V) = K(l, V) = \frac{V_i(R^i_k)V^k}{g(V,V)-g(l,V)^2}$, where $l = y/F$.

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Finsler metric with constant flag curvature

The complete classification is an open problem, no Hopf’s theorem!!!

In the class of Randers metrics there does exist a classification after a long story.

In 1977 Yasuda and Shimada publish a paper with a characterization of Randers metrics of scalar flag curvature. As a particular case they obtain the Randers metrics of constant flag curvature.


In summer 2000, P. Antonelli asks if Yasuda-Shimada theorem is indeed correct.
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Still no classification (solutions \( \sqrt{h} + h(W, v) \)) must have a \( h \)-Riemannian curvature related with the module of a \( h \)-Killing field. Finally, they perceive that when considering Zermelo expression of Randers metrics, the geometry comes out.
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Finally they perceive that when considering Zermelo expression of Randers metrics the geometry comes out
Flag constant curvature and stationary spacetimes

Zermelo metric:
$$\sqrt{\alpha g(v,v)} + \alpha^2 g(W,W)^2 - \sqrt{\alpha g(W,W)},$$
where \( \alpha = 1 - g(W,W) \).

Randers space forms are those Zermelo metrics having \( h \) of constant curvature and \( W \) a conformal Killing field.

Katok metrics are Randers space forms.

When the Fermat metric associated to a stationary spacetime is of constant flag curvature, then the spacetime is locally conformally flat.

Reciprocal is not true ($\sqrt{h + df}$).

What about scalar flag curvature?
Zermelo metric:

\[
\sqrt{\frac{1}{\alpha} g(v, v) + \frac{1}{\alpha^2} g(W, v)^2 - \frac{1}{\alpha} g(W, v)},
\]

where \( \alpha = 1 - g(W, W) \).
Flag constant curvature and stationary spacetimes

- **Zermelo metric:**
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What about scalar flag curvature?
Schur’s Lemma

**Theorem**

Let $M$ be a Riemannian manifold with dimension $\geq 3$. If for every point $x \in M$ the sectional curvature does not depend on the plane, then $M$ has constant sectional curvature.

Issai Schur (1875-1941)
Let $M$ be a Riemannian manifold with dimension $\geq 3$. If for every point $x \in M$ the sectional curvature does not depend on the plane, then $M$ has constant sectional curvature.

- It was established by Issai Schur (1875-1941)
Schur’s Lemma

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Let $M$ be a Riemannian manifold with dimension $\geq 3$. If for every point $x \in M$ the sectional curvature does not depend on the plane, then $M$ has constant sectional curvature.

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Theorem

Suppose $M$ is a 2-dim compact Riemannian manifold with boundary $\partial M$. Then

$$\int_{M} K \, dA + \int_{\partial M} k_{g} \, ds = 2\pi \chi(M),$$
Gauss-Bonnet Theorem

**Theorem**

Suppose $M$ is a 2-dim compact Riemannian manifold with boundary $\partial M$. Then

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- **Gauss** knew a version but never published it
Gauss-Bonnet Theorem

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Pierre O. Bonnet (1819-1892)
Gauss-Bonnet Theorem

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- **Allendoerfer-Weil-Chern** generalized Gauss-Bonnet to even dimensions using the Pfaffian in the mid-40’s

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S. S. Chern (1911-2004)

C. Allendoerfer (1911-1974)

André Weil (1906-1998)

M. A. Javaloyes (*)

Flag Curvature
Gauss-Bonnet Theorem

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*André Lichnerowitz (1915-1998)*
Gauss-Bonnet Theorem

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- **Bao-Chern** (Ann. Math. 1996) extend it to a wider class of Finsler manifolds
Bonnet-Myers Theorem

Theorem

If Ricci curvature of a complete Riemannian manifold $M$ is bounded below by $(n - 1)k > 0$, then its diameter is at most $\pi/\sqrt{k}$ and the manifold is compact.
Bonnet-Myers Theorem

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- **Louis Auslander** extended the result to the Finsler setting in 1955 (Trans AMS).
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- Bao-Chern-Chen assume just forward completeness in their book “Introduction to Riemann-Finsler geometry”
**Bonnet-Myers Theorem**

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- Causality reveals that completeness can be substituted by the condition

  $$B^+(x, r) \cap B^-(x, r) \text{ compact for all } x \in M \text{ and } r > 0$$

  (see Caponio-M.A.J.-Sánchez, preprint 09)
Synge’s Theorem

**Theorem**

If $M$ is an even-dimensional, oriented, complete and connected manifold, with all the sectional curvatures bounded by some positive constant, then $M$ is simply connected.


Louis Auslander (1928-1997) extended the result for Finsler manifolds in 1955. Again the completeness condition can be weakened.
Synge’s Theorem

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[Image of John Synge (1897-1995)]
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Theorem

If $M$ is a geodesically complete connected Riemannian manifold of non positive sectional curvature. Then

- Geodesics do not have conjugate points
- $\exp_p : T_p M \to M$ is globally defined and a local diffeomorphism
- If $M$ simply connected, then $\exp_p$ is a diffeomorphism
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- Obtained for surfaces in 1898 by Hadamard
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Rauch’s Comparison Theorem

**Theorem**

*For large curvature, geodesics tend to converge, while for small (or negative) curvature, geodesics tend to spread.*
Rauch’s Comparison Theorem

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A. D. ALEKSANDROV (1912-1999)
Rauch’s Comparison Theorem

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- Proved in the 40’s by **A. D. Aleksandrov** for surfaces
- Generalized to Riemannian manifolds in 1951 by **H. E. Rauch**
- Probably **P. Dazord** was the first one in giving the generalized Rauch theorem in 1968
Theorem

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Sphere Theorem

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- To obtain Rademacher’s result it is enough symmetrized compact balls and bounded reversivity index
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Victor A. Toponogov (1930-2004)
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Victor A. Toponogov (1930-2004)