On the interplay between Lorentzian Causality and Finsler metrics of Randers type

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Universidad de Granada

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Interplay between Randers metrics and stationary spacetimes
Interplay between Randers metrics and stationary spacetimes

\( (\mathbb{R} \times S, l) \) is a standard stationary spacetime

\( L_1 = \text{Observer} \)

A lightlike geodesic \((t, x)\)

\( S \) is naturally endowed with a Randers metric \( F \) called the Fermat metric
Interplay between Randers metrics and stationary spacetimes

Causal properties of \((\mathbb{R} \times S, I)\)

Hopf-Rinow properties of \((S, F)\)
Interplay between Randers metrics and stationary spacetimes

Global hyperbolicity of \((\mathbb{R} \times S, l)\)

\[\bar{B}^+(p, r) \cap \bar{B}^-(p, r)\] compact
\[\forall p \in S \text{ and } \forall r > 0 \text{ in } (S, F)\]

\(L_1 = \text{Observer}\)

lightlike geodesic \((t, x)\)

Fermat geodesic \(x\)

\(\downarrow\)
Interplay between Randers metrics and stationary spacetimes

Cauchy horizons of a subset $A$ contained in a slice $\{t_0\} \times S$

are the graph of the distance function to the complementary $A^c$ in $(S, F)$
Interplay between Randers metrics and stationary spacetimes

Differential properties of the Cauchy horizons in $(\mathbb{R} \times S, I)$

Differential properties of the distance function to a subset in $(S, F)$
Program of the talk

Preliminaries:
- Causality (the causal ladder)
- Standard stationary spacetimes and Fermat metrics
- Randers and Finsler metrics

First application of the Interplay: Causal properties in terms of Hopf-Rinow properties of the Fermat metric

Second application: equivalence of differentiability of Cauchy horizons and the distance function to a subset.
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- Second application: equivalence of differentiability of Cauchy horizons and the distance function to a subset.
Causal properties classify spacetimes depending on the behaviour of causal cones. A spacetime is:

- **Chronological** if \( p \not\in I^+ (p) \) for every \( p \in M \).
- **Distinguishing** if \( I^+ (p) = I^+ (q) \) or \( I^- (p) = I^- (q) \) implies \( p = q \).
- **Causally continuous** if it is distinguishing and the Chronological cones \( I^\pm (p) \) are continuous in \( p \in M \).
- **Causally simple** if the causal cones \( J^\pm (p) \) are closed for every \( p \in M \).
- **Globally hyperbolic** if it admits a Cauchy hypersurface (a subset \( S \) that meets exactly once every inextendible timelike curve).

\[ \text{Globally hyperbolic} \quad \Downarrow \quad \text{Causally simple} \quad \Downarrow \quad \text{Causally continuous} \quad \Downarrow \quad \text{Stably causal} \quad \Downarrow \quad \text{Strongly causal} \quad \Downarrow \quad \text{Distinguishing} \quad \Downarrow \quad \text{Causal} \quad \Downarrow \quad \text{Chronological} \quad \Downarrow \quad \text{Non-totally vicious} \]
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A spacetime is Stationary if it admits a timelike Killing field. Standard Stationary means that $M = \mathbb{R} \times S$ and 
\[
g((\tau, y), (\tau, y)) = g_0(y, y) + 2g_0(\delta, y)\tau - \beta(x)\tau^2,
\] 
where $(S, g_0)$ is Riemannian and $\beta(x) > 0$.

How restrictive is to consider standard stationary spacetimes rather than stationary?


E. Caponio, M. A. Javaloyes, M. Sánchez (*) Interplay between Lorentzian and Randers metrics 5 / 26
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Causal condition to have a standard splitting

**Theorem (M. A. J.- M. Sánchez)**

*If a stationary spacetime $L$ is distinguishing and the timelike Killing field is complete, then it is causally continuous and standard.*

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- A result of S. Harris $\Rightarrow L = \mathbb{R} \times Q$ (maybe $\{t_0\} \times Q$ is never spacelike).
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- Reflecting + Distinguishing $\Leftrightarrow$ Causally continuous
- Causally continuous $\Rightarrow$ Stably causal
- $\Rightarrow$ there exists a temporal function $t : L \to \mathbb{R}$
- $t^{-1}(0)$ is a section (it crosses all the orbits of the timelike Killing field)

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Relativistic Fermat Principle: lightlike pregeodesics are critical points of the arrival time function corresponding to an observer in a suitable class of lightlike curves. If you consider as observer $s \rightarrow L_1(s) = (s, x_1)$ in $(\mathbb{R} \times S, g)$, given a lightlike curve $\gamma = (t, x)$, the arrival time $\text{AT}(\gamma)$ is

$$t(b) = t(a) + R_{ba}^1 \beta g_0(\dot{x}, \delta) + q_1 \beta g_0(\dot{x}, \dot{x}) + \frac{1}{\beta^2} g_0(\dot{x}, \delta)^2 ds.$$ 

because

$$g_0(\dot{x}, \dot{x}) + 2 g_0(\delta(\dot{x}), \dot{x}) \dot{t} - \beta(x) \dot{t}^2 = 0 \quad (g(\dot{\gamma}, \dot{\gamma}) = 0).$$

Let us define the Fermat (Finslerian) metric in $S$ as

$$F(x, v) = \frac{1}{\beta} g_0(v, \delta) + q_1 \beta g_0(v, v) + \frac{1}{\beta^2} g_0(v, \delta)^2.$$
**Relativistic Fermat Principle:** Lightlike pregeodesics are critical points of the arrival time function corresponding to an *observer* in a suitable class of lightlike curves.
**Fermat principle in standard stationary spacetimes**

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Pierre de Fermat (1601-1665)
Theorem

A curve $s \rightarrow \gamma(s) = (s, x(s))$ is a lightlike pregeodesic of $(\mathbb{R} \times S, g)$ iff $s \rightarrow x(s)$ is a Fermat geodesic with unit speed.

Consequences:

Gravitational lensing can be studied from geodesic connectedness in Fermat metric

Existence of $t$-periodic lightlike geodesics is equivalent to existence of Fermat closed geodesics (Biliotti-M.A.J. to appear in Houston J. Math.)
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Randers metrics

$R: TM \to \mathbb{R}$ defined as:

$$R(x, v) = \sqrt{h(v, v) + \omega_x[v]}$$

where $h$ is Riemannian and $\omega$ a 1-form with $\|\omega_x\|_h < 1$ $\forall x \in M$, are basic examples of non-reversible Finsler metrics:

$R(x, -v) \neq R(x, v)$.

Named after the Norwegian physicist Gunnar Randers (1914-1992):

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Finsler metrics

Main reference:


DEFINITION: $F : TM \rightarrow [0, +\infty)$ continuous and

Positive homogeneous of degree one:

$F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$.

Fiberwise strictly convex square:

$g_{ij}(x, y) = \frac{1}{2} \partial^2 \left( F^2 \right) \partial y_i \partial y_j(x, y)$ is positively defined.

It can be showed that this implies:

$F$ is positive in $TM \setminus \{0\}$.

Triangle inequality holds in the fibers.

$F^2$ is $C^1$ on $TM$.
Finsler metrics

Main reference:


DEFINITION: $F : TM \to [0, +\infty)$ continuous and $C^\infty$ in $TM \setminus \{0\}$.
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**DEFINITION:** \( F : TM \rightarrow [0, +\infty) \) continuous and

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Paul Finsler (1894-1970)
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Paul Finsler (1894-1970)
Non-symmetric “distance”

We can define the length of a curve:

$$L(\gamma) = \int_{a}^{b} F(\gamma, \dot{\gamma}) \, ds$$

and then the distance between two points:

$$\text{dist}(p, q) = \inf_{\gamma \in C_{\infty}(p, q)} L(\gamma)$$

$\text{dist}$ is non-symmetric because $F$ is non-reversible.

The length of a curve $t \rightarrow \gamma(t)$ is different from the length of its reverse $t \rightarrow \gamma(t)$!!

We have to distinguish between forward and backward balls.
Non-symmetric “distance”

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We have to distinguish between forward and backward:
- balls
- Cauchy sequence
- topological completeness
- geodesical completeness
Causality through the Fermat metric

Let $d$ be the non-symmetric distance in $S$ associated to the Fermat metric

$$B^+ (x_0, s) = \{ p \in S : d(x_0, p) < s \}$$

forward balls

$$B^- (x_0, s) = \{ p \in S : d(p, x_0) < s \}$$

backward balls

Define the symmetrized distance

$$d_s (p, q) = \frac{1}{2} (d(p, q) + d(q, p))$$

and

$$B_s (x, r) = \{ p \in S : d_s (x, p) < r \}$$

Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime. Then

$$I^\pm (t_0, x_0) = \bigcup_{s > 0} \{ t_0 \pm s \} \times B^\pm (x_0, s)$$
Let $d$ be the non-symmetric distance in $S$ associated to the Fermat metric.
Causality through the Fermat metric

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Causality through the Fermat metric

Theorem

- Let $(\mathbb{R} \times S, g)$ be a standard stationary spacetime.
- Then $(\mathbb{R} \times S, g)$ is causally continuous and
  - (a) $(\mathbb{R} \times S, g)$ is causally simple iff the associated Finsler manifold $(S, F)$ is convex,
  - (b) it is globally hyperbolic if and only if $\bar{B}^+ \cap \bar{B}^- \cap (x, r) \cap (x, r')$ is compact for every $x \in S$ and $r > 0$,
  - (c) a slice $\{t_0\} \times S$, $t_0 \in \mathbb{R}$, is a Cauchy hypersurface if and only if the Fermat metric $F$ on $S$ is forward and backward complete.

**Diagram:**

Globally hyperbolic $\Downarrow$ Causally simple $\Downarrow$ Causally continuous $\Downarrow$ Stably causal $\Downarrow$ Strongly causal $\Downarrow$ Distinguishing $\Downarrow$ Causal $\Downarrow$ Chronological $\Downarrow$ Non-totally vicious
Causality through the Fermat metric

**Theorem**

Let \((\mathbb{R} \times S, g)\) be a standard stationary spacetime. Then \((\mathbb{R} \times S, g)\) is *causally continuous* and

- **Globally hyperbolic**
  - \(\Downarrow\)
  - **Causally simple**
    - \(\Downarrow\)
    - **Causally continuous**
      - \(\Downarrow\)
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        - \(\Downarrow\)
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E. Caponio, M. A. Javaloyes, M. Sánchez (*) Interplay between Lorentzian and Randers metrics
Causality through the Fermat metric

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Randers metrics with the same geodesics

Let $R$ and $R'$ be Randers metrics. They are associated to the same stationary spacetime if and only if

$$R' = R + df.$$  

Moreover, if $R \times S$ is the splitting associated to $R$, the splitting associated to $R'$ is $R \times S_f$, where

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Generalized Hopf-Rinow theorem

Theorem (Accurate Hopf-Rinow for Randers metrics)

Let $(S, R)$ a Randers manifold and given a function $f: S \rightarrow \mathbb{R}$ define $R_f(x, v) = R(x, v) - df_x(v)$. The following conditions are equivalent:

(A) the intersection $\overline{B} + (x, r) \cap \overline{B} - (x, r)$ of $(S, R)$ is compact for every $r > 0$ and $x \in S$.

(B) the symmetrized closed balls $\overline{B}_s(x, r)$ of $(S, R)$ are compact for every $r > 0$ and $x \in S$.

(C) there exists $f$ such that $R_f$ is geodesically complete.

(D) there exists $f$ and $p \in S$ such that the forward and the backward exponentials of $R_f$ are defined in $T_p S$.

(E) there exists $f$ such that the quasi-metric $d_f$ associated to $R_f$ is forward and backward complete.

In such a case, $(S, R)$ is convex.
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3. There exists \(f\) such that \(R_f\) is geodesically complete.
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E. Caponio, M. A. Javaloyes, M. Sánchez (*) Interplay between Lorentzian and Randers metrics
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In such a case, \((S, R)\) is convex.
In fact, condition (A) generalizes forward and backward completeness for any Finsler metric and it is enough to prove Palais-Smale condition of the energy functional \( (A) \Rightarrow \text{Convexity} \) holds for any Finsler metric. Morse theory can be developed assuming condition (A).

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In fact, condition $(A)$ generalizes forward and backward completeness for any Finsler metric and it is enough to prove Palais-Smale condition of the energy functional. "$(A) \implies \text{Convexity}" holds for any Finsler metric. Morse theory can be developed assuming condition $(A)$.

Convexity of Finsler metrics

- In fact, condition (A) generalizes forward and backward completeness for any Finsler metric and it is enough to prove Palais-Smale condition of the energy functional.
- "(A) \Rightarrow Convexity" holds for any Finsler metric.
- Morse theory can be developed assuming condition (A).


- As an application we obtain Morse theory for lightlike geodesics and timelike geodesics with fixed proper time from a point to a vertical line.
Cauchy developments and Cauchy horizons

A subset $A$ of a spacetime $M$ is achronal if no $x, y \in A$ satisfy $x \ll y$. The future (resp. past) Cauchy development of $A$ is $D_{+/-}(A) = \{ p \in M : \text{every past (resp. future) inextendible causal curve through } p \text{ meets } A \}$. The future (resp. past) Cauchy horizon is $H_{+/-}(A) = \{ p \in D_{+/-}(A) : I_{+/-}(p) \text{ does not meet } D_{+/-}(A) \}$.
A subset $A$ of a spacetime $M$ is achronal if no $x, y \in A$ satisfy $x \ll y$. 

The future (resp. past) Cauchy development of $A$ is $D^\pm(A) = \{ p \in M :$ every past (resp. future) inextendible causal curve through $p$ meets $A \}$. The future (resp. past) Cauchy horizon is $H^\pm(A) = \{ p \in D^\pm(A) : I^\pm(p)$ does not meet $D^\pm(A) \}$.
A subset $A$ of a spacetime $M$ is **achronal** if no $x, y \in A$ satisfy $x \ll y$.

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$D^+(A)$ is the red region
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The future (resp. past) Cauchy horizon is

$$H^\pm(A) = \{ p \in D^\pm(A) : l^\pm(p) \text{ does not meet } D^\pm(A) \}$$
Cauchy developments and distance function to a subset

Theorem

Let \((\mathbb{R} \times S, g)\) be a standard stationary spacetime such that \(\{t_0\} \times S\) is Cauchy, and \(A_{t_0} = \{t_0\} \times A\). Then

\[D^+ (A_{t_0}) = \{(t, y) : d(x, y) > t - t_0 \quad \forall x \notin A \text{ and } t \geq t_0}\],

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The function \( \partial \Omega \ni y \rightarrow \min(N, \ell(y)) \in \mathbb{R}^+ \) is Lipschitz-continuous on any compact subset. As a consequence \( h^{n-1}(\Sigma \cap B) < +\infty \), being \( B \) bounded.

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YanYan Li and Louis Nirenberg

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Cut loci of Randers metrics

A minimizing segment is a unit speed geodesic such that 

\[ \rho_C(\gamma(s)) = s \]

Cut \( C \) is the cut locus, the points \( x \in S \setminus C \) where the minimizing segment do not minimize anymore.

This function is studied when \( C \) is a \( C^2 \), 1 loc boundary in:


E. Caponio, M. A. Javaloyes, M. Sánchez (*) Interplay between Lorentzian and Randers metrics
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Cauchy horizons

Construct a standard stationary spacetime with \( \tilde{R} \) (the reverse metric of \( R \)) as Fermat metric.

If \( \tilde{R} = \sqrt{h + \omega} \Rightarrow g_0(v, w) = h(v, w) - \omega(v)\omega(w) \), \( \beta(x) = 1 \),

\( H = \{ (-\rho C(x), x) : x \in S \setminus C \} \) is a future horizon, that is, an achronal, closed, future null geodesically ruled topological hypersurface.

There are several results for the differentiability of future horizons:


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**J. K. Beem and A. Królak, Cauchy horizon end points and differentiability,**

**P. T. Chruściel, J. H. G. Fu, G. J. Galloway, and R. Howard, On fine differentiability properties of horizons and applications to Riemannian geometry,**
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$\rho_C$ is differentiable at $p \in S \setminus C$ iff it is crossed by exactly one minimizing segment.

**Corollary**

*The n-dimensional Haussdorf measure of $\text{Cut}_C$ is zero.*
Open problems

(1) Is there any relation between the flag curvature of the Fermat metric and the Weyl tensor of the spacetime?

(2) In the paper G. W. Gibbons, C. A. R. Herdeiro, C. M. Warnick, M. C. Werner, Stationary Metrics and Optical Zermelo-Randers-Finsler Geometry, Phys.Rev.D79: 044022, 2009, the authors show that Fermat metrics with constant flag curvature correspond with locally conformally flat stationary spacetimes, but the converse is not true.

(3) Which is the condition in the Fermat metric that characterizes conformally flatness for the stationary spacetime?

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More information in:


Further Bibliography about Fermat metrics


E. Caponio, M. A. Javaloyes, M. Sánchez (*), Interplay between Lorentzian and Randers metrics.
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