ON TOPOLOGY AND RENORMING OF A BANACH SPACE

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Introduction. The classification of Banach spaces atending to their weak topologies has been considered by several authors. The class of Banach spaces which are Cech-analytic when endowed with their weak topology includes WCG Banach spaces and Banach spaces having an equivalent Kadec norm, see definitions in [2]. Jayne, Namioka and Rogers [7] have shown that every Cech-analytic Banach space is σ-fragmentable. However all the known examples of σ-fragmentable Banach spaces satisfies a stronger property, namely the weak topology has $\|.\|$-SLD, which is defined as follows in a wider contex.

Definition 1. *Let (X, τ) be a topological space and let d be a metric on X. It is said that X has a countable cover by set of small local diameter (SLD) if for every $\varepsilon > 0$ there exists a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively τ-neighbourhood of d-diameter less than ε.

The following definition was introduced by Arkangel'skii in [1].

Definition 2. *Let (X, τ) be a topological space. A family Σ of subsets of X is said to be a network for τ if every open set is a union of sets from Σ.*

In the following $(X, \|.\|)$ denote a Banach space and X^* its dual. A subset of the dual unit ball B_{X^*} is said norming (resp. quasi-norming) if its w^*-closed convex envelope is B_{X^*} (resp. contains an open ball centered at 0). A linear subspace $Z \subset X^*$ is said norming (resp. quasi-norming) if $Z \cap B_{X^*}$ is a norming (resp. quasi-norming) set. We shall denote by $\sigma(X, Z)$ the topology on X of pointwise convergence on Z, but in the particular cases of the weak and the weak* topologies we shall use w and w^* respectively. For a norm $\|.\|$ it is equivalent to to be $\sigma(X, Z)$-lower semicontinuous ($\sigma(X, Z)$-lsc for short) and to have its unit ball $\sigma(X, Z)$-closed.
In this paper we shall relate some almost topological properties of Banach spaces with the existence of equivalent norms of the following types:

Definition 3. *Let X be a Banach space endowed with a norm $\|.\|$ and let S_X be the unit sphere. Then the norm is said to be*

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a) locally uniformly rotund (LUR), if for every $x, x_k \in X$*, with* $\|x\| = \|x_k\| = 1$*, and such that* $\lim_k \|x + x_k\| = 2$*, then* $\lim_k \|x - x_k\| = 0$.

b) rotund if for every $x, y \in X$*, with* $\|x\| = \|y\| = 1$ *and* $\|x+y\| = 2$ *then* $x = y$.

c) $\sigma(X, Z)$*-Kadec if* $\sigma(X, Z)$ *and the norm topologies coincide on* S_X*. If* $Z = X^*$ *we say simply that* $\|.\|$ *is Kadec.*

Let X be a Banach space. We call an open affine half space defined by an element $x^* \in X^*$ a set of the form $\{x \in X : \ x^*(x) < \alpha\}$ where $\alpha \in \mathbb{R}$. A nonempty intersection of a set A with some open half space is called a slice of A. If $Z \subset X^*$ is a linear subspace of the dual, we denote by $\mathbb{H}(Z)$ the set of the open affine half spaces defined by elements of Z.

Main Results. The purpose of this note is to announce some results concerning to Kadec and LUR renorming of a Banach space.
The first theorem tell us that the property SLD considered in [7] in the contex of a Banach space with its weak topology is almost equivalent to the existence of a Kadec renorming.

Theorem 4. *Let* X *be a Banach space and* $Z \subset X^*$ *a quasi-norming linear subspace. The following are equivalent:*

i) $(X, \sigma(X, Z))$ *has* $\|.\|$*-SLD.*

ii) There is a sequence (A_n) *of subsets of* X *such that* $\{A_n \cap \sigma(X, Z)\}$ *(i.e.* $\{A_n \cap U : \ n \in \mathbb{N}, \ U \in \sigma(X, Z)\}$*) is a network for the norm topology.*

iii) For every $c > 1$ *there is a non negative symetric homogeneous* τ*-lower semicontinuous function* F *on* X *with* $\|x\| \leq F(x) \leq c\|x\|$ *such that the norm topology and* τ *coincide on the set* $S = \{x \in X : \ F(x) = 1\}$

We can make the function F apearing in *iii*) of Theorem 4 to be a norm assuming the convexity of the sets (A_n) of statement *ii*), obtaining the following.

Theorem 5 *Let* X *be a Banach space and* $Z \subset X^*$ *a quasi-norming linear subspace. The following are equivalent:*

i) X *admits a* $\sigma(X, Z)$*-Kadec norm.*

ii) There is a sequence (A_n) *of convex subsets of* X *such that* $\{A_n \cap \sigma(X, Z)\}$ *is a network for the norm topology.*

Concerning to the existence of $\sigma(X, Z)$-lsc LUR norms we have the following theorem which extends previous results by Troyanski [2, pg. 148] and Molto, Orihuela and Troyanski [8].

Theorem 6 *Let* X *be a Banach space and* $Z \subset X^*$ *a quasi-norming linear subspace. The following are equivalent:*

i) *X admits a $\sigma(X,Z)$-lsc LUR norm.*

ii) *X admits both a rotund norm and a $\sigma(X,Z)$-Kadec norm.*

iii) *There is a sequence (A_n) of subsets of X such that $\{A_n \cap \mathbb{H}(Z)\}$ is a network for the norm topology.*

It should be noted that it is enough to check statement *iii)* above on the sphere S_X with the relative norm topology.
Applying Theorem 6 together with techniques from [8] we obtain the following result that answers a question of Haydon [5].

Corollary 7 *Let (K_n) be a sequence of closed subsets of a compact space K such that $K = \bigcup_{n=1}^{\infty} K_n$. Assume that $C(K_n)$ has an equivalent pointwise-lsc LUR norm for every $n \in \mathbb{N}$. Then $C(K)$ has an equivalent pointwise lsc LUR norm .*

A version of the preceding corollary without asking the LUR norms to be pointwise-lsc appears in [8].
A Banach space X is said weakly countably determined (WCD) if there exists a sequence (K_n) of w^*-compact sets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. A classic result of Vašak [9] shows that a WCD Banach space admits a LUR norm. It is possible to obtain from Theorem 6 the following.

Corollary 8 *Let X be a WCD Banach space, $Z \subset X^*$ a quasi-norming linear subspace. Then X admits an equivalent $\sigma(X,Z)$-lsc LUR norm.*

When X^* is a WCD dual space, we deduce the existence of a dual LUR norm. That result was obtained by M. Fabian in [3].
A dual Banach space with the Radon-Nikodym property (RNP) always admits a LUR norm by [4], but this norm is not necessary a dual norm. The following theorem gives some conditions equivalent to the existence of dual LUR norms.

Theorem 9 *Let X^* be a dual space. The following are equivalent:*

i) *X^* admits a dual LUR norm.*

ii) *X^* admits a w^*-Kadec norm.*

iii) *There is a sequence (A_n) of subsets of X^* such that $\{A_n \cap \mathbb{H}(X)\}$ is a network for the norm topology.*

iv) *(X^*, w^*) has $\|.\|$-SLD and X^* admits a dual rotund norm.*

Note that while the fact that the dual norm is w^*-Kadec implies dual-LUR renormability, Haydon gave an example of a Banach space having a Kadec norm but with no equivalent LUR norm neither rotund norm [2, pg. 325]. Haydon [6] also has built a Banach space X such that $(B_{X^{**}}, w^*)$ is a Corson compact (in particular X^* has RNP) and X^* admits no dual LUR norm. We deduce from Theorem 9 we have that this space X^* admits no w^*-Kadec norm.

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