Kadec norms and Borel sets in a Banach space

by
M. Raja*

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Abstract

We introduce a property for a couple of topologies that allows us to give simple proofs of some classic results about Borel sets in Banach spaces by Edgar, Schachermayer and Talagrand as well as some new results. We characterize the existence of Kadec type renormings in the spirit of the new results for LUR spaces by Moltó, Orihuela and Troyanski.

1 Introduction

Throughout this paper $(X, \|.\|)$ will denote a Banach space, X^* its dual, w and w^* the topologies weak and weak* respectively. B_X (resp. B_{X^*}) denotes the unit ball of X (resp. X^*). S_X will be the unit sphere of X. We shall also consider topologies on X of convergence on some subsets of the dual space. A subset of B_{X^*} is said to be norming (resp. quasi-norming) if its w^*-closed convex envelope is B_{X^*} (resp. contains an open ball centered at the origin).
A norm $\|.\|$ on X is said to have the Kadec property when the weak and

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the norm topologies coincide on its unit sphere. A norm is said to be locally uniformly rotund (LUR) if for every sequence (x_n) in the unit sphere and for every point x also in the unit sphere such that $\lim_n \|x_n + x\| = 2$ then (x_n) converges to x in norm. LUR norms have the Kadec property. For the proof of this fact and other properties of Banach spaces having an equivalent LUR norm we refer to the book [4]. There exists Banach spaces having a Kadec norm and admiting no equivalent LUR norm [11].

Edgar proved [5] that in a Banach space which admits an equivalent Kadec norm the Borel σ-algebras generated by the weak and the norm topologies coincide. He also noted that an analogous result also holds when the Kadec property happens for the weak* topology. Schachermayer [6] proved that a Banach space X that have an equivalent Kadec norm is a Borel set in (X^{**}, w^*). Talagrand, [26], showed that the two previous results are not true for general Banach spaces, but he proved, [25], that for subspaces of weakly compact generated spaces the Borel sets for the topology of pointwise convergence on a quasi-norming subset of the dual space and the norm Borel sets are the same.

Jayne, Namioka and Rogers [15] introduced the notion of countable cover by sets of local small d-diameter (SLD), see Definition 2, for a topological space with respect to some metric d and they noted that if a Banach space X has an equivalent Kadec norm then (X, w) has SLD with respect to the norm which implies the coincidence of Borel sets for the norm and the weak topology. In fact, property SLD implies coincidence of Borel sets for the original topology and the metric in a wider topological context. Oncina [22] has done a deep study of property SLD showing that a Banach space with SLD for the weak topology respect to the norm is a Borel set in its bidual. Another aproach to the coincidence of Borel sets and related properties has been given by Hansell in his unpublished preprint [10] using the notion of descriptive topological space. In the context of a Banach space endowed with its weak topology, Hansell's notion of descriptive space is equivalent to the property SLD, as pointed out by Moltó, Orihuela, Troyanski and Valdivia, [20].

Recently Moltó, Orihuela and Troyanski, [19], have characterized the Banach spaces which does admits an equivalent LUR norm in a Banach space as those spaces X such that (X, w) has special case of norm SLD: X has an equivalent LUR norm if and only if (X, w) satisfies Definition 2 and the weak neighbourhood there is a slice (the intersection with an open half space). See also the comments after Theorem 2.

Our aim in this paper is to show that all the above mentioned positive results on coincidence for Borel σ-algebras and the Borel nature of a Banach space in its bidual lie in a common topological principle which can to be used to characterize the existence of Kadec type norms when applied to a Banach space.
In section 2 we introduce a useful condition (Definition 1) for a couple of topologies that gives a natural aproach to the study of Borel sets (Proposition 3). When one of the topologies is given by a metric, our property is equivalent to property SLD (Definition 2, Proposition 2).
In section 3 we use the framework of topological vector spaces to study the relation between the property SLD and the existence of Kadec type equivalent norms. In particular we show that if X is a Banach space such that (X, w) has SLD then the weak and the norm topologies coincide on the level sets of some positive homogeneous function (Theorem 1). We also characterize the existence of an equivalent Kadec norms (Theorem 2) in the spirit of the recents results on LUR norms by Moltó, Orihuela and Troyanski [19].
In section 4 we apply the previous results to WCD Banach spaces taking advantage of the existence of a LUR norm to build Kadec norms for topologies weaker than the weak topology (Theorem 3) and to show the coincidence of Borel sets improving a result by Talagrand. As an application to non metric topologies we finish showing that if K is a Radon-Nikodym compact then $C(K)$ has a equivalent norm such that the weak and the pointwise topologies coincide on its unit sphere (Theorem 4).

Part of the results of this paper has been anounced in [23].

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2 Topological results

We begin with the main definition in this paper. Actually the idea is implicit in [25]. We recall that a network for some topology is a family of sets not necessary open such that every open set can be written as a union of sets in the family.

Definition 1 *Let X be a set, τ_1 and τ_2 two topologies on X. A subset*

$A \subset X$ is said to have the property $P(\tau_1, \tau_2)$ if there exists a sequence (A_n) of subsets of X such that the family of sets $(A_n \cap U)$ where $n \in \mathbb{N}$ and $U \in \tau_2$ is a network for τ_1, that is, for every $x \in A$ and every $V \in \tau_1$ with $x \in V$ there exists $n \in \mathbb{N}$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$.

Evidently, if $\tau_1 \subset \tau_2$ then X has $P(\tau_1, \tau_2)$, but this case is not interesting. The relevant case happens when $\tau_2 \subset \tau_1$, for instance, in applications to Banach spaces τ_1 and τ_2 will be the norm and the weak topology respectively. If τ_1 has a countable basis (V_n) then X has $P(\tau_1, \tau_2)$ for whatever topology τ_2, because we can take $A_n = V_n$. This happens in particular when (X, τ_1) is metrizable and separable. In fact, we shall use the property introduced in Definition 1 to extend results valid for separable spaces to nonseparable spaces.
Observe that if we take the sequence $(A_n \cap A)$ we can always suppose that $A_n \subset A$. That means that property $P(\tau_1, \tau_2)$ only depends on A equiped with the relative topologies.
To check $P(\tau_1, \tau_2)$ for a given A it is enough to verify the above set inclusion for all the V's belonging to a sub-basis of τ_1, because then A will have $P(\tau_1, \tau_2)$ with the countable family of the finite intersections of sets of the sequence (A_n).

The following proposition contains some other elementary consequences of the Definition 1.

Proposition 1 *Let X be a set, τ_1, τ_2 and τ_3 topologies on X and A a subset of X. Then*

i) If A has $P(\tau_1, \tau_2)$ and $B \subset A$ then B has $P(\tau_1, \tau_2)$.

ii) If A has $P(\tau_1, \tau_2)$ and $P(\tau_2, \tau_3)$ then A has $P(\tau_1, \tau_3)$.

iii) If every point of A has a τ_1-basis of neighbourhoods which is made up of τ_2-closed sets then the sequence (A_n) in Definition 1 can be taken of τ_2-closed sets.

iv) If every set A_n of Definition 1 is τ_2-Borel then for every $V \in \tau_1$ such that $A \subset V$, there is a τ_2-Borel set B satisfying $A \subset B \subset V$. In particular, if A is τ_1-open, or more generally, if A is a G_δ-set for the τ_1-topology, then A is τ_2-Borel.

Proof. i) The same sequence (A_n) satisfies Definition 1.
ii) If (B_m) is a sequence for $P(\tau_2, \tau_3)$ it is easy to check that $(A_n \cap B_m)$ satisfies the condition in Definition 1 for $P(\tau_1, \tau_3)$.
iii) Fix $x \in A$. Take $V \in \tau_1$ with $x \in V$. Take $V_0 \in \tau_1$ such that $x \in V_0$ and $\overline{V_0}^{\tau_2} \subset V$. There exists A_n and $U \in \tau_2$ such that $x \in A_n \cap U \subset V_0$. Thus

$$x \in \overline{A_n}^{\tau_2} \cap U \subset \overline{A_n \cap U}^{\tau_2} \subset \overline{V_0}^{\tau_2} \subset V$$

iv) For every $x \in A$ there exists $n_x \in \mathbb{N}$ and $U_x \in \tau_2$ such that $x \in A_{n_x} \cap U_x \subset V$. Now we have that

$$A = \bigcup_{x \in A} \{x\} \subset \bigcup_{x \in A} A_{n_x} \cap U_x = \bigcup_{n=1}^{\infty} (A_n \cap \bigcup_{n_x = n} U_x) = B \subset V$$

where B is clearly in $Borel(X, \tau_2)$.
If $A = \bigcap_{n=1}^{\infty} V_n$ where $V_n \in \tau_1$ we can take τ_2-Borel sets (B_n) such that $A \subset B_n \subset V_n$. Then $A = \bigcap_{n=1}^{\infty} B_n$.

A particularly interesting case occurs when τ_1 is metrizable. In this case the property introduced in Definition 1 agrees with the following one given by Jayne, Namioka and Rogers in [15] which is a special case of their σ-fragmentability.

Definition 2 *Let (X, τ) be a topological space and let d be a metric on X. It is said that X has a countable cover by set of small local diameter (SLD) if for every $\varepsilon > 0$ there exists a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively non empty τ-neighbourhood of diameter less than ε.

A Banach space X is said to have countable Szlenk index if for every $\varepsilon > 0$, there is a decreasing transfinite countable sequence of subsets (C_α) such that $B_X = \bigcup_\alpha (C_\alpha \setminus C_{\alpha+1})$ and every point of $C_\alpha \setminus C_{\alpha+1}$ has a relative weak neighbourhood in C_α of diameter less than ε. These spaces have been considered by Lancien in [18]. Clearly, if X has countable Szlenk index, then (X, w) has $\|.\|$-SLD. However, a separable Banach space X without the Point of Continuity Property does not have countable Szlenk index but (X, w) has $\|.\|$-SLD.

Proposition 2 *Let (X,τ) be a topological space and d a metric on X. Then X has countable cover by sets of small local diameter if and only if X has $P(d,\tau)$. Moreover, if the closed d-balls are τ-closed then the sets X_n^ε in Definition 2 can be taken as differences of τ-closed sets.*

Proof. If X_n^ε are the sets of Definition 2 it is easy to check that the sets (A_n) obtained arranging into a sequence by a diagonal proccess $(X_n^{\frac{1}{m}})_{n,m}$ satisfy Definition 1.
For the other implication, given $\varepsilon > 0$ just define

$$X_n^\varepsilon = \{x \in A_n : \exists U \in \tau,\ x \in U,\ diam(A_n \cap U) < \varepsilon\}$$

The moreover part is consequence of point *iii)* of Proposition 1.

The following result shows the good Borel behaviour of a topological space (X,τ) that has $P(d,\tau)$ for some adecuate metric d. The statement *a)* has been already noted by Jayne, Namioka and Rogers in [15] and [17], in terms of the property SLD.

Proposition 3 *Let (Y,τ) be a topological space and d a metric on Y stronger than τ and such that the closed d-balls are τ-closed. Let X be a subset of Y having $P(d,\tau)$.*

a) Considering X with the inherited topologies then

$$Borel(X,\tau) = Borel(X,d)$$

b) If X is d-closed in Y then $X \in Borel(Y,\tau)$.

Proof. a) Evidently every τ-Borel set is a d-Borel set. Conversely, if $V \subset X$ is a d-open set then it has $P(d,\tau)$. As the closed d-balls are τ-closed we can apply *iii)* and *iv)* of Proposition 1 to get that V is τ-Borel.
b) Being X a G_δ-set in (Y,d), the result follows from *iii)* and *iv)* of Proposition 1.

The next corollary contains the applications of property SLD to Banach spaces by Jayne, Namioka and Rogers [15], Oncina [22] and Hansell [10] (this last using the notion of descriptive space) that improve preceding ones by Edgar [5] and Schachermayer [6] on Banach spaces admiting

Kadec norms. We shall prove later that Banach spaces having $P(\|.\|,\tau)$ are not very different from the Banach spaces that admits an equivalent Kadec norm (Theorem 1).

Corollary 1 *Let X be a Banach space and τ a vector topology weaker than the norm and such that $\overline{B_X}^{\tau}$ is bounded.*

a) If X has $P(\|.\|,\tau)$, then $Borel(X,\|.\|) = Borel(X,\tau)$.

b) If X has $P(\|.\|,w)$, then $X \in Borel(X^{**},w^*)$.

Proof. Note that $\overline{B_X}^{\tau}$ is the unit ball of an equivalent norm on X whose closed balls are τ-closed. Then apply proposition 3.

Let us remark that $\overline{B_X}^{\tau}$ is bounded, for instance, when τ is a topology of convergence on a norming or a quasi-norming subset of X^*.

As a consequence of Proposition 3 we give an application to descriptive topology. Following Fremlin, see [16], a completely regular topological space X is Čech-analytic if for every finite sequence of positive integers s there is a set $A(s)$ open or closed in the Čech-Stone compactification of X such that

$$X = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} A(\sigma|n)$$

where $\sigma|n$ denotes the finite sequence made up from the first n terms of the sequence σ. The notion of Čech-analytic space has some interest in the context of nonseparable and nonmetrizable topological spaces (e.g. a Banach space endowed with its weak topology), where the classic descriptive set theory is not applicable in general. We address the interested reader to [16] and [10] for more information about Čech-analytic spaces and its applications to Banach spaces.

Corollary 2 *Let (X,τ) be a topological space. Suppose that there is a set T such that X can be identified as a subspace of $\mathbb{R}^T$ with the pointwise topology which is made up of bounded functions and that is complete for the metric d on X of uniform convergence on T. If X has $P(d,\tau)$, then X is a Borel subset of $\mathbb{R}^T$, in fact a pointwise $(F\cap G)_{\sigma\delta}$, and (X,τ) is Čech-analytic.*

Proof. We can assume that d is defined on $\mathbb{R}^T$ and it is stronger than the pointwise topology with pointwise d-closed balls. As X is complet for d, then it is d-closed in $\mathbb{R}^T$ and we finish applying the proofs of Propositions 1 and 3.
Acording to [16] a suficient condition for (X, τ) to be Čech-analytic is being homeomorphic to a Borel subset of some compact space. The reasoning above shows that $X \cap [-n, n]^T$ is Borel in $[-n, n]^T$, so it is Borel in $\overline{\mathbb{R}}^T$ where $\overline{\mathbb{R}}$ is the two points compactification of $\mathbb{R}$. Now, as $X = \cup_{n=1}^{\infty} X \cap [-n, n]^T$ it is a Borel set in the compact $\overline{\mathbb{R}}^T$.

Hansell proves in [10] that a descriptive topological space is always Čech-analytic, in particular, every Banach space X such that (X, w) is $\|.\|$-SLD is Čech-analytic, see [20]. Corollay 2 contains more information about the structure of X in that particular case.
Under the hypothesis of Corollary 2, it is easy to show that every d-Borel subset of X is pointwise Borel in $\mathbb{R}^T$ and analogously Čech-analitic.

3 Kadec norms

It is convenient for our purposes to give a more general definition of Kadec norm involving topologies different from the weak topology.

Definition 3 *Let X be a Banach space and τ a vector topology weaker than the norm topology. An equivalent norm $\|.\|$ is said τ-Kadec if the norm topology and τ coincide on the unit sphere of $\|.\|$.*

Next result appears in [1].

Proposition 4 *A τ-Kadec norm $\|.\|$ is τ-lower semicontinuous, that is, its unit ball is always τ-closed.*

Proof. Suppose that $\|.\|$ is not τ-lsc. Then there is a net (x_ω) on the unit sphere S_X and a point x out of the unit ball B_X such that $\tau\text{-}\lim_\omega x_\omega = x$. Take numbers $t_\omega > 1$ such that $\|x + t_\omega(x_\omega - x)\| = \|x\|$. Let $y_\omega = x + t_\omega(x_\omega - x)$. Note that $\{t_\omega\}$ is bounded because $\inf_\omega \|x_\omega - x\| > 0$. We deduce that $\tau\text{-}\lim_\omega y_\omega = x$. Since $\|y_\omega\| = \|x\|$ we should have that $\lim_\omega \|y_\omega - x\| = 0$, but this is impossible because $\|y_\omega - x\| \geq \|x_\omega - x\|$.

As said in the introduction, LUR norms provide us examples of norms with the Kadec property. In fact, it is not difficult to prove that a τ-lower semicontinuous LUR norm is τ-Kadec. At this point it is important to remark that if the unit ball of a Banach space is τ-closed for some vector topology τ, then the new unit ball after a renorming is no necessary τ-closed. For example, there exist a dual Banach space that admits an equivalent LUR norm but admiting no equivalent dual LUR norm, see the remarks after Theorem 3.

Given two topologies τ_1 and τ_2 on X and a family Σ of subsets of X we shall say that Σ is good at $x \in X$ if for every $V \in \tau_1$ with $x \in \tau_1$ there exists $S \in \Sigma$ and $U \in \tau_2$ such that $x \in S \cap U \subset V$. Good family means a family good at every point of X. It is easy to see that a family Σ covering X such that on every $S \in \Sigma$ the topologies τ_1 and τ_2 coincide is good and property $P(\tau_1, \tau_2)$ is equivalent to the existence of a countable good family.
The following lemma shows how to make a good family of "thick sets" from a good one made up of "thin sets".

Lemma 1 *Let X be a vector space, $\tau_2 \subset \tau_1$ vector topologies on X and Σ a family good at some $x \in X$. Then the family*

$$\{S + W : \ S \in \Sigma, \ 0 \in W \in \tau_1\}$$

is good at x.
Thus, if Σ and Π are families of subsets of X such that for every $S \in \Sigma$ and every $W \in \tau_1$ with $0 \in W$ there exists $P \in \Pi$ such that

$$S \subset P \subset S + W$$

then Π is good if and only if Σ is good.

Proof. Given $V \in \tau_1$ with $x \in V$ we shall find $S \in \Sigma$, $0 \in W \in \tau_1$ and $U \in \tau_2$ such that

$$x \in (S + W) \cap U \subset V$$

As $0 + x \in V$ we can take $W_1, V' \in \tau_1$ with $0 \in W_1$, $x \in V'$ and $W_1 + V' \subset V$. Since Σ is good at x we can take $S \in \Sigma$ and $U' \in \tau_2$ such that $x \in S \cap U' \subset V'$. As $0 + x \in U'$ we can take $W_2, U \in \tau_2$ with $0 \in W_2$, $x \in U$ and $W_2 + U \subset U'$. Now take $W = W_1 \cap (-W_2) \in \tau_1$. We shall show that U and W verify the above set inclusion. If $y \in (S + W) \cap U$

then there is $z \in S$ such that $y - z \in W \subset -W_2$ so $z = (z-y)+y \in U'$ thus $z \in S \cap U' \subset V'$. Now as $y - z \in W \subset W_1$ we have that $y = (y-z)+z \in V$.

The applications of Kadec type norms to the results developed in section 2 are contained in the following lemma.

Lemma 2 *Let $(X, \|.\|)$ a normed vector space, $\tau_2 \subset \tau_1$ vector topologies on X weaker than the norm topology. Suppose that there exists a positive homogeneous function F on X such that:*

a) $F(x) \geq c\|x\|$ for some $c > 0$.

b) τ_1 and τ_2 coincide on the set $S = \{x \in X : \; F(x) = 1\}$.

Then X has $P(\tau_1, \tau_2)$. In particular, if X is a Banach space that admits an equivalent τ-Kadec norm for some weaker vector topology τ then X has $P(\|.\|, \tau)$.

Proof. Consider the following families of sets: $\Sigma = \{S(t) : t \in [0, \infty)\}$ and the countable one $\Pi = \{A(r,s) : \; r, s \in \mathbb{Q}, \; 0 \leq r \leq s\}$ where

$$S(t) = \{x \in X : \; F(x) = t\}$$

$$A(r,s) = \{x \in X : \; r \leq F(x) \leq s\}$$

If $W \in \tau_1$ is neighbourhood of 0 then it contains some ball $B[0, \delta]$. It is easy to see that for δ small enough

$$S(t) \subset A(t - c\delta, t + c\delta) \subset S(t) + W$$

The result follows from Lemma 1.

Combining Proposition 1, Corollary 1 and the previous lemma we easily obtain the theorems of Edgar and Schachermayer. Note that a more direct proof of Edgar's theorem just needs a special case of lemma 1 and the idea of point *iv*) of Proposition 1. Schachermayer theorem besides needs point *iii*) of Proposition 1.

Corollary 3 *Let X be a Banach space that admits an equivalent Kadec norm. Then $Borel(X, \|.\|) = Borel(X, w)$ and $X \in Borel(X^{**}, w^*)$.*

Next theorem is the main result of this section. It provides us with a converse of Lemma 2 in the metric case. A partial similar result have been proved by Lancien in [18].

Theorem 1 *Let X be a Banach space and τ a vector topology coarser than the norm topology such that $\overline{B_X}^{\tau}$ is bounded. Then the following are equivalent:*

i) X has $P(\|.\|, \tau)$ (equivalently, (X, τ) has $\|.\|$-SLD).

ii) There exists a non negative symetric homogeneous τ-lower semicontinuous function F on X with $\|.\| \le F \le 3\|.\|$ such that the norm topology and τ coincide on the set

$$S = \{x \in X : \ F(x) = 1\}$$

Proof. *ii)* $\Rightarrow$ *i)* This is in fact Lemma 2.
i) $\Rightarrow$ *ii)* We shall assume that X is endowed with a τ-lower semicontinuous equivalent norm $\|.\|$. $B(0,a)$ and $B[0,a]$ will be the open and the closed balls of center 0 and radius a. As usual $B_X = B[0,1]$.
Suppose that X has $P(\|.\|, \tau)$ with the sequence (A_n). We can suppose every set A_n star shaped with respect to 0 and norm open. To see that we are going to modify the sequence in several steps.
First step. Take $A'_n = A_n \cap B_X$.
Second step. Take

$$A''_n = \{tx : \ 0 \le t \le 1, \ x \in A'_n\}$$

We are going to check that (A''_n) is good for the points of the unit sphere S_X. Let $x \in S_X$ and $\varepsilon > 0$. Appling Lemma 1 we can find $U \in \tau$, $n \in \mathbb{N}$ and $\delta > 0$ such that $x \in A'_n \cap U$ and $diam((A'_n + B(0,\delta)) \cap U) < \varepsilon$. Now it is clear that

$$A''_n \cap (U \setminus B[0, 1-\delta]) \subset (A'_n + B(0,\delta)) \cap U$$

Thus $U' = U \setminus B[0, 1-\delta] \in \tau$ verify $x \in A''_n \cap U'$ and $diam(A''_n \cap U') < \varepsilon$.
Third step. The family

$$\{rA''_n + B(0,\delta) : \ n \in \mathbb{N}, \ r \ge 0, \ \delta > 0, \ r, \delta \in \mathbb{Q}\}$$

is good for X after Lemma 1. Renumbering the index this family will be the desired (A_n).
Clearly the sets $\overline{A_n}^{\tau}$ are star shaped with respect to 0. Let f_n be the Minkowski's functional of $\overline{A_n}^{\tau}$. Since $\overline{A_n}^{\tau} = \{f_n \leq 1\}$ the function f_n is τ-lower semicontinuous. Let $\|f_n\|$ be the supremun of $|f_n(x)|$ with $x \in B_X$. The function F given by the formula

$$F(x) = \|x\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n(x)}{\|f_n\|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n(-x)}{\|f_n\|}$$

is τ-lower semicontinuous and symetric.
Let $(x_\omega) \subset S$ a net τ-converging to some $x \in S$. From the τ-lower semicontinuity of $\|.\|$ and f_n we have that

$$\|x\| \leq \liminf_{\omega} \|x_\omega\|$$

$$f_n(x) \leq \liminf_{\omega} f_n(x_\omega)$$

$$f_n(-x) \leq \liminf_{\omega} f_n(-x_\omega)$$

On the other hand, it is not difficult to see that

$$1 \geq \liminf_{\omega} \|x_\omega\| + \sum_{n=1}^{\infty} \frac{1}{2^n\|f_n\|} \liminf_{\omega} f_n(x_\omega) + \sum_{n=1}^{\infty} \frac{1}{2^n\|f_n\|} \liminf_{\omega} f_n(-x_\omega)$$

Since $F(x) = 1$, a simple reasoning with lim sup gives the following equalities and the existence of its left members

$$\lim_{\omega} \|x_\omega\| = \|x\|$$

$$\lim_{\omega} f_n(x_\omega) = f_n(x)$$

$$\lim_{\omega} f_n(-x_\omega) = f_n(-x)$$

for every $n \in \mathbb{N}$.
Fix $\varepsilon > 0$. After the proof of *iii)* of Proposition 1 there exists $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $diam(\overline{A_n}^{\tau} \cap U) \leq \varepsilon$. In particular, as A_n is norm open then $f_n(x) < 1$ so for ω big enough $f_n(x_\omega) < 1$ and thus $x_\omega \in \overline{A_n}^{\tau}$. Since for ω big enough we have that $x_\omega \in U$ we obtain that $\|x_\omega - x\| \leq \varepsilon$. This proves that the net (x_ω) converges to x in norm, so the

norm topology and τ coincide on S.

Clearly the constant 3 in the statement *ii*) of the preceding theorem can be changed by any constant greater than 1. In fact every function of the form $\|.\| + aF$ with $a > 0$ has the same property. This also shows that the norm can be aproximated uniformly by functions with the Kadec property provided the existence of one at least.
Note that S is a norm G_δ-set in $B = \{x \in X : \ F(x) \leq 1\}$, thus (S, τ) is completly metrisable.

A remarkable theorem of Kadec, see [2, pg. 177], shows that every separable Banach space has an equivalent τ-Kadec norm for the topology τ of convergence on a prefixed quasi-norming subset of its dual space. The following result characterizes the existence of τ-Kadec norms in general Banach spaces extending Kadec's theorem.

Theorem 2 *Let X be a Banach space and τ a weaker topology such that $\overline{B_X}^{\tau}$ is bounded. Then X has a equivalent τ-Kadec norm if and only if X has $P(\|.\|, \tau)$ where the sets (A_n) in Definition 1 are convex, in other words, if there exists convex sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and diam$(A_n \cap U) < \varepsilon$.*

Proof. If we begin with (A_n) convex in the proof of Theorem 1 it is easily checked that all the families of sets built there are still convex. Thus F is subadditive and it is so an equivalent τ-Kadec norm.
For the converse assume that the norm of X is τ-Kadec. The proof of Lemma 2 shows that X has $P(\|.\|, \tau)$ with a sequence of differences of closed balls centered at 0. As the closed balls are τ-closed we deduce that the sequence of closed balls with rational radius satisfies what is required.

We do not know if property $P(\|.\|, w)$ implies the existence of an equivalent Kadec norm.

Recently Moltó, Orihuela and Troyanski [19] have given a characterization of the existence of an equivalent LUR norm in a Banach space using a variant of Definition 2. Their result can be reformuled in similar terms to those of Definition 1 as follows: *a Banach space X admits a LUR norm if and only if there exists a sequence of sets $A_n \subset X$ such that for every*

$x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and an open semispace U such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$. Note that the topological counterpart of this result is Theorem 1 applied to the weak topology but to deduce that the function F is in fact a Kadec norm we did need a geometric asumption about the sets A_n.

4 Applications

A Banach space X is said to be weakly countably determined (WCD) if there exists a sequence (K_n) of w^*-compact sets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. WCD Banach spaces generalise in a natural way the weakly compact generated Banach spaces (WCG), that is, the spaces containing a total weakly compact set. A WCD Banach space admits a LUR norm, [28].
The coincidence of Borel families in the following theorem improves one by Talagrand [25] for subspaces of WCG Banach spaces.

Theorem 3 *Let X be a WCD Banach space and let τ be a Hausdorff vector topology weaker than the weak topology of X. Then X has $P(\|.\|, \tau)$. Moreover, if $\overline{B_X}^{\tau}$ is bounded then X also admits a τ-Kadec norm and*

$$Borel(X, \|.\|) = Borel(X, \tau)$$

Proof. We can assume without loss of generality that the sequence (K_n) is closed by finite intersections. Indeed, we claim that the sequence of w^*-closed convex hulls $\{\overline{co(K_n)}^{w^*}\}$ also satisfies the above definition. Indeed, fix $x \in X$ and $y \in X^{**} \setminus X$. The set $K = \bigcap_{x \in K_n} K_n$ is a weakly compact set of X containing x. Now, since $\overline{co(K)}^{w}$ is a weak*-compact convex set not containing y, there is a weak*-open half space H such that $x \in H$ and $y \notin \overline{H}^{w^*}$. By compactness, there is $n \in \mathbb{N}$ such that $x \in K_n \subset H$. As $\overline{co(K_n)}^{w^*} \subset \overline{H}^{w^*}$ we have that $x \in \overline{co(K_n)}^{w^*}$ and $y \notin \overline{co(K_n)}^{w^*}$. This ends the proof of the claim.
First we are going to check that X has $P(w, \tau)$. For every $x \in X$ define

$$S_x = \bigcap_{K_n \ni x} K_n$$

By definiton of WCD it is clear that S_x is weakly compact set of X. If we take $\{S_x\}$ as Σ and the traces on X of finite intersections of K_n's as a

countable family Π, we shall show that conditions in Lemma 1 are fulfilled. Indeed, Σ covers X and τ and w coincide on every S_x by compactness so it is good for (w, τ). Now let W be a weak neighbourhood of 0 and let W' be a weak* neighbourhood of 0 in X^{**} such that $W = X \cap W'$. For some increasing sequence of integers (n_j) we have $S_x = \bigcap_j K_{n_j}$. By compactness there is a finite number of K_{n_j}'s whose intersection is contained in $S_x + W'$. So X has convex $P(w, \tau)$.
Since a WCD banach space admits a Kadec norm, it has convex $P(\|.\|, w)$. Now X has $P(\|.\|, \tau)$ by *ii*) of Proposition 2 with convex sets. The existence of a τ-Kadec equivalent norm follows from Theorem 2 and the coincidence of Borel sets follows from Corollary 1.

Using the general definition of countably determined topological space (X, τ_1) by usco maps it is possible to prove that X has $P(\tau_1, \tau_2)$ for every weaker Hausdorff topolgy τ_2, but it is not clear if that implies the coincidence of Borel sets. For example, in the preceding theorem, if we want to prove the coincidence of Borel sets for τ and the weak topology directly from the fact that X has $P(w, \tau)$ we have to check that $X \cap K_n$ is τ-Borel which is not evident except in the case of a WCG space. Roughly speaking that was the argument of Talagrand [25], but WCD spaces were introduced some years later.
In the particular case of a dual WCD space when τ is the weak* topology it is known that the space admits a equivelent dual LUR norm [8]. Without the hypothesis of WCD the result may be no true: the space $J(\omega_1)$ is a dual with the Radon-Nikodym property, so it admits a equivalent LUR norm [9], but $Borel(J(\omega_1), w^*)$ is a proper subset of $Borel(J(\omega_1), w) = Borel(J(\omega_1), \|.\|)$ [7]. A natural generalization of dual WCD are the dual spaces X^* such that $(B_{X^{**}}, w^*)$ is a Corson compact but in this case there may be no dual LUR norm [12].

This corollary is inspired in a result of [5] for WCG spaces.

Corollary 4 *Let Y be a Banach space and τ a vector topology weaker than the weak topology of Y such that the unit ball $\overline{B_Y}^{\tau}$ is bounded. If X is a WCD norm closed subspace of Y then X is a τ-Borel set in Y.*

Proof. Note that τ is Hausdorff. We deduce from Theorem 3 that X has $P(\|.\|, \tau)$. Now apply *b*) of Proposition 3.

It is not difficult to see that in the conditions of Corollary 3 if X is $K_{\sigma\delta}$ in (X^{**}, w^*) (for example when X is WCG) then it is a $F_{\sigma\delta}$ in (Y, τ) while the proof of Corollary 4 gives that X is a $(F \cap G)_{\sigma\delta}$. It is not known if a WCD Banach space is always a $K_{\sigma\delta}$ in (X^{**}, w^*), see [4, Problem VI.3].

It is known that K-analytic topological spaces are Čech-analytic for every Hausdorff weaker topology. The same result is not true in general for WCD topological spaces. The next corollary gives a positive answer in the particular case of Banach spaces and "reasonable" topologies.

Corollary 5 *Let X be a WCD Banach space and τ the topology of convergence on a quasi norming subset of X^*. Then (X, τ) is Čech-analytic.*

Proof. Using an equivalent norm we can suppose that τ is given by a norming subset. Then apply Corollary 2.

Let us mention here that we also have as consequence of Proposition 2 and Theorem 3 that in the hypothesis of Corollary 5, (X, τ) is σ-fragmentable and, in particular, the τ-compact sets of X are fragmentable, see [3] for the definitions and some consequences.

A typical situation is the case of $C(K)$ spaces with the pointwise topology. There is a huge family of compact spaces K called Valdivia compacts such that $C(K)$ admits a LUR norm which makes pointwise closed the unit ball [27]. So the results above are applicable, in particular the Borel sets for the norm and the pointwise topologies coincide. Recently Haydon, Jayne, Namioka and Rogers [13] have showed that if K is a totally ordered set that is compact in its order topology then $C(K)$ admits a norm with the property of Kadec for the pointwise topology so the same coincidence of Borel sets happens.

A different class of compact spaces where we can check directly the coincidence of Borel sets in $C(K)$ for the weak and the pointwise topologies is the class of Radon-Nikodym compact spaces. Originally, a compact space is said Radon-Nikodym when it is homeomorphic to a w^*-compact of a dual with the Radon-Nikodym property. Equivalently, *a compact K is Radon-Nikodym if and only if there exists a stronger lower semicontinuous metric d on K such that every Radon measure on K is the restriction of a Radon measure on* (K, d), [21] and [14].

Theorem 4 *Let K be a Radon-Nikodym compact space. Then the $C(K)$ has a equivalent pointwise lower semicontinuous norm such that on its unit sphere the weak and the pointwise topologies coincide, $C(K)$ has $P(w, t_p(K))$ and*

$$Borel(C(K), w) = Borel(C(K), t_p(K))$$

Proof. A continuous function on K is d-uniformly continuous. Indeed, suppose not. Then we can take sequences (x_n) and (y_n) in K such that $\lim_n d(x_n, y_n) = 0$ while $|f(x_n) - f(y_n)| \geq \delta$ for some $\delta > 0$. By an ultrafilter we get the sequences to converge to the limits x and y respectly. But by the the lower semicontinuity of d we have $d(x, y) = 0$ so $x = y$ and this is a contradiction with the continuity of f.
Fix a d-dense set $(x_\alpha)_{\alpha\in\Gamma}$. Now we define the seminorns O_n as follows

$$O_n(f) = \sup_\alpha \sup \{|f(x) - f(x_\alpha)| : \ d(x, x_\alpha) \leq \frac{1}{n}\}$$

Clearly O_n is pointwise lower semicontinuous and since every $f \in C(K)$ is d-uniformly continuous for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $O_n(f) < \delta$.
Define a new norm by the formula

$$|||f||| = \|f\| + \sum_{n=1}^{\infty} \frac{1}{2^n} O_n(f)$$

Evidently $\|.\| \leq |||.||| \leq 3\|.\|$. Thus $|||.|||$ is an equivalent norm in $C(K)$. It is also no hard to check that the unit ball of $|||.|||$ is pointwise closed.
We are going to check that the weak and the pointwise topologies coincide on the set $S = \{f \in C(K) : |||f||| = 1\}$. Let (f_ω) be a net in S pointwise converging to $f \in S$. Take a Radon measure μ with $\|\mu\| \leq 1$ that we suppose already defined on $Borel(K, d)$ and take $\varepsilon > 0$.
From the pointwise lower semicontinuity of $\|.\|$ and O_n reasoning like in Theorem 1 we obtain that $\lim_\omega O_n(f_\omega) = O_n(f)$ for every $n \in \mathbb{N}$.
Now fix $n \in \mathbb{N}$ such that $O_n(f) \leq \varepsilon/8$. Then for ω big enough $O_n(f_\omega) \leq \varepsilon/6$.
Since μ has a d-separable d-support we can fix $F \subset \Gamma$ finite such that

$$|\mu|(\bigcup_{\alpha\in F} B[x_\alpha, \frac{1}{n}]) > |\mu|(K) - \frac{\varepsilon}{4}$$

If ω is big enough then $|f_\omega(x_\alpha)-f(x_\alpha)| \leq \varepsilon/6$ for $\alpha \in F$. So $|f_\omega(x)-f(x)| \leq \varepsilon/2$ for every $x \in \cup_{\alpha\in F} B(x_\alpha, 1/n)$.
Having in mind that $\|f\|$ and $\|f_\omega\|$ are bounded by 1, an easy calculus gives that

$$|\mu(f_\omega - f)| \leq \int |f_\omega - f| \, d|\mu| \leq \varepsilon$$

which implies that (f_ω) converges weakly to f.
Now apply Lemma 2 to deduce that $C(K)$ has $P(w, t_p(K))$. Since the unit ball is pointwise closed the weak and the pointwise topologies have the same Borel sets by *iv*) of Proposition 1, moreover, every weak open is a countable union of differences of pointwise closed sets.

Clearly Theorem 3 is still true for a continuous image of a RN-compact. We know no example of compact space with different Borel sets for the weak and the pointwise topology.
Note that if K is a RN-compact space such that $(C(K), w)$ has $\|.\|$-SLD, then $(C(K), t_p(K))$ has $\|.\|$-SLD. In particular, K has the Namioka property, see [15].

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Matias Raja
Mathématiques Pures
Université Bordeaux 1
351, cours de la Libération
33405 TALANCE Cedex
FRANCE
e-mail: raja@math.u-bordeaux.fr