Locally uniformly rotund norms

by
M. Raja*

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Abstract

Given a Banach space X and a norming subspace $Z \subset X^*$ we introduce a geometrical method to characterize the existence of an equivalent $\sigma(X,Z)$-lsc LUR norm on X. A new simple proof of the Theorem of Troyanski: *every rotund space with a Kadec norm is LUR renormable*, and a generalization of the Moltó, Orihuela and Troyanski characterization of the LUR renormability are proved without probability arguments. Among other applications, we obtain that a dual Banach space with a w^*-Kadec norm admits a dual LUR norm.

1 Introduction

Throughout this paper $(X, \|.\|)$ will denote a Banach space and X^* its dual. A subset of the dual unit ball B_{X^*} is said to be norming (resp. quasi-norming) if its w^*-closed convex envelope is B_{X^*} (resp. contains an open ball centered at 0). A linear subspace $Z \subset X^*$ is said norming (resp. quasi-norming) if $Z \cap B_{X^*}$ is a norming (resp. quasi-norming) set. We shall denote by $\sigma(X,Z)$ the topology on X of pointwise convergence on Z, but in the particular cases of the weak and the weak* topologies we shall use w and w^* respectively. For a norm $\|.\|$ it is equivalent to to be $\sigma(X,Z)$-lower semicontinuous ($\sigma(X,Z)$-lsc for short) and to have its unit ball $\sigma(X,Z)$-closed.

Definition 1 *Let X be a Banach space endowed with a norm $\|.\|$ and let S_X be the unit sphere. Then the norm is said to be*

a) locally uniformly rotund (LUR), if for every $x, x_k \in X$, with $\|x\| = \|x_k\| = 1$, and such that $\lim_k \|x + x_k\| = 2$, then $\lim_k \|x - x_k\| = 0$.

b) rotund if for every $x, y \in X$, with $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$ then $x = y$, in other words, $\|.\|$ is rotund if and only if every point of S_X is an extreme point in B_X.

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c) $\sigma(X,Z)$-Kadec if $\sigma(X,Z)$ and the norm topology coincide on S_X. If $Z = X^*$ we say simply that $\|.\|$ is Kadec.

It is known that a LUR norm which is $\sigma(X,Z)$-lsc is also $\sigma(X,Z)$-Kadec. We include a proof of that fact as part of Theorem 2.

Let X be a Banach space. We call an open affine half space defined by an element $x^* \in X^*$ a set of the form $\{x \in X : \ x^*(x) < \alpha\}$ where $\alpha \in \mathbb{R}$. A relatively open subset of a set defined by an intersection with some open half space is called slice. If $Z \subset X^*$ is a linear subspace of the dual, we shall denote by $\mathbb{H}(Z)$ the set of the open affine half spaces defined by elements of Z.

The study of the existence of equivalent LUR norms plays a central role in the geometric theory of Banach spaces. See for example the book [7] which contains the main advances on this subject until 1993. One of the deepest results on LUR renorming is given by Moltó, Orihuela and Troyanski in [16], who characterizes the existence of an equivalent LUR norm in a Banach space in terms of a variant of the following topological property introduced by Jayne, Namioka and Rogers in [13] in a more general situation under the name of *having a countable cover by sets of local small diameter*. The variant in brackets is the property considered in [16].

Definition 2 *Let $(X, \|.\|)$ be a Banach space. It is said that X has JNR (resp. s-JNR) if for every $\varepsilon > 0$ there exists a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a weak neighbourhood (resp. a slice) in X_n^{ε} of diameter less than ε.
A dual Banach space X^* is said to have *JNR (resp. s^*-JNR) if above the neighbourhoods are weak*-open (resp. if the slices are given by elements of the predual X).

In [16] it is shown that a Banach space X has an equivalent LUR norm if and only if it has s-JNR. The proof of [16] is based in some martingale arguments of Troyanski. This technique does not allow, a priori, to build LUR norms satifying a condition of lower semicontinuity with respect to topologies coarser than the weak topology of the Banach space. One can realize how lower semicontinuity is lost in Troyanski's formula, see [7, p. 144]. It seems that to avoid that difficulty should be interesting in the case of dual spaces or $C(K)$ spaces. It is our aim here to show how to do it.

In this paper we give a self contained geometric proof of the theorem of Moltó, Orihuela and Troyanski. Moreover, our method allows us to control the lower semicontinuity of the new norm with respect to the topology of convergence on a quasi-norming subspace. In particular we prove, according to the intuition, that a dual Banach space X^* has an equivalent dual LUR norm if and only if it has s*-JNR.

The following definition was introduced by Arkangel'skii in [2] and it will play a fundamental role along this paper.

Definition 3 *Let (X, τ) be a topological space. A family Σ of subsets of X is said to be a network for τ if every open set is a union of sets belonging to Σ.*

One can realize that a network made up of open sets is a basis for the topology. It is shown in [18] that a Banach space has JNR if and only if there is a sequence (A_n) of subsets of X such that the family of the intersections of sets from the sequence (A_n) and the weak open sets of X, $\{A_n \cap w\}$ for short, is a network for the norm topology. Analogously, it can be proved that X (resp. a dual X^*) has s-JNR (resp. s*-JNR) if and only if there is a sequence (A_n) of subsets of X such that $\{A_n \cap \mathbb{H}(X^*)\}$ (resp. $\{A_n \cap \mathbb{H}(X)\}$) is a network for $(X, \|.\|)$ (resp. $(X^*, \|.\|)$).

We shall use the following characterization of the existence of equivalent Kadec norms from our previous work (see Remark 3).

[18, Theorem 2] *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

i) X admits a $\sigma(X, Z)$-Kadec norm.

ii) There is a sequence (A_n) of convex subsets of X such that $\{A_n \cap \sigma(X, Z)\}$ (i.e. $\{A_n \cap U : n \in \mathbb{N}, U \in \sigma(X, Z)\}$) is a network for the norm topology.

Our main theorem provides us with characterizations of the existence of equivalent LUR norms which are $\sigma(X, Z)$-lsc.

Theorem A *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

i) X admits a $\sigma(X, Z)$-lsc LUR norm.

ii) X admits both a rotund norm and a $\sigma(X, Z)$-Kadec norm.

iii) There is a sequence (A_n) of subsets of X such that $\{A_n \cap \mathbb{H}(Z)\}$ is a network for the norm topology.

It should be noted that it is enough to apply statement *iii*) above to check it on the sphere S_X with the relative norm topology, see Remark 2. After this observation, in the particular case when $Z = X^*$, our Theorem A is the Main Theorem of [16] together the Theorem of Troyanski, [7, p. 148]. Some other equivalent statements to those of Theorem A will be given along the paper.

Applying Theorem A together with techniques from [16] we obtain the following result that answers a question of Haydon [10].

Corollary 1 *Let (K_n) be a sequence of closed subsets of a compact space K such that $K = \bigcup_{n=1}^{\infty} K_n$. Assume that $C(K_n)$ has an equivalent pointwise-lsc LUR norm for every $n \in \mathbb{N}$. Then $C(K)$ has an equivalent pointwise lsc LUR norm.*

A version of the preceding corollary without asking the LUR norms to be pointwise-lsc appears in [16].

A dual Banach space with the Radon-Nikodym property (RNP) always admits a LUR norm by [9], but this norm is not necessary a dual norm. The following theorem gives some conditions equivalent to the existence of dual LUR norms.

Theorem B *Let X^* be a dual space. The following are equivalent:*

i) *X^* admits a dual LUR norm.*

ii) *X^* admits a w^*-Kadec norm.*

iii) *X^* has s^*-JNR.*

iv) *X^* has *JNR and it admits a dual rotund norm.*

Note that while the fact that the dual norm is w^*-Kadec implies dual-LUR renormability, there exists Banach spaces having a Kadec norm but with no equivalent LUR norm neither rotund norm [12]. Haydon [11] also has built a Banach space X such that $(B_{X^{**}}, w^*)$ is a Corson compact (in particular X^* has RNP) and X^* admits no dual LUR norm. After Theorem B we have that this space X^* admits no w^*-Kadec norm.

The rest of this paper is divided into four sections. Section 2 is devoted to Troyanski's Theorem. We give a self contained direct proof which motives the construction for the general case in the following section. In Section 3 we prove Theorem A under the additional assumption that the sets (A_n) in statement *iii)* are convex. In Section 4 we remove the convexity hypothesis ending the proof of Theorem A. Finally, in Section 5 we prove Theorem B and the results are applied to $C(K)$ spaces, WCD Banach spaces, etc.

Part of the results of this paper has been anounced in [19].

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2 A Theorem of Troyanski

In this section we shall prove a famous theorem of Troyanski, see [7, p. 148]. Althought it is a particular case of Theorem 2 in the following section, we provide a simple proof before. Recall that a point x of a convex set C is said to be a denting point if there are slices of C containing x of arbitrarily small diameter. If the slices are given by the elements of some subspace Z of the dual we say that x is Z-denting.

Theorem 1 *Let X be a Banach space. The following are equivalent:*

i) X admits a LUR norm.

ii) X admits both a rotund norm and a Kadec norm.

iii) X admits a norm such that every point of the unit sphere is a denting point of the unit ball.

Proof. *i)* $\Rightarrow$ *ii)* is standard. It can be found for example in [7]. We include a proof of a slightly more general result in the next section.

ii) $\Rightarrow$ *iii)*. This a lemma of Lin-Lin-Troyanski, [15], which shows that that the points of a bounded closed convex set which are both extreme points and points of continuity are denting points. Nevertheless, we include the proof for sake of completness.

First we can asume that X is endowed with a norm $\|.\|$ which is both rotund and Kadec. Indeed, if $\|.\|_1$ is rotund and $\|.\|_2$ is Kadec, then the norm $\|.\|$ defined by

$$\|.\|^2 = \|.\|_1^2 + \|.\|_2^2$$

shares both properties.

Take $x \in S_X$. We claim that x is an extreme point of $B_{X^{**}}$. Indeed, suppose that $x = (x_1 + x_2)/2$, where $x_i \in B_{X^{**}}$ for $i = 1, 2$. Fix $\varepsilon > 0$ and set U a w^*-open neighbourhood of x such that $diam(B_{X^{**}} \cap U) < \varepsilon/2$. This is possible because of Goldstine's Theorem and the w^*-lower semicontinuity of the metric. Take U_i a w^*-open neighbourhood of x_i for $i = 1, 2$ such that $U_1 + U_2 \subset 2U$. If $y \in B_{X^{**}} \cap U_1$ then

$$\frac{y + x_2}{2} \in B_{X^{**}} \cap U$$

because of the convexity. We deduce that $diam(B_{X^{**}} \cap U_1) < \varepsilon$. Since $B_X \cap U_1$ is non empty, we obtain that x_1 can be aproximated uniformly by points of B_X. As B_X is norm complete, $x_1 \in B_X$, and in consequence $x_2 \in B_X$. As x is an extreme point in B_X, we have that $x = x_1 = x_2$. Now, since x is an extreme point of the w^*-compact set $B_{X^{**}}$, we can apply Choquet's Lemma, see [5, Proposition 25.13], to conclude that the slices of $B_{X^{**}}$ given by elements of X^* are a local basis for the norm topology at x, and thus x is denting in B_X.

iii) $\Rightarrow$ *i)*. We shall need the following lemma relative to convex functions.

Lemma 1 *Let f be a convex function on a Banach space X. Consider the symmetric function*

$$Q_f(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f(\frac{x+y}{2})^2$$

Then the following properties are verified:

1) $Q_f \geq 0$.

2) If (f_n) is a sequence of convex functions such that $\sum_{n=1}^{\infty} f_n^2$ is convergent then the positive function f defined by $f^2 = \sum_{n=1}^{\infty} f_n^2$ is convex and

$$Q_f = \sum_{n=1}^{\infty} Q_{f_n}$$

3) Given $x, x_k \in X$ the following are equivalent

a) $\lim_k f(x_k) = f(x)$ *and* $\lim_k f(\frac{x+x_k}{2}) = f(x)$.

b) $\lim_k Q_f(x, x_k) = 0$.

Proof. Statement 2) is a straightforward computation. To prove 1) and 3) just consider the inequalities

$$Q_f(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f(\frac{x+y}{2})^2 \geq \frac{f(x)^2 + f(y)^2}{2} - (\frac{f(x)+f(y)}{2})^2$$
$$= \frac{(f(x) - f(y))^2}{4} \geq 0$$

Let us go on with the proof of the theorem. Let $\|.\|$ be a norm as in *iii*). Consider the sets

$$B_\varepsilon = \{x \in B_X : \ \forall H \in \mathbb{H}(X^*), \ x \in H, \ diam(B_X \cap H) > \varepsilon\}\}$$

The set B_ε is what remains of the unit ball of the unit ball after we have removed all slices of diameter at most ε. If $\varepsilon < 2$ then B_ε is a closed symmetric convex set with nonempty interior. Let f_n be the Minkowski fuctional of $B_{1/n}$ for $n \in \mathbb{N}$. Since $B_{1/n}$ contains $\frac{1}{2}B_X$ we have that $f_n \leq 2\|.\|$. Define an equivalent norm $|||.|||$ on X by the formula

$$|||x|||^2 = \|x\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x)^2$$

We claim that $|||.|||$ is LUR. Let $x \in X$ and let $(x_k) \subset X$ be such that $|||x_k||| = |||x|||$ and $\lim_k |||x + x_k||| = 2\|x\|$. We have that

$$\lim_k (\frac{|||x|||^2 + |||x_k|||^2}{2} - |||\frac{x+x_k}{2}|||^2) = 0$$

Now, using Lemma 1 it follows that

$$\lim_k \|x_k\| = \lim_k \|\frac{x+x_k}{2}\| = \|x\|$$

$$\lim_k f_n(x_k) = \lim_k f_n(\frac{x+x_k}{2}) = f_n(x)$$

for every $n \in \mathbb{N}$. We deduce that in order to prove that (x_k) converges to x, we can suppose $\|x_k\| = \|x\| = 1$ by taking the sequence $(x_k/\|x_k\|)$.
Fix $\varepsilon > 0$. Since every point in S_X is a denting point of the unit ball, there is $H \in \mathbb{H}(X^*)$ such that $x \in H$ and

$$diam(B_X \cap H) \leq \varepsilon/2$$

Take $\varepsilon/2 \leq 1/n \leq \varepsilon$, then $x \notin B_{1/n}$. We have that $f_n(x) > 1$, so for $k \in \mathbb{N}$ big enough $f_n((x+x_k)/2) > 1$, and thus $(x+x_k)/2 \in B_X \setminus B_{1/n}$. By definition of $B_{1/n}$ there is $H' \in \mathbb{H}(X^*)$ such that $(x+x_k)/2 \in H'$ and $diam(B_X \cap H') \leq \varepsilon/2$. But, by convexity of $B_X \setminus H'$ at least one of x and x_k belongs to $B_X \cap H'$. Since

$$\|x_k - x\| = 2\,\|x_k - \frac{x+x_k}{2}\| = 2\,\|x - \frac{x+x_k}{2}\|$$

we deduce that $\|x_k - x\| \leq \varepsilon$ for k big enough. Consequently x_k converges to x and the proof is finished.

Remark 1 *The idea of considering the set obtained from a convex set after removing al the points of ε-dentability in renorming of Banach spaces was considered by Lancien in [14].*

3 Convexity arguments

In this section we shall prove the part of Theorem A which is contained in Theorem 2 below. For the systematic study, it seems to be convenient to introduce the following definition:

Definition 4 *Let Σ_1 and Σ_2 families of subsets of a given set X. We say that X has $P(\Sigma_1, \Sigma_2)$ if there is a sequence (A_n) of subsets of X such that for every $x \in X$ and every $V \in \Sigma_1$ with $x \in V$ there is $n \in \mathbb{N}$ and $U \in \Sigma_2$ such that*

$$x \in A_n \cap U \subset V$$

We have already used that definition in [18] in the particular case when Σ_1 and Σ_2 are topologies. In that case the Definition 4 means that the family of sets $\{A_n \cap \Sigma_2\}$ is a network for Σ_1. One can easily realize that property P is transitive, that is, if X has $P(\Sigma_1, \Sigma_2)$ and $P(\Sigma_2, \Sigma_3)$ then X has $P(\Sigma_1, \Sigma_3)$. Most of the "combinatorial type" results of the paper are based on this fact. When X is a vector space and when the sets A_n in Definition 4 can be chosen convex we shall say that X has convex-$P(\Sigma_1, \Sigma_2)$. If Σ_1 is the metric topology of X and Σ_2 is a family of half spaces $\mathbb{H}(Z)$ we shall write $P(\|.\|, Z)$ instead of $P(\Sigma_1, \Sigma_2)$. With this notation we have that JNR (resp. *JNR) is equivalent to $P(\|.\|, w)$ (resp. $P(\|.\|, w^*)$. Analogously, s-JNR (resp. s*-JNR) is equivalent to $P(\|.\|, X^*)$ (resp. $P(\|.\|, X^*)$).

Lemma 2 *Let X be a vector space and $\tau_2 \subset \tau_1$ be locally convex vector topologies on X. Denote by $\mathbb{S}(\tau_i)$ a sub-basis of τ_i given by a family of sublinear functions for $i = 1, 2$. Fix any point $x \in X$ and let Δ be a family of sets of X with the property: for every $V \in \mathbb{S}(\tau_1)$ with $x \in V$ there exists $A \in \Delta$ and $U \in \mathbb{S}(\tau_2)$ such that*

$$x \in A \cap U \subset V$$

Then for every $V \in \mathbb{S}(\tau_1)$ with $x \in V$ there exists $A \in \Delta$, $W \in \tau_1$ and $U \in \mathbb{S}(\tau_2)$ such that

$$x \in (A + W) \cap U \subset V$$

Proof. Suppose that $x \in X$ and $V \in \tau_1$ with $x \in V$ are given. We claim that there is $W_1, V' \in \mathbb{S}(\tau_1)$ with $0 \in W_1$, $x \in V'$ and $W_1 + V' \subset V$. Indeed, if $V = \{y \in X : f(x - y) < \varepsilon\}$ where f is a sublinear function and $\varepsilon > 0$ then just take the sets $W_1 = \{y \in X : f(y) < \varepsilon/2\}$ and $V' = \{y \in X : f(x - y) < \varepsilon/2\}$. By the property of Δ of the hypothesis we can take $A \in \Delta$ and $U' \in \mathbb{S}(\tau_2)$ such that $x \in A \cap U' \subset V'$.
As above, we can take $W_2, U \in \mathbb{S}(\tau_2)$ with $0 \in W_2$, $x \in U$ and $W_2 + U \subset U'$.
Now take $W = W_1 \cap (-W_2) \in \tau_1$. We shall show that $(A + W) \cap U \subset V$. If $y \in (A + W) \cap U$ then there is $z \in A$ such that $y - z \in W \subset -W_2$ so $z = (z - y) + y \in U'$ thus $z \in A \cap U' \subset V'$. Now as $y - z \in W \subset W_1$ we have that $y = (y - z) + z \in V$.

Remark 2 *If S_X has $P(\|.\|, \sigma(X, Z))$ with some sequence of sets $A_n \subset S_X$ then an easy consequence of Lemma 2 is that X has $P(\|.\|, \sigma(X, Z))$ with the countable family of the sets*

$$A_{n,r,s} = \{tx : \ r < t < s, \ x \in A_n\}$$

where $n \in \mathbb{N}$ and $0 \leq r < s$ are rational numbers.

Theorem 2 *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

1) X admits a $\sigma(X, Z)$-lsc LUR norm.

2) X admits a norm such that every point of the unit sphere is Z-denting.

3) X admits both a rotund norm and a $\sigma(X, Z)$-Kadec norm.

4) X admits a LUR norm and X has convex-$P(w, \sigma(X, Z))$.

5) X has convex-$P(\|.\|, Z)$.

Proof. 1) $\Rightarrow$ 2). Fix $x_0 \in S_X$ and $\varepsilon > 0$. As $\|.\|$ is a LUR norm, there is $\delta > 0$ such that if $x \in S_X$ and $\|\frac{x_0+x}{2}\| > 1-\delta$ then $\|x - x_0\| < \varepsilon$. We can suppose $\delta < 1$. Since $(1-\delta)^{-1}x_0 \notin B_X$ and B_X is $\sigma(X,Z)$-closed, by Hahn-Banach Theorem there is $x^* \in Z$ such that

$$\sup\{x^*(x) : \ x \in B_X\} < x^*(x_0)$$

We can suppose $\|x^*\| = 1$ and thus $(1-\delta)^{-1}x^*(x_0) > 1$. Now, define $H = \{x \in X : \ x^*(x) > 1-\delta\}$. Then $x_0 \in H$ and if $x \in B_X \cap H$ then

$$\|\frac{x_0+x}{2}\| \geq x^*(\frac{x_0+x}{2}) > 1-\delta$$

and thus $\|x - x_0\| < \varepsilon$.

2) $\Rightarrow$ 5). We shall check that the set of balls centered at 0 with rational radius satisfies condition 5). Fix $x \in X$, and without loss of generality suppose that $x \neq 0$. Let $B[0, \|x\|]$ be the closed ball of center 0 and radius $\|x\|$. By hypothesis, for every $\varepsilon > 0$ there is $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$B[0, \|x\|] \cap H \subset B(x, \varepsilon/2)$$

Fix $\varepsilon > 0$. By Lemma 2, there is $\delta > 0$ and $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$(B[0, \|x\|] + B(0,\delta)) \cap H \subset B(x, \varepsilon/2)$$

Now we can find a rational $r > 0$ such that

$$B[0, \|x\|] \subset B[0, r] \subset B[0, \|x\|] + B(0,\delta)$$

and thus $x \in B[0,r] \cap H$ and $diam(B[0,r] \cap H) < \varepsilon$.

1) $\Rightarrow$ 3). We shall check the Kadec property for $\sigma(X,Z)$. Take $(x_\omega) \subset S_X$ a net $\sigma(X,Z)$-converging to $x \in S_X$. Since $\sigma(X,Z)$-$\lim_\omega(x + x_\omega) = 2x$ and $\|.\|$ is $\sigma(X,Z)$-lower semicontinuous we have that $\liminf_\omega \|x + x_\omega\| \geq 2\|x\| = 2$. On the other hand, $\|x + x_\omega\| \leq 2$ for every ω. Thus, $\lim_\omega \|x + x_\omega\| = 2$. Applying the fact that $\|.\|$ is LUR we have that $\|x - x_\omega\| = 0$.

3) $\Rightarrow$ 4). If X admits a $\sigma(X,Z)$-Kadec norm then it has convex-$P(\|.\|, \sigma(X,Z))$ by [18]. A direct proof can be obtained from the idea in 2) $\Rightarrow$ 5) above. We deduce that, in particular, X has convex-$P(w, \sigma(X,Z))$. As a $\sigma(X,Z)$-Kadec norm is Kadec, by Theorem 1 we have that X admits a LUR norm.

4) $\Rightarrow$ 5). The proof is a consequence of the following lemma and the transitivity of property P.

Lemma 3 *Convex-*$P(w, \sigma(X,Z))$ *implies convex-*$P(X^*, Z)$

Proof. Suppose that X has $P(w, \sigma(X,Z))$ with a sequence of convex sets (A_n). We shall prove that X has $P(X^*, Z)$ with the sequence (A_n). Fix $x \in X$ and $H \in \mathbb{H}(X^*)$. Take $n \in \mathbb{N}$ and $U \in \sigma(X,Z)$ such that $x \in A_n \cap U \subset H$. Since U and the convex set $A_n \setminus H$ are disjoint, the set $\overline{A_n \setminus H}^{\sigma(X,Z)}$ cannot contain x. By Hahn-Banach Theorem we obtain $H' \in \mathbb{H}(Z)$ such that $x \in H'$ and $\overline{A_n \setminus H}^{\sigma(X,Z)} \cap H' = \emptyset$. Thus we have $x \in A_n \cap H' \subset H$.

Now, to finish the proof of 4) $\Rightarrow$ 5), note that if X admits a LUR norm then it has convex-$P(\|.\|, X^*)$ by Theorem 1 and 2) $\Rightarrow$ 5) for the weak topology. This property together with convex-$P(X^*, Z)$ entails convex-$P(\|.\|, Z)$.

5) $\Rightarrow$ 1). Without loss of generality we can suppose that X is already endowed with a norm $\|.\|$ which is $\sigma(X,Z)$-lower semicontinuous.
Suppose that X has $P(\|.\|, Z)$ with a sequence of convex sets (A_n). Then by Lemma 2 the countable family of convex sets

$$\{A_n + B(0,r) : \ n \in \mathbb{N}, \ r > 0, \ r \in \mathbb{Q}\}$$

also satisfies 5), so we can suppose that the sets A_n are norm open. Fix $a_n \in A_n$ for every $n \in \mathbb{N}$ and let f_n be the functional of Minkowski with respect to the point a_n of the set $\overline{A_n}^{\sigma(X,Z)}$, that is, if g is the Minkowski functional of $\overline{A_n - a_n}^{\sigma(X,Z)}$ then $f_n(x) = g(x - a_n)$. Clearly the fuction f_n is convex, Lipschitz and $\sigma(X,Z)$-lsc.
Now, for $m \in \mathbb{N}$ we define the sets

$$A_{n,m} = \{x \in \overline{A_n}^{\sigma(X,Z)} : \ \forall H \in \mathbb{H}(Z), \ x \in H, \ diam(A_n \cap H) > 1/m\}$$

It is easy to see that the sets $A_{n,m}$ are empty or $\sigma(X,Z)$-closed convex. For every $p \in \mathbb{N}$ consider the sets

$$A_{n,m,p} = \overline{(A_{n,m} + B(0,1/p))}^{\sigma(X,Z)}$$

We claim that $A_{n,m} = \bigcap_{p=1}^{\infty} A_{n,m,p}$ when $A_{n,m}$ is non empty. Indeed, if $x \notin A_{n,m}$ since that set is convex and $\sigma(X,Z)$-closed we can take $H \in \mathbb{H}(Z)$ such that $x \in H$ and $A_{n,m} \cap H = \emptyset$. Taking the halfspace given by a parallel hyperplane, we can assume that the distance between $A_{n,m}$ and H is positive. Then for some $p \in \mathbb{N}$ we have

$$(A_{n,m} + B(0,1/p)) \cap H = \emptyset$$

and thus $x \notin A_{n,m,p}$.
Now, for every $n, m \in \mathbb{N}$ such that $A_{n,m} \neq \emptyset$ take $a_{n,m} \in A_{n,m}$. Let $f_{n,m,p}$ be the functional of Minkowski with respect to $a_{n,m}$ of the set $A_{n,m,p}$, which is convex, Lipschitz and $\sigma(X,Z)$-lsc. If $A_{n,m} = \emptyset$ we take $f_{n,m,p} = 0$ for every $p \in \mathbb{N}$.
We define a symmetric convex function F on X by the formula

$$F(x)^2 = \|x\|^2 + \sum_n \alpha_n \, f_n(x)^2 + \sum_{n,m,p} \beta_{n,m,p} \, f_{n,m,p}(x)^2$$

$$+\sum_n \alpha_n\, f_n(-x)^2 + \sum_{n,m,p} \beta_{n,m,p}\, f_{n,m,p}(-x)^2$$

where (α_n) and $(\beta_{n,m,p})$ are positive constants taken in such a way that the series converges uniformly on bounded subsets of X, so F is uniformly continuous on bounded sets, and that the absolutely convex set

$$B = \{x \in X : \; F(x) \leq 1\}$$

contains 0 as a interior point. As a series of $\sigma(X,Z)$-lsc fuctions is $\sigma(X,Z)$-lsc as well, we deduce that B is $\sigma(X,Z)$-closed. Let $|||.|||$ be the Minkowski fuctional of B. Then $|||.|||$ is an equivalent $\sigma(X,Z)$-lsc norm on X. We claim that for every sequence $(x_k) \subset B$ with $\lim_k |||x_k||| = 1$, then $\lim_k F(x_k) = 1$. Since F is $\|.\|$-continuous we deduce that

$$\{x \in X : \; F(x) < 1\} \subset \{x \in X : \; |||x||| < 1\}$$

because the first set is $\|.\|$-open and contained in B. Let us consider $x'_k = x_k/|||x_k|||$ and we should have $F(x'_k) = 1$. We have that $\lim_k \|x'_k - x_k\| = 0$. Since F is uniformly continuous on bounded sets we deduce that

$$\lim_k F(x_k) = 1$$

Finally, we shall prove that $|||.|||$ is a LUR norm. Let $x \in X$ and let $(x_k) \subset X$ such that $|||x_k||| = |||x||| = 1$ and $\lim_k |||x + x_k||| = 2$. As we have seen above this entails that $F(x_k) = F(x) = 1$ and $\lim_k F(\frac{x+x_k}{2}) = 1$. We have that

$$\lim_k \left(\frac{F(x)^2 + F(x_k)^2}{2} - F(\frac{x + x_k}{2})^2\right) = 0$$

Now, using Lemma 1 we have that for every $n, m, p \in \mathbb{N}$

$$\lim_k f_n(x_k) = \lim_k f_n(\frac{x + x_k}{2}) = f_n(x)$$

$$\lim_k f_{n,m,p}(x_k) = \lim_k f_{n,m,p}(\frac{x + x_k}{2}) = f_{n,m,p}(x)$$

Fix $1/2 > \varepsilon > 0$. There is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n \cap H$ and

$$diam(\overline{A_n}^{\sigma(X,Z)} \cap H) \leq \varepsilon/3$$

because H is $\sigma(X,Z)$-open and $\|.\|$ is $\sigma(X,Z)$-lsc. Since A_n is $\|.\|$-open, we have that $f_n(x) < 1$, so for $k \in \mathbb{N}$ big enough $f_n(x_k) < 1$, and thus $x_k \in \overline{A_n}^{\sigma(X,Z)}$. Analogously, for $k \in \mathbb{N}$ big enough, $(x + x_k)/2 \in \overline{A_n}^{\sigma(X,Z)}$.
Take $m \in \mathbb{N}$ such that $2/\varepsilon < m < 3/\varepsilon$, thus $x \notin A_{n,m}$. If $A_{n,m} \neq \emptyset$ then for some $p \in \mathbb{N}$ we have that $f_{n,m,p}(x) > 1$, so for $k \in \mathbb{N}$ big enough $f_{n,m,p}(\frac{x+x_k}{2}) > 1$ and thus $(x + x_k)/2 \notin A_{n,m,p}$. If $A_{n,m} = \emptyset$ we have simply that $(x + x_k)/2 \notin A_{n,m}$. Therefore, for $k \in \mathbb{N}$ big enough we have that

$$\frac{x + x_k}{2} \in \overline{A_n}^{\sigma(X,Z)} \setminus A_{n,m}$$

By definition of $A_{n,m}$ there is $H' \in \mathbb{H}(Z)$ such that $(x+x_k)/2 \in \overline{A_n}^{\sigma(X,Z)} \cap H'$ and $diam(\overline{A_n}^{\sigma(X,Z)} \cap H') \leq \varepsilon/2$. But either x or x_k belongs to H', and thus, to $\overline{A_n}^{\sigma(X,Z)} \cap H'$. Since

$$\|x_k - x\| = 2\,\|x_k - \frac{x+x_k}{2}\| = 2\,\|x - \frac{x+x_k}{2}\|$$

we deduce that $\|x_k - x\| \leq \varepsilon$. This ends the proof of Theorem 2.

Remark 3 *If we have the hypothesis convex-*$P(\|.\|, \sigma(X,Z))$ *instead of* 5) *the same proof with* F *defined by*

$$F(x)^2 = \|x\|^2 + \sum_n \alpha_n\, f_n(x)^2 + \sum_n \alpha_n\, f_n(-x)^2$$

suffices to prove that $|||.|||$ *is an equivalent* $\sigma(X,Z)$*-Kadec norm on* X*. This provides another proof for our Theorem [18, Th. 2] stated in the introduction. In [18] we study the general case when the sets* A_n *are not convex giving a characterization of the JNR property in terms of the existence of a positive homogeneous function* F*, not necessary convex, with the Kadec property, that is, the coincidence of* $\sigma(X,Z)$ *and the norm topology on the set* $\{x \in X : F(x) = 1\}$.

4 On Bourgain-Namioka Lemma

Here we shall complete the proof of Theorem A by proving the following result.

Theorem 3 *Let* X *be a Banach space and let* $Z \subset X^*$ *a quasi-norming linear subspace. To the list of equivalent properties given in Theorem 2, we may add the following:*

6) X *has* $P(\|.\|, Z)$.

7) X *has both* $P(\|.\|, w)$ *(equivalently, it has JNR) and* $P(w, Z)$.

The Bourgain-Namioka Lemma (see [6] or [3]) is the master key to prove that if a set has a point of ε-dentability, then its convex envelope also has points of 3ε-dentability. But the original point may not be a point of "small" dentability in the convex envelope. The following lemma establishes that with some kind of iteration "eating" in each step points of small dentability in convex envelopes we can reach a prefixed point of ε-dentability.

Lemma 4 *Let* M *and* ε *be positive constants and* $H \in \mathbb{H}(Z)$ *an open half space. For every* $E \subset X$ *such that* $diam(E) \leq M$ *and* $diam(E \cap H) < \varepsilon$ *there is a set* $E[H] \subset E$ *with the following properties:*

i) If $F \subset E$*, then* $F[H] \subset E[H]$.

ii) For every $x \in E \setminus E[H]$*, there is* $H' \in \mathbb{H}(Z)$ *such that* $x \in H'$*,* $E[H] \cap H' = \emptyset$ *and* $diam(co(E) \cap H') < 3\varepsilon$.

iii) If $(E[H^n])$ is the sequence defined inductively by $E[H^1] = E[H]$ and $E[H^{n+1}] = E[H^n][H]$, then $\bigcap_{n=1}^{\infty} E[H^n] \cap H = \emptyset$.

Proof. Suppose that $H = \{x^* > 1\}$ with $x^* \in Z$ and $E \cap H \subset B$ where B is a closed ball of diameter less than 2ε. Define the convex sets $C = \overline{co(E)}^{\sigma(X,Z)}$, $C_0 = C \cap B$ and $C_1 = C \setminus H$.
For every $0 \leq r \leq 1$ consider the set

$$D_r = \{(1-\lambda)x_0 + \lambda x_1 : \ r \leq \lambda \leq 1, \ x_0 \in C_0, \ x_1 \in C_1\}$$

Now, note that if $x \in D_0 \setminus D_r$ then $x = (1-\lambda)x_0 + \lambda x_1$ with $x_0 \in C_0$, $x_1 \in C_1$ and $0 \leq \lambda < r$, thus $\|x - x_0\| = \lambda \|x_0 - x_1\| \leq Mr$. Since $C = \overline{D_0}^{\sigma(X,Z)}$ we have

$$diam(C \setminus \overline{D_r}^{\sigma(X,Z)}) \leq diam(D_0 \setminus D_r) \leq 2Mr + diam(C_0) \leq 2Mr + 2\varepsilon$$

Fix $r > 0$ small enough to have $diam(C \setminus \overline{D_r}^{\sigma(X,Z)}) \leq 3\varepsilon$. Note that r depends only on M and ε.
Take $E[H] = E \cap \overline{D_r}^{\sigma(X,Z)}$. It is clear that *i)* holds. Now we shall check point *ii)*. If $x \in E \setminus E[H]$ then $x \in C \setminus \overline{D_r}^{\sigma(X,Z)}$. This set has diameter less than 3ε. By Hahn-Banach Theorem, we can find $H' \in \mathbb{H}(Z)$ such that $x \in H'$ and $H' \cap \overline{D_r}^{\sigma(X,Z)} = \emptyset$. Thus $diam(C \cap H') < 3\varepsilon$.
Finally we shall check point *iii)*. Note that

$$\sup\{x^*(x) : \ x \in E[H]\} \leq \sup\{x^*(x) : \ x \in D_r\}$$

$$\leq (1-r)\sup\{x^*(x) : \ x_0 \in C_0\} + r \sup\{x^*(x_1) : \ x_1 \in C_1\}$$

$$\leq (1-r)\sup\{x^*(x) : \ x \in E\} + r$$

So if $s_n = \sup\{x^*(x) : \ x \in E[H^n]\}$. We have that $s_{n+1} \leq (1-r)s_n + r$. This shows that s_n converges to 1. If $x \in H$, as $x^*(x) > 1$, then $x \notin E[H^n]$ for some n big enough.

Lemma 5 *Let $A \subset X$ be a bounded set and $\varepsilon > 0$. There exists a sequence of convex sets (C_n) such that, for every point x contained in a Z-slice of A of diameter less than ε, then there is $n \in \mathbb{N}$ such that x is contained in a Z slice of C_n of diameter less than 3ε.*

Proof. Given $E \subset A$, for every $H \in \mathbb{H}(Z)$ such that $diam(E \cap H) < \varepsilon$, let $E[H]$ the set given by the preceding lemma and take $E' = \bigcap_H E[H]$. We claim that for every $x \in E \setminus E'$ there is $H \in \mathbb{H}(Z)$ such that $x \in co(E) \cap H$ and it has diameter less than 3ε. Indeed, if $x \in E \setminus E'$ then there is $H \in \mathbb{H}(Z)$ such that $x \in E \setminus E[H]$. Then, apply *ii)* of Lemma 4.
Now we define inductively (E_n) by $E_0 = A$ and $E_{n+1} = (E_n)'$ if $E_n \neq \emptyset$. We claim that for every point x contained in a Z-slice of A of diameter less than ε there is some n such that $x \notin E_n$. Indeed, suppose that $x \in A \cap H$ with $H \in \mathbb{H}(Z)$ and $diam(H \cap A) < \varepsilon$. Clearly we have $E_n \subset E[H^n]$. Condition *iii)*

of Lemma 4 shows that for some n then $x \notin E_n$.
Finally, we have just to define $C_n = co(E_n)$.

Proof of Theorem 3. The equivalence of statements 6) and 7) follows from the transitive property of P. Clearly, statement 5) of Theorem 2 implies statement 6). We only have to prove the converse. Suppose that X has $P(\|.\|, Z)$ with a sequence of sets (A_n). It is not a restriction to assume that every A_n is bounded. For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ consider the sequence of convex sets $(C^{\varepsilon}_{n,m})_{m=0}^{\infty}$ given by Lemma 5 for $A = A_n$. The thesis in Lemma 5 entails that the countable family of convex sets

$$\{C^{1/k}_{n,m} : \ k, n, m \in \mathbb{N}\}$$

satisfies statement 5) of Theorem 2.

5 Applications and related results

We give a adaptation of the transfer technique developed in [16] which involves topologies of type $\sigma(X, Z)$. If $T : X \to Y$ is linear operator between Banach spaces we can consider on X the seminorm $\|x\|_T = \|T(x)\|$.

Theorem 4 *Let X and Y be Banach spaces, $Z \subset X^*$ a quasi norming subspace and $T : X \to Y$ a bounded one to one operator. Suppose that Y is LUR, X has $P(\|.\|, \|.\|_T)$ and that there is a norming subspace $W \subset Y^*$ such that T is $\sigma(X, Z)$-$\sigma(Y, W)$-continuous. Then X admits an equivalent $\sigma(X, Z)$-lsc LUR norm.*

Proof of Theorem 3. Suppose that $W \subset Y^*$ is a norming subspace such that T is $\sigma(X, Z)$-$\sigma(Y, W)$-continuous. Then Y has $P(\|.\|, W)$ with some sequence $A_n \subset Y$. Since the elements of $\mathbb{H}(W)$ can be lifted by T to elements of $\mathbb{H}(Z)$ we deduce that X has $P(\|.\|_T, Z)$. If X also has $P(\|.\|, \|.\|_T)$, then X has $P(\|.\|, Z)$ so it is renormable by a $\sigma(X, Z)$-lsc LUR norm.

It is shown in [16] that a sufficient condition to X have $P(\|.\|, \|.\|_T)$ is that for every bounded sequence (x_n) in X such that $\|x_n - x\|_T$ we have that $x \in \overline{sp(x_n)}^{\|\cdot\|}$ (in particular when (x_n) converges weakly to x).

Proof of Corollary 1. The proof is a modification of the proof of Corollary 8 in [16]. For every $n \in \mathbb{N}$ let $\|.\|_n$ be an equivalent norm on $C(K_n)$ with $\|.\|_n \leq \|.\|_\infty$. Let $(Y, \|.\|)$ be the l^2-sum of the family of Banach spaces $\{(C(K_n), \|.\|_n)\}_{n\in\mathbb{N}}$. Consider the operator $T : C(K) \to Y$ defined by $T(x) = (\frac{1}{n}x|_{K_n})_{n=1}^{\infty}$. In [16] it is shown that $C(K)$ has $P(\|.\|, \|.\|_T)$. On the other hand, one can realize that T is continuous from the pointwise topology of $C(K)$ to the topology on Y of pointwise convergence on each coordinate. To finish the proof apply Theorem 3.

Proposition 1 *Let X^* be a dual Banach space. Then X^* admits a dual rotund norm if and only if X^* has convex-$P(w^*, X)$.*

Proof. If X^* has a dual rotund norm then every point of S_{X^*} is a extreme point of B_{X^*}. By Choquet's Lemma the X-slices at every point of S_{X^*} are a neighbourhood basis for the weak* topology in B_{X^*}. We can apply Lemma 2, as we have done before, to obtain that X^* has $P(w^*, X)$ with the balls having center at the origin and rational radius.
Suppose now that X^* has $P(w^*, X)$ with some sequence of convex sets (A_n). Proceding like in the proof of the Theorem 2 we can suppose that the sets (A_n) are norm open. Take a point $a_n \in A_n$ and consider f_n the Minkowski functional of $\overline{A_n}^{w^*}$ with respect to a_n. Define a positive convex function F by the formula

$$F(x)^2 = \|x\|^2 + \sum_{n=1}^{\infty} \alpha_n f_n(x)^2 + \sum_{n=1}^{\infty} \alpha_n f_n(-x)^2$$

where (α_n) are positive numbers chosen in such a way that the series converges uniformly on bounded subsets of X^* and such that the symmetric convex set $B = \{x \in X : F(x) \leq 1$ contains 0 as an interior point. From the w^*-lsc of F we deduce that B is the unit ball of an equivalent dual norm $|||.|||$ on X^*. We claim that $|||.|||$ is a rotund norm. Suppose that $|||x_1||| = |||x_2||| = |||(x_1+x_2)/2||| = 1$. We have that

$$\frac{F(x_1)^2 + F(x_2)^2}{2} - F(\frac{x_1 + x_2}{2})^2 = 0$$

The proof of Lemma 1 entails that

$$f_n(x_1) = f_n(x_2) = f_n(\frac{x_1 + x_2}{2})$$

for every $n \in \mathbb{N}$. If x_1 and x_2 were different then we would find a w^*-closed w^*-neighbourhood V of $(x_1 + x_2)/2$ that not contains x_1 neither x_2. By property P we can find $n \in \mathbb{N}$ and $H \in \mathbb{H}(X)$ such that

$$\frac{x_1 + x_2}{2} \in A_n \cap H \subset V$$

In particular $f_n((x_1 + x_2)/2) < 1$. Now, x_1 or x_2 belongs to H. Suppose that x_1 does. Since $\overline{A_n}^{w^*} \cap H \subset V$ we deduce that x_1 cannot belong to $\overline{A_n}^{w^*}$, and thus, $f_n(x_1) > 1$. This contradiction shows that $x_1 = x_2$.

Proposition 2 *If X^* is a dual Banach space such that the weak and the weak* topologies coincide on the unit sphere, then X^* admits a dual LUR norm.*

Proof. If $(S_{X^*}, w^*) = (S_{X^*}, w)$ then X^* has convex-$P(w, w^*)$. This can be proved using a similar argument to the proof of 2) $\Rightarrow$ 5) of Theorem 2. In the other hand, it is well known that the coincidence of the weak and the weak* topologies on the unit sphere implies that X^* has RNP, so by [9], X^* admits a LUR norm. Statement 4) of Theorem 2 entails that X^* admits a dual LUR norm.

Proof of Theorem B.
$i) \Leftrightarrow iii) \Rightarrow ii)$. It follows from Theorem A.
$ii) \Rightarrow i)$. It is a consequence of Proposition 2.
$i) \Rightarrow iv)$. It is obvious.
$iv) \Rightarrow i)$ Since X^* admits a dual rotund norm, by Lemma 4 it has $P(w^*, X)$. This together with $P(\|.\|, w^*)$ gives that X^* has $P(\|.\|, X)$ and thus it admits a dual LUR norm by Theorem A.

Given a Banach space X and a total subspace $Z \subset X^*$ consider the following construction. For a convex set B and $\varepsilon > 0$ take

$$(B)'_\varepsilon = \{x \in B : \ \forall x \in H \in \mathbb{H}(Z), \ diam(B \cap H) > \varepsilon\}$$

Now we define by transfinite induction the sets (B^α_ε) as follows

$$B^0_\varepsilon = B_X$$

$$B^{\alpha+1}_\varepsilon = (B^\alpha_\varepsilon)'_\varepsilon$$

and for α a limit ordinal

$$B^\alpha_\varepsilon = \bigcap_{\beta<\alpha} B^\beta_\varepsilon$$

Now take $\delta_Z(X, \varepsilon) = \inf\{\alpha : \ B^\alpha_\varepsilon = \emptyset$, if it exists, and $\delta_Z(X) = \sup_{\varepsilon>0} \delta_Z(X, \varepsilon)$. When defined, $\delta_Z(X)$ is called the Z-dentability index of X, see [1].
Lancien has shown in [14] that for a Banach space the condition $\delta_{X^*}(X) < \omega_1$, where ω_1 is the first uncountable ordinal, entails the existence of a LUR norm on X. He also proved that in the case of a dual space X^*, if $\delta_X(X^*) < \omega_1$ then X^* admits a dual LUR norm. His method involves distances to sets and it seems difficult to apply it to other topologies.

Proposition 3 *Let X be a Banach space, $Z \subset X^*$ a quasi-norming linear subspace. If $\delta_Z(X) < \omega_1$ then X admits an equivalent $\sigma(X, Z)$-LUR norm.*

Proof. The countable family of convex sets $\bigcup_{n=1}^\infty \{B^\alpha_{1/n} : \alpha < \delta_Z(X, 1/n)\}$ satisfies statement 5) of Theorem 2.

A Banach space X is said weakly countably determined (WCD) if there exists a sequence (K_n) of w^*-compact sets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. A classic result of Vašak [20] shows that a WCD Banach space admits a LUR norm. We proved in [18] that a WCD Banach space admits $\sigma(X, Z)$-Kadec norms for every quasi norming Z. As consequence of point $ii)$ of Theorem A we obtain the following.

Corollary 2 *Let X be a WCD Banach space, $Z \subset X^*$ a quasi-norming linear subspace. Then X admits an equivalent $\sigma(X, Z)$-lsc LUR norm.*

When X^* is a WCD dual space, we deduce the existence of a dual LUR norm. That result was obtained by M. Fabian in [8].

Clearly, a separable Banach space satisfies statement 5) in Theorem 2, and we obtain a well known result of Kadec, see [4, p. 178]. If we revise the proof of Theorem 2, we shall realize that in the case of a separable space the slices has no role. That allows us to improve the Theorem of Kadec obtaining τ-lsc of the LUR norm for more general topologies.

Proposition 4 *Let X be a separable Banach space and τ a vector topology on X coarser than the norm topology such that $\overline{B_X}^\tau$ is bounded. Then X admits an equivalent τ-lower semicontinuous LUR norm.*

Proof. Taking the Minkowski functional of $\overline{B_X}^\tau$ we can suppose X endowed with a τ-lower semicontinuous norm. Let (a_n) be a norm dense subset of X. Consider the τ-lower semicontinuous convex function F defined by

$$F(x)^2 = \frac{1}{3}\|x\|^2 + \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|a_n - x\|^2}{\|a_n\|^2+1} + \frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|a_n + x\|^2}{\|a_n\|^2+1}$$

Let $B = \{x \in X : F(x) \le 1\}$. An easy calculus shows that $B_X \subset B \subset 2B_X$. Let $|||.|||$ be the functional of Minkowski of B, then $|||.|||$ is a τ-lower semicontinuous equivalent norm. We claim that $|||.|||$ is a LUR norm. Indeed, let $x, x_k \in X$ with $|||x||| = |||x_k||| = 1$ and $\lim_k |||x + x_k||| = 2$. It is not difficult to see that $F(x) = F(x_k)$ and $\lim_k F(\frac{x+x_k}{2}) = 1$. From Lemma 1 we deduce for every $n \in \mathbb{N}$ that $\lim_k \|a_n - x_k\| = \|a_n - x\|$. Since x can be aproximated by points of the sequence (a_n), this shows that $\lim_k \|x - x_k\| = 0$.

Remark 4 *G. Godefroy has pointed out us that property * JNR in a dual Banach space implies the existence of an equivalent dual LUR norm. The proof, which depends on the results of this paper, will appear elsewhere.*

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Matias Raja
Mathématiques Pures
Université Bordeaux 1
351, cours de la Libération
33405 TALANCE Cedex
FRANCE
e-mail: raja@math.u-bordeaux.fr