On some class of Borel measurable maps and absolute Borel topological spaces

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Abstract

We introduce a class of Borel measurable maps between topological spaces which is stable under usual operations. We characterize those completely regular topological spaces which are Borel sets in every regular embedding.

1 Introduction

In the following (X, τ) will denote a Hausdorff topological space. The Borel σ-algebra of X will be denoted $Borel(X, \tau)$, that is, the smallest family of subsets of X which contains the topology τ and is closed by countable unions and complementaries in X. Given another topological space (Y, δ), a map $f : X \to Y$ is said to be Borel (measurable) if $f^{-1}(V)$ belongs to $Borel(X, \tau)$ for every open V from δ.

It is well known that all the Borel sets in a metric space can be generated by an alternative application (indexed by the countable ordinal numbers) of the operations "δ" (countable intersections) and "σ" (countable unions) starting with the open sets (or the closed sets). For nonmetrizable topologies a little change is required and the suitable definition is as follows.

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Definition 1.1 *Let (X, τ) be a topological space. The family of the additive sets of class α ($\mathcal{A}_\alpha$) and the family of the multiplicative sets of class α ($\mathcal{M}_\alpha$) are constructed for every countable ordinal α by the following inductive process:*

i) $\mathcal{A}_0$ consists of all the τ-open sets and $\mathcal{M}_0$ consists of all the τ-closed sets.

ii) If $\alpha > 0$ then the sets of $\mathcal{A}_\alpha$ are of the form $\bigcup_{n=1}^{\infty}(A_n \cap B_n)$ and the sets of $\mathcal{M}_\alpha$ are of the form $\bigcap_{n=1}^{\infty}(A_n \cup B_n)$, where $A_n \in \mathcal{A}_{\alpha_n}$ and $B_n \in \mathcal{M}_{\alpha_n}$ with $\alpha_n < \alpha$.

Every Borel set $A \in Borel(X, \tau)$ belongs to some additive (resp. multiplicative) family $\mathcal{A}_\alpha$ (resp. $\mathcal{M}_\alpha$). Then we say that A is of additive (resp. multiplicative) class α. The classification of Borel subsets allows to introduce a classification of Borel functions.

Definition 1.2 *Let (X, τ) and (Y, δ) be topological spaces. A map $f : X \to Y$ is said to be Borel of class α if $f^{-1}(V) \in \mathcal{A}_\alpha$ for every $V \in \delta$. A Borel map is said to be classifiable if it is of class α for some countable ordinal.*

We may regard Borel sets as the "countably constructible" sets starting from the sets of the topology. But this intuitive idea disappears when considering Borel functions (even classifiable). In this paper we propose a class of maps between topological spaces which try to fill the idea of "countably constructibity" for Borel maps. We call p-Borel maps these maps and the definition is as follows:

Definition 1.3 *Let (X, τ) and (Y, δ) be topological spaces. A map $f : X \to Y$ is said to be p-Borel if there is sequence (A_n) of τ-Borel sets in X such that for every $x \in X$ and every $V \in \delta$ with $f(x) \in V$ there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $f(A_n \cap U) \subset V$.*

This definition is motivated by a suficient condition for coincidence of σ-algebras employed in [14], which generalizes arguments of Edgar [2] and Talagrand [15]. There are also conexions with the maps having a σ-relatively discrete base considered by Hansell in [7]. The properties of p-Borel maps are studied in Section 2. The first important fact is that every p-Borel map is Borel measurable of bounded class. We study the behaviour of p-Borel maps with respect to composition, sums and limits among other properties. We also show that p-Borel maps are Lusin measurable for every Radon measure on the domain. In Section 3 we give some examples of p-Borel maps. A deep result of Hansell is used to show that every Borel map of bounded class from a complete metric space into a metric space is p-Borel.

In Section 4 we shall use the notion of p-Borel map to give an intrinsic characterization of the completely regular absolute Borel spaces.

Definition 1.4 *A topological space X is said to be absolute Borel if for every embedding $i : X \to Z$ into a regular topological space, then $i(X)$ is a Borel subset of Z.*

It is well known that a metrizable space is absolute Borel, if and only if, it is homeomorphic to a Borel subset of a complete metric space [10]. More recently, metrizable absolute Borel spaces have been characterized internally in terms of complete sequences of covers [11]. Our main result uses the notion of Čech-complete topological space and it can be stated as follows.

Theorem 1.5 *Let (X, τ) a completely regular topological space. Then the following statements are equivalent:*

i) *(X, τ) is absolute Borel.*

ii) *(A, τ) is absolute Borel for every $A \in Borel(X, \tau)$.*

iii) *There is a Čech-complete topology δ on X finer than τ such that the identity map $\mathbb{I} : (X, \tau) \to (X, \delta)$ is p-Borel.*

A consequence of the theorem is an extension of a classical result of Lusin and Souslin about injective Borel measurable maps from a Polish space into a metrizable one [8]. We also show the following: if X is a Banach space and X^{**} denotes its bidual, then the property of being X a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$ in X^{**} endowed with the weak*-topology is a weak invariant of X. The last result of the paper states that Borel absolute spaces are preserved by p-Borel isomorphisms.

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2 Properties of p-Borel maps

Given families $\Sigma_1, \Sigma_2, \ldots$ of subsets of X, we shall denote $\text{top}(\Sigma_1, \Sigma_2, \ldots)$ the topology on X generated by these families, that is, the smallest topology on X containing the sets of each family. The next result shows that we may think of the p-Borel maps as a kind of "continuous" maps. The proof is left to the reader (we shall omit the proofs if they are very easy or rutinary).

Proposition 2.1 *Let (X, τ) and (Y, δ) be topological spaces. A map $f : X \to Y$ is p-Borel if and only if there is a countable collection $\{A_n : n \in \mathbb{N}\}$ of Borel subsets of X such that f is continuous when X is endowed with* $\text{top}(\tau, \{A_n : n \in \mathbb{N}\})$.

The following lemma contains some properties of the additive and multiplicative families of Borel sets in a topological space.

Lemma 2.2 *Let (X, τ) be a topological space, $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$ the families defined above. Then the following holds:*

i) $A \in \mathcal{A}_\alpha$ *if and only if* $X \setminus A \in \mathcal{M}_\alpha$.

ii) $\mathcal{A}_\alpha$ *is stable under countable unions and* $\mathcal{M}_\alpha$ *is stable under countable intersections.*

iii) $\mathcal{A}_\alpha$ *is stable under finite intersections and* $\mathcal{M}_\alpha$ *is stable under finite unions.*

iv) If $\alpha < \beta$ *then* $\mathcal{A}_\alpha \cup \mathcal{M}_\alpha \subset \mathcal{A}_\beta \cap \mathcal{M}_\beta$.

v) If $\alpha > 0$ *then* $\mathcal{A}_{\alpha+1} = (\mathcal{M}_\alpha)_\sigma$ *and* $\mathcal{M}_{\alpha+1} = (\mathcal{A}_\alpha)_\delta$.

vi) $\bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha = Borel(X, \tau)$.

Theorem 2.3 *Every p-Borel map is Borel measurable of bounded class.*

Proof. Let V be a δ-open set in Y. We shall check that $f^{-1}(V)$ is a τ-borel set in X. For every $x \in f^{-1}(V)$ we can find $n_x \in \mathbb{N}$ and $U_x \in \tau$ such that $f(A_{n_x} \cap U_x) \subset V$. Now note that

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} A_{n_x} \cap U_x = \bigcup_{n=1}^{\infty} \bigcup_{n_x = n} A_{n_x} \cap U_x = \bigcup_{n=1}^{\infty} A_n \cap (\bigcup_{n_x = n} U_x)$$

is a τ-Borel set in X. Note that if α is a countable ordinal which bounds the additive class of the sets (A_n), then for every $V \in \delta$ we have that $f^{-1}(V)$ is of additive class α.

The following results establishes some properties of the class of p-Borel maps. Note that some of them are well known properties of the class of Borel maps.

Proposition 2.4 *In order to a given map $f : X \to Y$ be p-Borel, it is enough to check the condition in Definition 1.3 for V belonging to a subbasis of δ.*

Proposition 2.5 *If $f : X \to Y$ is p-Borel, and $X_0 \subset X$ and $Y_0 \subset Y$ are such that $f(X_0) \subset Y_0$. Then $f|_{X_0} : X_0 \to Y_0$ is p-Borel for the relative topologies.*

Proposition 2.6 *Let (X_i, τ_i), $i = 1, 2, 3$ be topological spaces, and $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are p-Borel, then $g \circ f : X_1 \to X_3$ is also p-Borel.*

Proof. Let $(A_n) \subset X_1$ a sequence of sets satisfying Definition 1.3 for f and let $(B_n) \subset X_2$ satisfying Definition 1.3 for g. After Theorem 2.3, the sets $f^{-1}(B_n)$ are τ_1-Borel. It is easy to see that the countable familly $(A_n \cap f^{-1}(B_m))$ of τ_1-Borel sets satisfies Definition 1.3 for the map $g \circ f$.

Proposition 2.7 *Let (X, τ) and (Y_i, δ_i) topological spaces for $i \in I$ where I is finite or countable. Let $f_i : X \to Y_i$ be p-Borel maps for $i \in I$. Then the map $f : X \to \prod_{i \in I} Y_i$ defined by $f(x) = (f_i(x))_{i \in I}$ is p-Borel.*

Proof. Let (A^i_n) a sequence of τ-Borel sets satisfying Definition 1.3 for f_i. Now, for every finite subset $F \subset I$ and every finite sequence $(n_i) \subset \mathbb{N}$ for $i \in F$ consider the set $\bigcap_{i \in F} A^i_{n_i}$. In this way we obtain a countable familly of τ-Borel sets. We claim that this familly satisfies Definition 1.3. Let $x \in X$ and $V \subset \prod_{i \in I} Y_i$ an open neigbourhood of $f(x)$ that we can suppose that is of the form $\prod_{i \in I} V_i$ where $V_i = Y_i$ for $i \in I \setminus F$ and F is finite. For every $i \in F$ we set $A^i_{n_i}$ and $U_i \in \tau$ such that $f_i(A^i_{n_i} \cap U_i) \subset V_i$. Then we have that

$$f((\bigcap_{i \in F} A^i_{n_i}) \cap (\bigcap_{i \in F} U_i)) \subset \prod_{i \in I} V_i = V$$

which finish the proof of the claim.

Corollary 2.8 *Let (X, τ) be a topological space, $(Y, *, \delta)$ be a topological group, and $f, g : X \to Y$ be p-Borel maps. Then $f * g$ is p-Borel as well.*

Proof. The map $x \to (f(x), g(x))$ is p-Borel after Proposition 2.7 and the composition with the product map is p-Borel after Proposition 2.6.

In similar way to Proposition 2.7, we can prove the following.

Proposition 2.9 *Let (X_i, τ_i) and (Y_i, δ_i) topological spaces for $i \in I$ finite or countable. Let $f_i : X_i \to Y_i$ be p-Borel maps for $i \in I$. Then the map $f : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ defined by $f((x_i)_{i \in I}) = (f_i(x_i))_{i \in I}$ is p-Borel.*

Corollary 2.10 *If $f : X \to Y$ is a p-Borel map between topological spaces, then the set*

$$\mathbf{graf}(f) = \{(x, y) \in X \times Y : f(x) = y\}$$

is Borel in $X \times Y$ (with the product topology).

Proposition 2.11 *Let (X, τ) be a topological space, Y a set and $f : X \to Y$ a map. Let (δ_n) be a sequence of topologies on Y such that for every $n \in \mathbb{N}$, when endowed Y with δ_n, the map f is p-Borel. Then f is p-Borel when Y is endowed with* $\mathrm{top}(\{\delta_n : n \in \mathbb{N}\})$.

The proof of the properties considered above depends mainly upon the similarity between continuity and property p. The next result shows that, when Y is metrizable, p-Borel functions, like Borel functions, are stable under pointwise limits of sequences.

Theorem 2.12 *Let $f : X \to Y$ be a map from a topological space X to a metric space Y. If f is the pointwise limit of a sequence (f_n) of p-Borel maps, then f is also a p-Borel map.*

Proof. Let (A_n^i) be a sequence satifying Definition 1.3 for f_i. Fix a compatible metric d on Y. From Proposition 2.7 and the continuity of d we deduce that the map $d(f_i(x), f_j(x))$ is p-Borel, and thus the sets

$$E_k^i = \{x \in X : \ d(f_i(x), f_j(x)) \leq 1/k, \ \forall\, j \geq i\}$$

are τ-Borel. We claim that the countable family of sets $\{A_n^i \cap E_k^i : \ i, n, k \in \mathbb{N}\}$ satisfies Definition 1.3 for f. Fix $x \in X$, $\varepsilon > 0$ and $k > 3/\varepsilon$. There is $i \in \mathbb{N}$

such that $d(f_j(x), f(x)) < 1/2k$ for every $j \geq i$, thus $x \in E_k^i$. Now take $U \in \tau$ and $n \in \mathbb{N}$ such that $x \in A_n^i \cap U$ and $diam(f_i(A_n^i \cap U)) \leq \varepsilon/3$. If $y \in A_n^i \cap E_k^i \cap U$ then

$$d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)) < \varepsilon$$

which proves the claim.

Remark 2.13 *The pointwise limit of a sequence of continuous functions satisfies Definition 1.3 with a sequence (A_n) of τ-closed sets.*

Remark 2.14 *With the notation of the proof, if f is limit uniformly of the sequence (f_n), then it is easy to see that f is a p-Borel map with the countable family of Borel sets $(A_n^i)_{n,i}$. In particular, this shows that a uniform limit of real Borel measurable maps of class α is also of class α.*

Corollary 2.15 *Let (X, τ) be a topological space and (Y, d) a metric space. Then the smallest family of maps from X to Y which contains the continuous maps and is closed by pointwise limits of sequences is composed of p-Borel maps.*

The following results show that p-Borel maps preserve some good properties of measures. All the measures will be supposed to be positive and finite.

Definition 2.16 *A Borel measure μ is said to be smooth if for every family (U_α) of open sets there is a countable sets of indexes (α_n) such that*

$$\mu(\bigcup_{\alpha} U_\alpha) = \mu(\bigcup_{n=1}^{\infty} U_{\alpha_n})$$

A Borel measure μ is said to be a Radon measure if for every Borel set $A \subset X$ and every $\varepsilon > 0$ we can find a compact set $K \subset A$ such that

$$\mu(A) < \mu(K) + \varepsilon$$

Theorem 2.17 *Let $f : X \to Y$ be p-Borel. If μ is a smooth Borel measure on X then the image measure $f(\mu)$ is smooth on Y.*

Proof. Suppose that f is p-Borel with with some sequence (A_n) of subsets of X. Let $(V_i)_{i\in I}$ a family of open sets in Y. Reasoning as in the proof of Theorem 2.3, for every index $i \in I$ and every $n \in \mathbb{N}$ there is a open set $U_{n,i}$ in X such that

$$f^{-1}(V_i) = \bigcup_{n=1}^{\infty} A_n \cap U_{n,i}$$

For every fixed $n \in \mathbb{N}$ consider the family $(U_{n,i})_{i\in I}$ of open sets in X. Since μ is smooth we can take a countable subfamily $I_n \subset I$ such that

$$\mu(\bigcup_{i\in I} U_{n,i}) = \mu(\bigcup_{i\in I_n} U_{n,i})$$

thus

$$\mu(A_n \cap \bigcup_{i\in I} U_{n,i}) = \mu(A_n \cap \bigcup_{i\in I_n} U_{n,i})$$

and we deduce that

$$\mu(\bigcup_{n=1}^{\infty} \bigcup_{i\in I} A_n \cap U_{n,i}) = \mu(\bigcup_{n=1}^{\infty} \bigcup_{i\in I_n} A_n \cap U_{n,i})$$

Taking $I_0 = \bigcup_{n=1}^{\infty} I_n$ we obtain

$$f(\mu)(\bigcup_{i\in I} V_i) = f(\mu)(\bigcup_{i\in I_0} V_i)$$

and thus $f(\mu)$ is smooth.

Theorem 2.18 *Let $f : X \to Y$ be p-Borel. For every regular measure μ on X and every $\varepsilon > 0$ there is a closed set $F \subset X$ with $\mu(F) > \mu(X) - \varepsilon$ such that f restricted to F is continuous. Moreover, if μ is Radon the closed set F can be chosen compact.*

Proof. Without loss of generality we can suppose that $\mu(X) = 1$. Let τ be the topology on X and suppose that f is p-Borel with some sequence (A_n) of subsets of X. Fixed $\varepsilon > 0$, for every $n \in \mathbb{N}$ take τ-closed sets $F_n \subset A_n$ and $F'_n \subset X \setminus A_n$ such that

$$\mu(F_n \cup F'_n) > 1 - 2^{-n}\varepsilon$$

Take $F = \bigcap_{n=1}^{\infty}(F_n \cup F'_n)$. Then we have that F is a τ-closed set with $\mu(K) > 1 - \varepsilon$. By construction, the topologies τ and $\mathrm{top}(\tau, \{A_n : n \in \mathbb{N}\})$ coincide on F. Since f is continuous for the topology $\mathrm{top}(\tau, \{A_n : n \in \mathbb{N}\})$, then f is τ-continuous when restricted to F. When μ is Radon, the set F can be made compact just taken F_n and F'_n compact.

Definition 2.19 *It is said that f is Lusin μ-measurable if for every $\varepsilon > 0$ there is a compact set $K \subset X$ with $\mu(X \setminus K) < \varepsilon$ such that $f|_K$ is continuous.*

Corollary 2.20 *A p-Borel map $f : X \to Y$ is Lusin μ-measurable for every Radon measure μ on X. In particular the image measure $f(\mu)$ on Y is Radon.*

The sets of a topological space (X, τ) which are measurable for every Radon measure are called universally measurable. The family of all universally measurable subsets of (X, τ), denoted $Univ(X, \tau)$ is a σ-algebra containing the Borel subsets.

Corollary 2.21 *Let X be a set and $\tau_2 \subset \tau_1$ be topologies on X. If the identity map*

$$\mathbb{I} : (X, \tau_2) \to (X, \tau_1)$$

is p-Borel, then

$$Univ(X, \tau_1) = Univ(X, \tau_2)$$

The last result of the section allows us to make continuous p-Borel maps without loss of certain good properties of the topology. For the proof we shall use the following lemma, which will also be useful in the last section.

Lemma 2.22 *Let X be a set and let $\mathfrak{C}$ be a class of topologies on X which satisfies these two properties:*

i) If $\tau \in \mathfrak{C}$ and S is τ-closed, then $\mathrm{top}(\tau, \{S\}) \in \mathfrak{C}$.

ii) If $\{\tau_n\} \subset \mathfrak{C}$ is a sequence, then $\mathrm{top}(\{\tau_n : n \in \mathbb{N}\}) \in \mathfrak{C}$.

Then the following holds:

1) Suppose that (τ_α) is a transfinite sequence such that $\tau_1 \in \mathfrak{C}$, $\tau_{\alpha+1} =$ $\mathrm{top}(\tau_\alpha, \{F_n^\alpha : n \in \mathbb{N}\})$ where every F_n^α is τ_α-closed and if α a limit ordinal $\tau_\alpha = \mathrm{top}(\tau_\beta : \beta < \alpha)$. Then $\tau_\gamma \in \mathfrak{C}$ and

$$\tau_\gamma = \mathrm{top}(\tau_1, \{F_n^\alpha : n \in \mathbb{N}, \alpha < \gamma\})$$

for every countable ordinal γ. The sets $\{F_n^\alpha : n \in \mathbb{N}\}$ are of additive class α in (X, τ_1).

2) Given $\tau_1 \in \mathfrak{C}$ *and* $(A_n) \subset Borel(X, \tau_1)$ *of additive class* γ *there is a transfinite sequence* $(\tau_\alpha)_{\alpha \leq \gamma} \subset \mathfrak{C}$ *as in 1) such that every* A_n *is* τ_γ*-open. The sets* $\{F_n^\alpha : n \in \mathbb{N}, \alpha < \gamma\}$ *are of additive class* γ *in* (X, τ_1)*.*

Proof. It is not difficult using transfinite induction.

The following result is well known in the context of Polish spaces.

Proposition 2.23 *Let* (X, d) *be a complete metric space,* Y *a topological space and* (f_n) *a sequence of* p*-Borel maps from* X *to* Y*. Then there is a complete metric* d_0 *on* X *finer than* d *such that every* f_n *is* d_0*-continuous and the identity map* $\mathbb{I} : (X, d) \to (X, d_0)$ *is* p*-Borel. In particular,* (X, d) *and* (X, d_0) *have the same density character.*

Proof. Let $(A_m^n)_m$ be a sequence of Borel sets satisfying Definition 1.3 for f_n. The class of completely metrizable topologies finer than d satisfies the hypothesis of Lemma 2.22. Indeed, if d_1 is a complete metric finer than d and F is d_1-closed, then take d_2 and d_3 complete metrics bounded by 1 defined respectively on F and $X \setminus F$. Take $d_0(x, y)$ equal to $d_1(x, y)$, $d_2(x, y)$ or 1 depending if the points x and y lies in F, $X \setminus F$ or different sets respectively. It is easy to check that d_0 is a complete metric compatible with $\text{top}(d_1, \{F\})$. This proves condition $i)$ of Lemma 2.22. To verify condition $ii)$ of the lemma, suppose that (d_n) is a sequence of complete metrics on X finer than d. It is not difficult to show that $d_0 = \sum_{n=1}^{\infty} 2^{-n} \min\{d_n, 1\}$ is complete metric compatible with $\text{top}(\{d_n : n \in \mathbb{N}\})$. To finish the proof of the proposition, apply Lemma 2.22 to the countable family $(A_m^n)_{n,m}$ in order to get a completely metrizable topology τ_γ finer than d which makes every f_n continuous. The afirmation about the density character is consequence of Proposition 3.6.

3 Examples of p-Borel maps

In this section we shall give some suficient conditions for a map between topological spaces to be p-Borel. The first and second examples are just trivial remarks.

Example 3.1 *Let* $f : X \to Y$ *a map between topological spaces such that there is a sequence Borel sets* $A_n \subset X$ *such that* $f|_{A_n}$ *is continuous and* $X = \bigcup_{n=1}^{\infty} A_n$*. Then* f *is* p*-Borel.*

We call the maps as above σ-continuous. In particular, Borel measurable maps with countable range are σ-continuous, and so p-Borel (if the range is finite, we have the simple maps). When the space Y is metric, it is possible to show that a p-Borel map $f : X \to Y$ is uniform limit of a sequence of σ-continuous maps.

Example 3.2 *Let $f : X \to Y$ a Borel measurable map. Suppose that $f(X)$ has a countable network of relatively Borel sets. Then f is p-Borel.*

It is well known that if X is a Polish space, Y metrizable space and $f : X \to Y$ a Borel measurable map, then $f(X)$ is separable. Thus we have the following.

Corollary 3.3 *Any Borel measurable map from a Polish space into a metric space is p-Borel.*

In a complete metric space a subset is called analytic if it can be obtained by Souslin's operation applied to closed sets. This notion of analytic subset coincides with the classical one if the metric space is separable. An indexed family $\{H_i : i \in I\}$ is said σ-discretely decomposable if for each $i \in I$ there is a decomposition $H_i = \bigcup_{n=1}^{\infty} H_{i,n}$ such that the family $\{H_{i,n} : i \in I\}$ is discrete for every $n \in \mathbb{N}$. A deep result of Hansell [6] allows us to remove the separability hypothesis from the metric case.

Theorem 3.4 (Hansell) *Let X be a complete metric space. Then every disjoint family with the property that arbitrary unions of sets from the family are analytic is σ-discretely decomposable.*

The following is a reformulation of Hansell's result about the existence of σ-discrete bases for Borel measurable map between metric spaces.

Example 3.5 *If X is a complete metric space and Y is metrizable, then every Borel map of bounded class $f : X \to Y$ is p-Borel.*

Proof. Suppose that f is of class α. Let $\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ a basis of Y where every $\mathfrak{B}_n$ is disjoint. Since f is Borel measurable, $f^{-1}(\mathfrak{B}_n)$ satisfy the hypothesis of the theorem above, so it is σ-discretely decomposable. For every $V \in \mathfrak{B}_n$ put $f^{-1}(V) = \bigcup_{m=1}^{\infty} H(m, V)$ where $\{H(m, V) : V \in \mathfrak{B}_m\}$ is discrete. Define

$$A(n, m) = \bigcup_{V \in \mathfrak{B}_n} f^{-1}(V) \cap \overline{H(m, V)}$$

A discrete union of sets of additive class α is also of class α (see [10]), in particular, the sets $A(n, m)$ are Borel. It is easy to check that f has property p with the countable family $\{A(n, m) : n, m \in \mathbb{N}\}$, thus f is p-Borel.

Fleissner's Axiom (see [4]) implies that every disjoint family in a metric space with the property that arbitrary unions of sets from the family are analytic is σ-discretely decomposable. Assuming this, the proof of Example 3.5 shows that every Borel map of bounded class between metric spaces is p-Borel.

Proposition 3.6 *Let $f : X \to Y$ be a p-Borel map between topological spaces and let $A \subset X$ be a subset. If $\aleph$ is a cardinal which bounds the density of all the subsets of A, the $\aleph$ bounds the density of all the subsets of $f(A)$.*

Proof. Take $S \subset f(A)$. We have to construct a dense subset of S of cardinality not greater than $\aleph$. Suppose that f satisfies Definition 1.3 with a sequence of sets $(A_n) \subset X$. For every $n \in \mathbb{N}$, let D_n be a dense subset of $A_n \cap f^{-1}(S)$ of cardinality not greater than $\aleph$. We claim that $f(\bigcup_{n=1}^{\infty} D_n)$ is dense in S. Indeed, let $f(x) \in S$ any point. Take V a neighbourhood of $f(x)$. There is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $f(A_n \cap U) \subset V$. Since $A_n \cap f^{-1}(S) \cap U$ is nonempty, it contains points of D_n, and thus V contains points of $f(D_n)$.

The preceding result implies when X and Y are metrizable that the p-Borel image of separable subsets are separable.

Example 3.7 *Let τ be the usual topology on $\mathbb{R}$ and let δ be the discrete topology on $\mathbb{R}$. Under Martin's Axiom and the negation of the Continuum Hypothesis there exists a uncountable subset X of $\mathbb{R}$ such that every subset is a relative $\mathcal{F}_\sigma$ (see [12]). Consequently, the identity map $\mathbb{I} : (X, \tau) \to (\mathbb{R}, \delta)$ is first Borel class but it is not p-Borel.*

The following is a topological version of renorming results from [14].

Proposition 3.8 *Let (X, τ) and (Y, δ) be topological spaces, and let $f : X \to Y$ be a map. Then f is p-Borel if and only if there is a Borel measurable real function h on X such that $\delta\text{-}\lim_\omega f(x_\omega) = f(x)$ for every net (x_ω) such that $\tau\text{-}\lim_\omega x_\omega = x$ and $\lim_\omega h(x_\omega) = h(x)$.*

Proof. Suppose that f is p-Borel with a sequence (A_n). Let χ_A be the characteristic function of the set A. Consider the series

$$h(x) = \sum_{n=1}^{\infty} 3^{-n} \chi_{A_n}(x)$$

Let (x_ω) be a net with τ-$\lim_\omega x_\omega = x$ and $\lim_\omega h(x_\omega) = h(x)$. We want to show that δ-$\lim_\omega f(x_\omega) = x$. Firstly, we claim that

$$\lim_\omega \chi_{A_n}(x_\omega) = \chi_{A_n}(x)$$

for every $n \in \mathbb{N}$. Indeed, consider the map $T : \{0,1\}^{\mathbb{N}} \to [0,1]$ defined by $T((a_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} 3^{-1} a_n$. This map is continuous, from the product topology to the standard one on $[0,1]$, and one-to-one as a consequence of the inequality $1 > \sum_{n=1}^{\infty} 3^{-n} = 2^{-1}$. By compactness, the inverse $T^{-1} : T(\{0,1\}^{\mathbb{N}}) \to \{0,1\}^{\mathbb{N}}$ is continuous. The proof of the claim follows easily from the continuity of the projections on coordinates. Now, for every δ-neighbourhood V of $f(x)$ there is n and $U \in \tau$ such that $x \in A_n \cap U \subset V$. Since $\chi_{A_n}(x_\omega)$ must be constant for ω big enough, we deduce that $x_\omega \in A_n$. Also, for ω big enough $x_\omega \in U$, thus $f(x_\omega) \in V$ which shows the δ-convergence of $(f(x_\omega))$.
To see the converse suppose that there exists such function h. Let (B_n) be a countable basis of $\mathbb{R}$. The condition means that for every $x \in X$ and every $V \in \delta$ with $f(x) \in V$, there is $U \in \tau$ and $n \in \mathbb{N}$, with $x \in U$ and $h(x) \in B_n$, such that if $y \in U$ and $h(y) \in B_n$, then $f(y) \in V$. Clearly, this implies that f is p-Borel with the sequence $A_n = F^{-1}(B_n)$.

In [14] we have studied equivalent conditions on a Banach space for the identity map $\mathbb{I} : (X, weak) \longrightarrow (X, \|.\|)$ to be p-Borel. For the sake of completness we include the following.

Example 3.9 *If X is normed space which has an equivalent Kadec norm $\|.\|$, that is, the norm topology and the weak topology coincide on the unit sphere $S_{\|.\|} = \{x \in X : \|x\| = 1\}$, then the identity map $\mathbb{I} : (X, weak) \longrightarrow (X, \|.\|)$ is p-Borel.*

Proof. Using the homogenity of the norm it is easy to see that if $\|.\|$ is Kadec, then a net $(x_\omega) \subset X$ is norm convergent to a point x if and only if (x_ω) converges weakly to x and $\lim_\omega \|x_\omega\| = \|x\|$. Thus the function $h(x) = \|x\|$ satisfies Proposition 3.8.

As a consequence we obtain the result of Edgar [2], if a Banach space admits an equivalent Kadec norm then the weak Borel sets coincide with the norm Borel sets. An analogous result can be proved for a dual Banach space X^* having a w^*-Kadec norm, that is, the norm and the weak* topologies coincide on the unit sphere. In this case, $\mathbb{I} : (X^*, weak^*) \longrightarrow (X^*, \|.\|)$ is p-Borel, and this implies that $Univ(X^*, \|.\|) = Univ(X^*, weak^*)$ by Corollary 2.21. It is well known that for a Banach space X, it is always true that $Univ(X, \|.\|) = Univ(X, weak)$. But in the case of a dual Banach space X^*, a theorem of Edgar [2] states that the identity $Univ(X^*, \|.\|) = Univ(X^*, weak^*)$ is equivalent to the Radon-Nikodym property of X^*.

4 Absolute Borel spaces

For covenience we introduce the following definition from [14].

Definition 4.1 *Let Z be a set, τ_1 and τ_2 two topologies on Z. We say that a subset $X \subset Z$ has property $P(\tau_1, \tau_2)$ with a sequence $A_n \subset Z$ of sets, if for every $x \in X$ and every $V \in \tau_1$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$.*

Lemma 4.2 *Let X be a set and $\tau_2 \subset \tau_1$ two topologies on X. If X has $P(\tau_1, \tau_2)$ with τ_2-Borel sets, then*

$$Borel(X, \tau_1) = Borel(X, \tau_2)$$

Proof. The identity map $\mathbb{I} : (X, \tau_2) \to (X, \tau_1)$ is p-Borel.

In order to characterize the absolute Borel completely regular topological spaces we shall use the notion of Čech-complete topological space. The definition of Čech-complete space that we shall use is in fact a result by Frolik [5]. We prefer this definition because it is formulated in terms of the topology of the space.

Definition 4.3 *A completely regular topological space (X, δ) is said to be Čech-complete if it has a complete sequence of open covers, that is, there are δ-open covers $(\mathcal{S}_n)$ of X such that every filter $\mathfrak{F}$ in X has a cluster point provided that $\mathfrak{F} \cap \mathcal{S}_n \neq \emptyset$ for every $n \in \mathbb{N}$.*

Next lemma is part of Frolik's characterizations of Čech-complete topological spaces. The proof uses ideas that come from Sierpinski's results about completely metrizables subspaces of a regular topological space (see [9]).

Lemma 4.4 *A Čech-complete topological space (X,τ) is a $\mathcal{F}\cap\mathcal{G}_\delta$-set in every regular embedding. Conversely, if (X,τ) is a $\mathcal{F}\cap\mathcal{G}_\delta$-set in some compact space, then (X,τ) is Čech-complete.*

Proof. Suppose that (X,τ) is a dense subspace of a regular space $(Z,\widetilde{\tau})$. Let $(\mathcal{S}_n)$ be a complete sequence of open covers of X. We define for every $n\in\mathbb{N}$ the open sets

$$G_n=\{z\in Z:\exists U_{n,z}\in\widetilde{\tau}, z\in U_{n,z}, X\cap U_{n,z}\in\mathcal{S}_n\}$$

Clearly we have that $X\subset G_n$ for every $n\in\mathbb{N}$. We claim that $X=\bigcap_{n=1}^{\infty}G_n$. Indeed, take $z\in\bigcap_{n=1}^{\infty}G_n$. We have for every $n\in\mathbb{N}$ that $z\in U_{n,z}$ and $X\cap U_{n,z}\in\mathcal{S}_n$. Let $\mathfrak{F}$ be the filter of neighbourhoods of z. The regularity of $(Z,\widetilde{\tau})$ implies that $\bigcap_{U\in\mathfrak{F}}\overline{U}^{\widetilde{\tau}}=\{z\}$. By the density of X in Z, we have that $\mathfrak{F}|_X$ is also a filter and $X\cap U_{n,z}\in\mathfrak{F}|_X\cap\mathcal{S}_n$ for every $n\in\mathbb{N}$. Applying that $\mathcal{S}_n$ is a complete sequence of covers, we have that

$$\emptyset\neq\bigcap_{U\in\mathfrak{F}}\overline{(X\cap U)}^{\tau}\subset X\cap\bigcap_{U\in\mathfrak{F}}\overline{U}^{\widetilde{\tau}}$$

This implies that $z\in X$.
Now suppose that (X,τ) is $\mathcal{F}\cap\mathcal{G}_\delta$-set in a compact space $(Z,\widetilde{\tau})$. Changing Z by $\overline{X}^{\widetilde{\tau}}$, we may assume without loss of generality that X is a dense $\mathcal{G}_\delta$-set in Z. Put $X=\bigcap_{n=1}^{\infty}G_n$, where every G_n is $\widetilde{\tau}$-open. For every $n\in\mathbb{N}$ define $\mathcal{S}_n$ as the collection of the sets of the form $X\cap U$ where $U\in\widetilde{\tau}$ and $\overline{U}^{\widetilde{\tau}}\subset G_n$. Every filter $\mathfrak{F}$ must have a cluster point in Z by the compactness. If $\mathfrak{F}\cap\mathcal{S}_n\neq\emptyset$ for every $n\in\mathbb{N}$, then the cluster points must belong to G_n for every $n\in\mathbb{N}$, so $\mathfrak{F}$ has its cluster points in X and this shows that $(\mathcal{S}_n)$ is a complete sequence of open covers of X.

A well known consequence is that $\mathcal{F}\cap\mathcal{G}_\delta$ subsets of Čech-complete spaces are also Čech-complete.

Lemma 4.5 *The class of Čech-complete topologies on some set X which are finer than a prefixed Hausdorff topology verify the properties of Lemma 2.22.*

Proof. Let τ be a Čech-complete topology on X. Let $F \subset X$ a τ-closed set. Assume that X is a $\mathcal{G}_\delta$-set in some compact space $(Z, \widetilde{\tau})$. It is easy to see that the following map

$$i : X \longrightarrow \{0,1\} \times Z$$

defined by $i(x) = (0, x)$ if $x \in F$ and $i(x) = (1, x)$ if $x \in X \setminus F$ is an embedding of X endowed with $\mathrm{top}(\tau, \{F\})$ and $i(X)$ is a $\mathcal{F} \cap \mathcal{G}_\delta$-set in the compact space $\{0,1\} \times Z$, and thus $\mathrm{top}(\tau, \{F\})$ is a Čech-complete topology.
Now suppose that (τ_n) is a sequence of Čech-complete topologies on X. Assume that there is a Hausdorff topology τ_0 such that $\tau_0 \subset \tau_n$ for every $n \in \mathbb{N}$. Take a compact space $(Z_n, \widetilde{\tau}_n)$ containing (X, τ_n) as a $\mathcal{G}_\delta$ subspace for each $n \in \mathbb{N}$. Let $\tau = \mathrm{top}(\{\tau_n : n \in \mathbb{N}\})$ and consider the map

$$i : X \longrightarrow \prod_{n=1}^{\infty} Z_n$$

defined by $i(x) = (x)_{n=1}^{\infty}$. It is easy to see that i is an embedding of (X, τ) into a compact space and $\prod_{n=1}^{\infty} X$ is a $\mathcal{G}_\delta$ subset of $\prod_{n=1}^{\infty} Z_n$. We claim that $i(X)$ is closed in $\prod_{n=1}^{\infty}(X, \tau_n)$. It is enough to show that $i(X)$ is closed in the coarser topology of $\prod_{n=1}^{\infty}(X, \tau_0)$. Indeed, if $(x_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} X \setminus i(X)$, then $x_{n_1} \neq x_{n_2}$. As τ_0 is Hausdorff, take $U_1, U_2 \in \tau_0$ disjoint neighbourhoods of x_{n_1} and x_{n_2} respectively. Let π_n be the projection on the n'th coordinate. It is easy to see that $\pi_{n_1}^{-1}(U_1) \cap \pi_{n_2}^{-1}(U_2)$ is a neighbourhood of $(x_n)_{n=1}^{\infty}$ that does not meet $i(X)$. We have that (X, τ) is homeomorphic to a relatively closed subset of a $\mathcal{G}_\delta$ subset of a compact space, and thus τ is Čech-complete.

An internal characterization for absolute Borel metrizable spaces in terms of complete sequences of covers was obtained in [11]. The main result of this section provides a characterization of those completely regular spaces which are absolute Borel and a sufficient condition in the regular case.

Theorem 4.6 *Let (X, τ) a regular topological space. Consider the following statements:*

i) (X, τ) *is absolute Borel.*

ii) $(A, \tau|_A)$ *is absolute Borel for every* $A \in \mathit{Borel}(X, \tau)$.

iii) There is a Čech-complete topology δ on X finer than τ such that X has $P(\delta, \tau)$ with a sequence (A_n) of τ-Borel sets.

iv) There is a complete metric d on X finer than τ such that X has $P(d,\tau)$ with a sequence (A_n) of τ-Borel sets.

Then iv) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i). If (X,τ) is completely regular, then i), ii) and iii) are equivalent. If (X,τ) is metrizable, then all the statements are equivalent.
Moreover, if $\gamma > 0$ is countable ordinal, then (X,τ) is of multiplicative class $\gamma+1$ in every regular embedding if (for (X,τ) completely regular, if and only if) the sets (A_n) of iii) and iv) can be taken of additive class γ.

Proof. It is clear that *iv)* $\Rightarrow$ *iii)* and *ii)* $\Rightarrow$ *i)*.
iii) $\Rightarrow$ *i)* Assume that (X,τ) is a subspace of a regular topological space $(Z,\widetilde{\tau})$. Put $A_n = A'_n \cap X$ where $A'_n \in Borel(Z,\widetilde{\tau})$ are of additive class γ. Applying Lemma 2.22, there is a transfinite sequence of regular topologies $(\widetilde{\tau}_\alpha)_{\alpha\leq\gamma}$ with $\widetilde{\tau}_1 = \widetilde{\tau}$ and $\widetilde{\tau}_{\alpha+1} = \mathrm{top}(\widetilde{\tau}_\alpha, \{F_n^\alpha : n \in \mathbb{N}\})$ where F_n^α is $\widetilde{\tau}_\alpha$-closed, and every A'_n is $\widetilde{\tau}_\gamma$-open. Since δ is stronger than τ, the transfinite sequence defined by $\delta_1 = \delta$, $\delta_{\alpha+1} = \mathrm{top}(\delta_\alpha, \{X \cap F_n^\alpha : n \in \mathbb{N}\})$ and for α a limit ordinal $\tau_\alpha = \mathrm{top}(\tau_\beta : \beta < \alpha)$ is made up of Čech-complete topologies by Lemmata 2.22 and 4.5. We claim that $\delta_\gamma = \widetilde{\tau}_\gamma|_X$. Indeed, it is easy to see by induction that $\widetilde{\tau}_\alpha|_X \subset \delta_\alpha$ for $\alpha \leq \gamma$. In particular, $\widetilde{\tau}_\gamma|_X \subset \delta_\gamma$. On the other hand, every A'_n is $\widetilde{\tau}_\gamma$-open, and the inclusion $\delta_\gamma \subset \widetilde{\tau}_\gamma|_X$ follows.
We have that $(X,\widetilde{\tau}_\gamma|_X)$ is Čech-complete, so by Lemma 4.4, X is a $\mathcal{F}\cap\mathcal{G}_\delta$-set in $(Z,\widetilde{\tau}_\gamma)$. Note that a $\widetilde{\tau}_\gamma$-open set has additive class γ in $(Z,\widetilde{\tau})$, and thus X has multiplicative class $\gamma+1$ in $(Z,\widetilde{\tau})$.
iii) $\Rightarrow$ *ii)* It is enough to see that $(A,\tau|_A)$ satisfies the condition *iii)*. If δ is a Čech-complete topology on X such that X has $P(\delta,\tau)$ with τ-Borel sets, by Lemma 2.22, we can construct a Čech-complete topology δ_γ on X such that A is δ_γ-open and X has $P(\delta_\gamma,\tau)$ with τ-Borel sets. Now $(A,\delta_\gamma|_A)$ is Čech-complete and A has $P(\delta_\gamma|_A,\tau|_A)$ with $\tau|_A$-Borel sets.
i) $\Rightarrow$ *iii)* Consider (X,τ) as subset of its Čech-Stone compactification $(\beta X,\widetilde{\tau})$. If X is of multiplicative class $\gamma+1$ in βX then it can be written $X = \bigcap_{n=1}^\infty X_n$ where every X_n is of additive class γ. By Lemma 2.22, there is a Čech-complete topology on βX obtained from $\widetilde{\tau}$ by adding a countable sequence of sets of additive class γ making every X_n open. Thus X is a $\mathcal{G}_\delta$-set in a Čech-complete space and so it is Čech-complete too for that finer topology.
i) $\Rightarrow$ *iv)* It is enough to consider (X,τ) into some complete metric space and reasoning like in *i)* $\Rightarrow$ *iii)*.

Since any compact space is absolute Borel, we obtain as a corollary that Borel subsets of a compact space are absolute Borel spaces. We also have the following.

Corollary 4.7 *A completely regular topological space is absolute Borel if and only if it is a Borel subset in its Čech-Stone compactification.*

The following extends a result of Oncina [13] about Banach spaces with a countable cover by sets of local small diameter.

Corollary 4.8 *Let (Z, τ) be a regular topological space. Let X be a subset of Z such that there is metric d on X stronger than the restriction of τ. Suppose that X has $P(d, \tau)$ and the closed d-balls are τ-closed in X. If (X, d) is complete, then X is $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$-set in (Z, τ).*

Proof. Assume that X has $P(d, \tau)$ with a sequence (A_n). We claim that X has $P(d, \tau)$ with $(\overline{A_n}^{\tau})$. Fix $x \in A$. Take $\varepsilon > 0$. There exists A_n and $U \in \tau$ such that $x \in A_n \cap U \subset B(x, \varepsilon/2)$. Thus

$$x \in \overline{A_n}^{\tau} \cap U \subset \overline{A_n \cap U}^{\tau} \subset \overline{B(x, \varepsilon/2)}^{\tau} \subset B(x, \varepsilon)$$

Since closed sets are of first additive class, by applying Theorem 4.6 we deduce that X is a subset of the second multiplicative class of Z, that is, a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$ subset.

An example where the above Corollary can be applied is a Banach space having an equivalent Kadec norm. Following result appears in [3].

Corollary 4.9 (Schachermayer) *If X is a Banach space having an equivalent Kadec norm, then $(X, weak)$ is an absolute Borel space. In particular, X is weak*-$(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$-set in X^{**}.*

Notice that if a Banach space X is a weak*-Borel subset in its bidual X^{**}, then (X, w) is absolute Borel. Indeed, (X^{**}, w^*) is absolute Borel because it is σ-compact.

We can get as corollaries some classic results, see Kuratowski [10]. The first one tell us that the Borel subsets of complete metric spaces are absolute Borel spaces.

Corollary 4.10 (Sierpinski) *Let X be a metrizable space. Then X is Borel subset in every embedding into a regular space if and only if X is homeomophic to a Borel subset of a complete metric space.*

Lemma 4.11 *Let (X, d) be a Polish space and let τ be a coarser Hausdorff topology on X. Then X has $P(d, \tau)$ with τ-Borel sets. In particular, (X, τ) is an absolute Borel space.*

Proof. Since d has a countable basis, it is enough to prove that any d-open subset is τ-Borel. Observe that the d-open and d-closed subsets can be regarded as continuous images from polish spaces to (X, τ), so they are Souslin subsets. By the Separation Theorem [1, IX §6], disjoint Souslin subsets in a Hausdorff space can be separed by a Borel subset. In particular, a d-open subset of X must be τ-Borel.

Corollary 4.12 (Lusin-Souslin) *Let X be a Polish space, Z a regular space and let $f : X \to Z$ be a one-to-one p-Borel map. Then $f(A)$ is a Borel set in Z for every Borel subset A of X. If Z is metrizable, it is enough to ask f to be a Borel measurable one-to-one map to get the same conclusion.*

Proof. Since any Borel subset of X is a Polish space with some stronger topology, Proposition 2.23, we only have to prove that $f(X)$ is Borel. Again by Proposition 2.23, we can take a metric d such that (X, d) is complete separable and f is continuous. If τ is the topology of Z, then $f^{-1}(\tau)$ is a regular topology on X coarser than d. By Lemma 4.11, X has $P(d, f^{-1}(\tau))$ with Borel sets. Now apply Theorem 4.6 to the embedding of $(X, f^{-1}(\tau))$ into (Z, τ) to get the conclusion. If Z is metrizable, then f is p-Borel by Example 3.3.

We shall say that an one-to-one map between topological spaces is a p-Borel isomorphism if the map and its inverse are p-Borel. Theorem 4.6 says that a topological space is absolute Borel if there is a particular p-Borel isomorphism to a Čech-complete space. The following result shows that the property of being absolute Borel is preserved under p-Borel isomorphism.

Theorem 4.13 *A topological space is absolute Borel if it is p-Borel isomorphic to an absolute Borel space.*

Proof. Let $f : X \to Y$ be a p-Borel isomorphism between topological spaces, and assume that (X, τ) is absolute Borel. Suppose that f has satisfies Definition 1.3 with a sequence (A_n) of τ-Borel sets. By Theorem 4.6 there is a Čech complete topology δ on X finer than τ such that X has $\mathbb{I} : (X, \delta) \to (X, \tau)$ is p-Borel. By Lemma 2.22 we may assume that every A_n is δ-open. The topology $f(\delta)$ on Y is Čech-complete and statement $iii)$ of Theorem 4.6 is verified, so Y is an absolute Borel space.

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