Weak* locally uniformly rotund norms and descriptive compact spaces

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Abstract

We prove that a dual Banach space X^* has an equivalent W*LUR norm if and only the weak* topology has a σ-isolated network. We give sufficient conditions for the existence of equivalent norms with various kinds of differentiability.

1 Introduction

Differentiability properties of convex functions on a Banach space X are closely related to properties of the weak* topology of the dual space X^*. A Banach space X is said to be Asplund (resp. weak Asplund) if every convex continuous function defined on X has a $\mathcal{G}_\delta$ dense set of points of Fréchet (resp. Gâteaux) differentiability. It is well known that X is weak Asplund if B_{X^*} is a fragmentable compactum (in the weak* topology), and X is Asplund if and only if B_{X^*} is fragmented by the norm of X^*. Recall that fragmentable means fragmented by some metric, and a topological space (Z, τ) is said to be fragmented by a metric (or pseudometric) d defined on Z, if for every $\varepsilon > 0$ and every nonempty $A \subset Z$, there is $U \in \tau$ such that $A \cap U \neq \emptyset$ and $diam(A \cap U) < \varepsilon$. Asplund properties can be generalized using the following notion of differentiability. Let $f : X \to \mathbb{R}$ be a convex function defined on a Banach space X and let $M \subset X$ be a bounded subset. It is said that f is M-differentiable at $x \in X$ if the limit

$$\lim_{t \to 0} t^{-1}(f(x + th) + f(x - th) - 2f(x)) = 0$$

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exists uniformly in $h \in M$. A bounded subset $M \subset X$ is an Asplund set if any convex continuous function on X is M-differentiable on a dense $\mathcal{G}_\delta$ subset. A bounded subset $M \subset X$ is an Asplund set if and only if B_{X^*} is fragmented by the pseudometric of uniform convergence on M. For the proof of all these facts and further references see the books [3, 6].

If the Banach space X has an equivalent Gâteaux smooth norm, then B_{X^*} is a fragmentable compactum [20, 6]. In this paper we give general conditions for the existence of an equivalent Gâteaux norm on a Banach space X in terms of the weak* topology of its dual X^*. For this aim, we shall use a topological property related to the fragmentability. Some terminology is needed. Let $\{H_i : i \in I\}$ be a family of subsets of a topological space (Z, τ). The family is said to be isolated if it is discrete in its union endowed with the relative topology, or in other words, if for every $i \in I$ we have

$$H_i \cap \overline{\bigcup_{j \in I, j \neq i} H_j} = \emptyset.$$

If there is a decomposition $I = \bigcup_{n=1}^\infty I_n$ such that every family $\{H_i : i \in I_n\}$ is isolated, then the family $\{H_i : i \in I\}$ is said to be σ-isolated. A family $\mathfrak{N}$ of subsets of Z is said to be a network if every open set is a union of members of $\mathfrak{N}$. Topological spaces are supposed to be Hausdorff.

Definition 1.1 *A compact space K is said to be descriptive if its topology has a σ-isolated network.*

Descriptive topological spaces were introduced by Hansell in [10]. A particular class of compact spaces having a σ-isolated network was studied in [18] under the name of Namioka-Phelps compacta. The main (and simplest) examples of descriptive compact spaces are the Eberlein compacta and the scattered compacta K with $K^{(\omega_1)} = \emptyset$, see [18]. We shall also prove that Gul'ko compacta are descriptive too, Corollary 2.4. Recall that a compact space K is said to be Gul'ko if the Banach space $C(K)$ is weakly countably determined, see [3, 6].

Let X be a Banach space such that its dual unit ball B_{X^*} is a descriptive compactum. We shall prove in the paper that X has an equivalent Gâteaux smooth norm. By Šmulyan criterion, see [3], it is enough to build an equivalent strictly convex dual norm on X^*, but our construction will actually give a stronger property.

Definition 1.2 *A dual norm $\|.\|$ on X^* is said to be weak* locally uniformly rotund (W*LUR) if* $w^*\text{-}\lim_n x_n^* = x^*$, *whenever* x^*, x_n^* *are such that* $\|x_n^*\| = \|x^*\| = 1$ *and* $\lim_n \|x^* + x_n^*\| = 2$.

The corresponding definition for the weak topology, that is, weak locally uniformly rotund norms (WLUR) has been investigated by several authors. The most striking result from the point of view of renorming was given by Moltó, Orihuela, Troyanski and Valdivia [16]; they proved that a WLUR Banach space has an equivalent LUR norm. A version of their result for dual Banach spaces is our Corollary 4.2.

Next theorem enumerates the main results of the paper.

Theorem 1.3 *Let X be a Banach space and let K be a compact space. Then the following affirmations hold:*

*(a) X^* has an equivalent W*LUR norm if and only if B_{X^*} is descriptive.*

*(b) Assume that B_{X^*} is descriptive. For every Asplund subset $M \subset X$, there is an M-differentiable equivalent norm on X such that the dual norm is W*LUR.*

*(c) K is descriptive if and only if $C(K)^*$ has an equivalent W*LUR norm.*

The first consequence of Theorem 1.3 is that the existence of an equivalent W*LUR norm on a dual Banach space X^* is a non linear topological property. Theorem 1.3 also has as corollaries some well known results about Gâteaux or Fréchet renormability of certain clases of Banach spaces X due to: Amir and Lindenstrauss, if B_{X^*} is Eberlein compact [9]; Mercourakis, if X is WCD [15, 3] (in this case B_{X^*} is Gul'ko); Fabian, if X is Asplund and WCD [4, 3]; and Deville, for $C(K)$ if K is scattered and $K^{(\omega_1)} = \emptyset$ [2, 3].

We shall briefly describe the contains of the remaining sections of the paper. Section 2 contains some results about topological spaces with σ-isolated network which are essential for the proof of Theorem 1.3. Descriptive compact spaces has much more interesting topological properties that will be published elsewere. In Section 3 we deal with Radon measures on a descriptive compact space K in order to prove a renorming result for $C(K)^*$. The proof of Theorem 1.3 is completed in Section 4 using a transfer technique.

2 The topological arguments

Lemma 2.1 *A descriptive compact space is fragmentable.*

Proof. Let $\{H^n_\alpha : n \in \mathbb{N}, \alpha \in [1, \xi_n)\}$ a network of K where every family $\{H^n_\alpha : \alpha \in [1, \xi_n)\}$ is isolated. We have that $\{\overline{H^n_\alpha} : \alpha \in [1, \xi_n), n \in \mathbb{N}\}$ is also a network by regularity. Take open sets U^n_α for $\alpha \in [1, \xi_n)$ such that $H^n_\alpha \subset U^n_\alpha$ and $U^n_\alpha \cap \bigcup\{H_\beta : \beta \in [1, \xi_n), \beta \neq \alpha\} = \emptyset$. Put $A_n = \overline{\bigcup_{\alpha\in[1,\xi_n)} H^n_\alpha}$ and $W^n_\alpha = A_n \cap U^n_\alpha$ for $\alpha \in [1, \xi_n)$. Since U^n_α is open, we have $N^n_\alpha \subset W^n_\alpha \subset \overline{N^n_\alpha}$. This implies that $\{W^n_\alpha : n \in \mathbb{N}, \alpha \in [1, \xi_n)\}$ is also a network of K. Take $W^n_0 = K \setminus A_n$ and $W^n_{\xi_n} = A_n \setminus \bigcup_{\alpha\in[1,\xi_n)} U^n_\alpha$. For every $n \in \mathbb{N}$, the well ordered family $\{W^n_\alpha : \alpha \in [0, \xi_n]\}$ is a relatively open partitioning of K [6], i.e. W^n_α is a relatively open subset of $\bigcup_{\beta\geq\alpha} W^n_\beta$ for every $\alpha \in [0, \xi_n]$. The proof finishes by applying a result of Ribarska [6] which states that a compact space K is fragmentable if and only if there is a sequence of relatively open partitionings which separates the points of K. ■

We need to introduce a kind of covering property, see [1]. A topological space Z is said to be weakly θ-refinable (also called weakly submeta compact) if every open cover of Z has a σ-isolated (non necessary open) refinement. If every subspace of Z is weakly θ-refinable, then it is said that Z is hereditarily weakly θ-refinable. Clearly, a topological space with a σ-isolated network is hereditarily weakly θ-refinable.

Lemma 2.2 *Let (Z, τ) be a hereditarily weakly θ-refinable regular topological space. Assume that Z is fragmented by a finer metric d. Then for every $n \in \mathbb{N}$ there is a family $\mathfrak{N}_n$ of disjoint relatively open subsets of a closed subset $A_n \subset Z$, such that $\mathfrak{N} = \bigcup_{n=1}^\infty \mathfrak{N}_n$ is a network for (Z, d). Moreover, (Z, τ) has a σ-isolated network.*

Proof. Using the fragmentability by d and the regularity of τ, for every $n \in \mathbb{N}$ we can take a transfinite sequence of open sets $\{U^n_\alpha : \alpha < \xi_n\}$ covering Z, such that $\overline{U^n_\alpha} \cap (Z \setminus \bigcup_{\beta<\alpha} U^n_\beta)$ is nonempty and has diameter less than n^{-1}. Put $D^n_\alpha = U^n_\alpha \cap (Z \setminus \bigcup_{\beta<\alpha} U^n_\beta)$. Then we have

$$diam(\overline{D^n_\alpha}) \leq diam(\overline{U^n_\alpha} \cap (Z \setminus \bigcup_{\beta<\alpha} U^n_\beta)) \leq n^{-1}.$$

Since (Z, τ) is hereditarily weakly θ-refinable, by [11, Lemma 6.20], the family $\{D^n_\alpha : \alpha < \xi_n\}$ is σ-isolatedly decomposable, that is, for every α we can write

$D^n_\alpha = \bigcup_{m=1}^\infty H^{n,m}_\alpha$ and the family $\{H^{n,m}_\alpha : \alpha < \xi_n\}$ is isolated for every $m \in \mathbb{N}$. Thus we can take open sets $U^{n,m}_\alpha$ such that $H^{n,m}_\alpha \subset U^{n,m}_\alpha$ and $U^{n,m}_\alpha \cap \bigcup\{H^{n,m}_\beta : \beta < \xi_n, \beta \neq \alpha\} = \emptyset$. Take $A_{n,m} = \overline{\bigcup_{\alpha<\xi_n} H^{n,m}_\alpha}$. Then we have

$$U^{n,m}_\alpha \cap A_{n,m} = U^{n,m}_\alpha \cap \overline{H^{n,m}_\alpha} \subset \overline{D^n_\alpha}.$$

Take $\mathfrak{N}_{n,m} = \{U^{n,m}_\alpha \cap A_{n,m} : \alpha < \xi_n\}$ and $\mathfrak{N} = \bigcup_{n,m\in\mathbb{N}} \mathfrak{N}_{n,m}$. The last set inclusion shows that $\mathfrak{N}$ is a network for d. The proof is finished up to a change of the indexes. ■

Proposition 2.3 *A compact space is descriptive if and only if it is fragmentable and hereditarily weakly θ-refinable.*

Proof. Apply Lemmata 2.1 and 2.2. ■

Corollary 2.4 *Gul'ko compact spaces are descriptive.*

Proof. Gul'ko compact spaces are fragmentable [19, 6] and hereditarily weakly θ-refinable [8]. ■

We shall also need the following definition, see [7]. A symmetric on a set Z is a function $\rho : Z \times Z \to \mathbb{R}$ satisfiying the following conditions: $\rho(x, y) \geq 0$; $\rho(x, y) = 0$ if and only if $x = y$; and $\rho(x, y) = \rho(y, x)$. A topology τ on Z is said to be semi-metrizable if there is a symmetric ρ such that for every point $x \in Z$, the family of "open balls" of radius $r > 0$

$$B_\rho(x, r) = \{y \in Z : \ \rho(x, y) < r\}$$

is a basis of τ at x. A semi-metrizable topological space is hereditarily weakly θ-refinable, see [7, Theorem 9.8, Theorem 5.11] and [1, Diagram 4.1].

Lemma 2.5 *Let X^* be a dual Banach space satifying that either the unit sphere S_{X^*} is weak* metrizable or the norm is W*LUR. Then the weak* topology of X^* has a σ-isolated network.*

Proof. Acording to [10, Theorem 7.2] it suffices to show that (S_{X^*}, w^*) has a σ-isolated network. The first case follows from the fact that any metrizable space has a σ-discrete base [14]. Assume that the norm is W*LUR. We claim that (S_{X^*}, w^*) is semi-metrizable. Consider the symmetric given by the formula $\rho(x^*, y^*) = 1 - 2^{-1}\|x^* + y^*\|$. Fix a point $x^* \in S_{X^*}$. From the definition of W*LUR is clear that every neighbourhood of x^* contains a "ball" $B_\rho(x^*, r)$ for some $r > 0$. To have that S_{X^*} is semi-metrized by ρ it remains to show that $B_\rho(x^*, r)$ is a neighbourhood of x^*. By the Hanh-Banach Theorem find a w^*-open halfspace U containing x^* and disjoint with $(1-r)B_{X^*}$. If $y^* \in S_{X^*} \cap U$, then $(x^* + y^*)/2 \in U$ and thus $2^{-1}\|x^* + y^*\| > 1 - r$. So $\rho(x^*, y^*) < r$ and $S_{X^*} \cap U \subset B_\rho(x, r)$. Now, by a well known result of Ribarska, see [6], (B_{X^*}, w^*) is fragmentable. Lemma 2.2 implies that (S_{X^*}, w^*) has a σ-isolated network. A different proof can be obtained adapting results from [16] which are formulated for WLUR norms. ■

3 Renorming of $C(K)^*$

Notice that if K is a descritive compactum fragmented by a finer metric d, then every bounded d-continuous map $f : K \to X$ with values in a normed space X is Bochner integrable with respect to each Radon measure μ on K. Indeed, from Lemma 2.2 follows easily that the d-open sets are Borel, thus f is Borel measurable. The fragmentability implies that there is a d-separable subset where μ is concentrated [13], so f has essentially separable range.

Lemma 3.1 *Let K a descriptive compact space, let d a finer metric fragmenting K. Then d has a network $\mathfrak{N}$ with the following properties:*

1) there are closed sets A_n such that $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ where $\mathfrak{N}_n$ is a disjoint family of relatively open subsets of A_n for every $n \in \mathbb{N}$,

2) for every $n_1, n_2 \in \mathbb{N}$ there is $n_3 \in \mathbb{N}$ such that

$$\mathfrak{N}_{n_3} = \{H_1 \cap H_2 : H_1 \in \mathfrak{N}_{n_1}, H_2 \in \mathfrak{N}_{n_2}\}.$$

Proof. Let $\{A_n \cap U_i : n \in \mathbb{N}, i \in I_n\}$ be the network given by Theorem 2.2 and put $\mathfrak{N}'_n = \{A_n \cap U_i : i \in I_n\}$. For a given k-uple of integers $(n_1, \ldots, n_k)$ the sets of the form $H_1 \cap \ldots \cap H_k$, where $H_j \in \mathfrak{N}'_{n_j}$, is a family of disjoint relatively open subsets of the closed set $A_{n_1} \cap \ldots \cap A_{n_k}$. Call this family $\mathfrak{N}'_{n_1,\ldots,n_j}$ and finally take $\mathfrak{N} = \bigcup\{\mathfrak{N}'_{n_1,\ldots,n_j} : n_1, \ldots, n_j \in \mathbb{N}\}$. ■

Lemma 3.2 *Let K be a descriptive compact space, let d be a metric on K and assume that $\mathfrak{N}$ is a network for d verifying the conditions of Lemma 3.1. Let μ, μ_ω be positive measures satisfying that* *i)* $\lim_\omega \mu_\omega(K) = \mu(K)$.

ii) $\lim_\omega \sum_{H\in\mathfrak{N}_n} |\mu_\omega(H) - \mu(H)| = 0$ *for every* $n \in \mathbb{N}$.

Then, for every bounded d-continuous map $f : K \to X$ *with values into a normed space,*

$$\lim_\omega \| \int f \, d\mu_\omega - \int f \, d\mu \| = 0$$

Proof. First of all we notice that $\lim_\omega \mu_\omega(H) = \mu(H)$ for every $H \in \mathfrak{P}_n$ where $\mathfrak{P}_n$ is the partition generated by $\mathfrak{N}_1, \ldots, \mathfrak{N}_n$. To see that, just remark that the measure of every member of $\mathfrak{P}_n$ can be expresed as a finite linear combination of measures of unions of sets from $\mathfrak{N}$, where the sets in each union lies in some $\mathfrak{N}_m$. Since the canonical norm of $l^1(\mathfrak{P}_n)$ is pointwise Kadec, the net (μ_ω) converges to μ in the norm of $l^1(\mathfrak{P}_n)$.
Given $\varepsilon > 0$, we claim that for $n \in \mathbb{N}$ large enough

$$\sum_{E\in\mathfrak{P}_n} diam(f(E)) \, \mu(E) < \varepsilon$$

To see that, take points x_E^1, x_E^2 where $E \in \mathfrak{P}_n$ such that

$$\|f(x_E^1) - f(x_E^2)\| > 2^{-1} diam(f(E)).$$

Define functions $f_n^1(x) = f(x_E^1)$ and $f_n^2(x) = f(x_E^2)$ if $x \in E$. Clearly

$$\sum_{E\in\mathfrak{P}_n} diam(f(E)) \, \mu(E) \le 2 \int \|f_n^1 - f_n^2\| \, d\mu.$$

Since $\mathfrak{N}$ is a network, we have that $\lim_n \|f_n^1(x) - f_n^2(x)\| = 0$ for every $x \in X$. Thus, by Lebesgue's Theorem, we have

$$\int \|f_n^1 - f_n^2\| \, d\mu < \varepsilon/2.$$

for n big enough and this proves the claim. Let us go on with the proof of the lemma. Take $\varepsilon > 0$ and $n \in \mathbb{N}$ such that

$$\sum_{E\in\mathfrak{P}_n} diam(f(E)) \, \mu(E) < \varepsilon/2$$

Since (μ_ω) converges to μ in $l^1(\mathfrak{P}_n)$ we deduce that

$$\sum_{E\in\mathfrak{P}_n} diam(f(E))\,\mu_\omega(E) < \varepsilon/2$$

for ω large enough. Fix points $x_E \in E \in \mathfrak{P}_n$. Then we have

$$\| \int f\,d\mu - \sum_{E\in\mathfrak{P}_n} f(x_E)\| \leq \sum_{E\in\mathfrak{P}_n} diam(f(E))\,\mu(E)$$

$$\| \int f\,d\mu_\omega - \sum_{E\in\mathfrak{P}_n} f(x_E)\| \leq \sum_{E\in\mathfrak{P}_n} diam(f(E))\,\mu_\omega(E)$$

and we deduce that

$$\| \int f\,d\mu_\omega - \int f\,d\mu\| \leq \varepsilon$$

which finishes the proof of the claim. ■

Theorem 3.3 *Let K be a descriptive compact space and let d be a finer metric fragmenting K. Then there is an equivalent dual norm $|\!|\!|.|\!|\!|$ on $C(K)^*$ such that for every bounded d-continuous function $f : K \to X$ with values into a normed space*

$$\lim_\omega \| \int f\,d\mu_\omega - \int f\,d\mu\| = 0$$

whenever the measures $\mu, \mu_\omega \in C(K)^*$ are such that $\lim_\omega |\!|\!|\mu_\omega|\!|\!| = |\!|\!|\mu|\!|\!|$ and $\lim_\omega |\!|\!|\mu + \mu_\omega|\!|\!| = 2|\!|\!|\mu|\!|\!|$. In particular, $|\!|\!|.|\!|\!|$ is a W^*LUR norm.

Proof. Let $\mathfrak{N}$ be a network satisfying Lemma 3.1. In particular, $\mathfrak{N} = \bigcup_{n=1}^\infty \mathfrak{N}_n$ and every $\mathfrak{N}_n$ is a disjoint family of relatively open subsets of a closed subset $A_n \subset K$. Clearly, $A'_n = A_n \setminus \bigcup \mathfrak{N}_n$ is also closed.
Construction for positive measures. For any open subset $U \subset K$ the evaluation $|\mu|(U)$ is w^*-lower semicontinuous. Indeed, we have

$$|\mu|(U) = \sup\{\int f\,d\mu : |f| \leq 1, supp(f) \subset U\}.$$

Now we proceed to the construction of a dual norm on $C(K)^*$. For every $n \in \mathbb{N}$ define a function F_n on $C(K)^*$ by

$$F_n(\mu)^2 = \sum_{H\in\mathfrak{N}_n} |\mu|(H)^2.$$

The function F_n is convex and when restricted to $C(A_n)^*$ is w^*-lower semicontinuous. For $m, n \in \mathbb{N}$ define a seminorm $\|.\|_{m,n}$ on $C(K)^*$ by the formula

$$\|\mu\|_{m,n}^2 = \inf\{m^{-1}F_n(\nu)^2 + \|\mu - \nu\|^2 : supp(\nu) \subset A_n\}.$$

Notice that taking $\nu = \mu\restriction_{A_n}$ (that is, the restriction of μ to A_n) we deduce the following inequality

$$\|\mu\|_{m,n}^2 \leq m^{-1}\|\mu\|^2 + |\mu|(K \setminus A_n)^2.$$

Using the w^*-lower semicontinuity of F_n on $C(A_n)^*$ and compactness it is easy to see that, for every $\mu \in C(K)^*$ there is $\nu \in C(A_n)^*$ such that

$$\|\mu\|_{m,n}^2 = m^{-1}F_n(\nu)^2 + \|\mu - \nu\|^2.$$

Define an equivalent norm $|||.|||_+$ on $C(K)^*$ by

$$|||\mu|||_+^2 = \|\mu\|^2 + \sum_{m,n} 2^{-m-n}\|\mu\|_{m,n}^2 + \sum_n 2^{-n}|\mu|(K \setminus A_n)^2 + \sum_n 2^{-n}|\mu|(K \setminus A_n')^2.$$

The norm $|||.|||_+$ is dual because is w^*-lower semicontinuous. Suppose we are given measures $\mu, \mu_\omega \in \mathfrak{M}^+(K)$ satisfying $\lim_\omega |||\mu_\omega|||_+ = |||\mu|||_+$ and $\lim_\omega |||\mu + \mu_\omega|||_+ = 2|||\mu|||_+$. We claim that (μ_ω) converges to μ in $l^1(\mathfrak{N}_n)$ for every $n \in \mathbb{N}$. A standar convexity argument [3, Fact II.2.3] shows that

$$\lim_\omega \mu_\omega(K \setminus A_n) = \mu(K \setminus A_n),$$

$$\lim_\omega \mu_\omega(K \setminus A_n') = \mu(K \setminus A_n'),$$

$$\lim_\omega \|\mu_\omega\|_{m,n} = \lim_\omega \|\frac{\mu + \mu_\omega}{2}\|_{m,n} = \|\mu\|_{m,n}$$

for every $m, n \in \mathbb{N}$. From the two first equalities we deduce that

$$\lim_\omega \sum_{H \in \mathfrak{N}_n} |\mu_\omega|(H) = \sum_{H \in \mathfrak{N}_n} |\mu|(H).$$

Since the canonical norm of $l^1(\mathfrak{N}_n)$ is pointwise Kadec it is enough to show that $\lim_\omega \mu_\omega(H) = \mu(H)$ for every $H \in \mathfrak{N}_n$. Take measures $\nu^{m,n}, \nu_\omega^{m,n}$ supported by A_n such that

$$\|\mu\|_{m,n}^2 = m^{-1}F_n(\nu^{m,n})^2 + \|\mu - \nu^{m,n}\|^2,$$

$$\|\mu_\omega\|^2_{m,n} = m^{-1}F_n(\nu_\omega^{m,n})^2 + \|\mu_\omega - \nu_\omega^{m,n}\|^2.$$

Again a convexity argument shows that

$$\lim_\omega F_n(\nu_\omega^{m,n}) = \lim_\omega F_n(\frac{\nu^{m,n} + \nu_\omega^{m,n}}{2}) = F_n(\nu^{m,n}),$$

$$\lim_\omega \|\mu_\omega - \nu_\omega^{m,n}\| = \|\mu - \nu^{m,n}\|.$$

From the first equality we obtain that $\lim_\omega \nu_\omega^{m,n}(H) = \nu^{m,n}(H)$ for $H \in \mathfrak{N}$. On the other hand we have

$$\|\mu_\omega - \nu_\omega^{m,n}\| = \mu_\omega(K \setminus A_n) + \|\mu_\omega \restriction_{A_n} - \nu_\omega^{m,n}\|,$$

$$\|\mu - \nu^{m,n}\| = \mu(K \setminus A_n) + \|\mu \restriction_{A_n} - \nu^{m,n}\|.$$

We deduce that

$$\lim_\omega \|\mu_\omega \restriction_{A_n} - \nu_\omega^{m,n}\| = \|\mu \restriction_{A_n} - \nu^{m,n}\|.$$

Take any $\varepsilon > 0$. We have

$$\|\mu \restriction_{A_n} - \nu^{m,n}\| + \mu(K \setminus A_n) = \|\mu - \nu^{m,n}\|$$

$$\leq \|\mu\|_{m,n} \leq [\, m^{-1}\|\mu\|^2 + \mu(K \setminus A_n)^2\,]^{1/2}.$$

Thus we can take $m \in \mathbb{N}$ such that $\|\mu \restriction_{A_n} - \nu^{m,n}\| < \varepsilon/3$. For ω big enough we have $\|\mu_\omega|_{A_n} - \nu_\omega^{m,n}\| < \varepsilon/3$ and $|\, \nu_\omega^{m,n}(H) - \nu^{m,n}(H)\,| < \varepsilon/3$. Then we deduce that

$$|\, \mu_\omega(H) - \mu(H)\,| < \varepsilon$$

which proves the claim.

Construction of $|||.|||$. Let $|||.|||_+$ the norm constructed in the former step. Define an equivalent norm on $C(K)^*$ by the formula

$$|||\mu|||^2 = \inf\{|||\mu^1|||_+^2 + |||\mu^2|||_+^2 : \mu^1, \mu^2 \in \mathfrak{M}^+(K), \mu^1 - \mu^2 = \mu\}.$$

A compactness argument shows that $|||.|||$ is a dual norm and that for every $\mu \in C(K)^*$ there are $\mu^1, \mu^2 \in \mathfrak{M}^+(K)$ such that $\mu = \mu^1 - \mu^2$ and

$$|||\mu|||^2 = |||\mu^1|||_+^2 + |||\mu^2|||_+^2.$$

Suppose we are given measures $\mu, \mu_\omega \in C(K)^*$ satisfying $\lim_\omega |||\mu_\omega||| = |||\mu|||$ and $\lim_\omega |||\mu + \mu_\omega||| = 2|||\mu|||$. We can take measures $\mu_\omega^1, \mu_\omega^2 \in \mathfrak{M}^+(K)$ such that $\mu_\omega = \mu_\omega^1 - \mu_\omega^2$ and

$$|||\mu_\omega|||^2 = |||\mu_\omega^1|||_+^2 + |||\mu_\omega^2|||_+^2.$$

A standar convexity argument shows that

$$\lim_\omega |||\mu_\omega^i|||_+ = \lim_\omega |||\frac{\mu^i + \mu_\omega^i}{2}|||_+ = |||\mu^i|||_+$$

for $i = 1, 2$. By the former part, (μ_ω^1) and (μ_ω^2) converge to μ^1 and μ^2 respectively on every set $H \in \mathfrak{N}$. By Lemma 3.2 we have

$$\lim_\omega \| \int f \, d\mu_\omega^i - \int f \, d\mu^i \| = 0$$

for $i = 1, 2$, which easily implies that

$$\lim_\omega \| \int f \, d\mu_\omega - \int f \, d\mu \| = 0$$

for every bounded d-continuous map $f : K \to X$ with values into a normed space. This ends the proof of Theorem 3.3. ■

4 The general case

Notice that at this point we have already proved Theorem 1.3 (c). Indeed, one way follows from Theorem 3.3. The other one follows from Lemma 2.5 and the fact that K embeds into $(C(K)^*, w^*)$.

Proof of Theorem 1.3 (a) If X^* is a dual Banach space with an equivalent W*LUR norm, then B_{X^*} is a descriptive compact space after Lemma 2.5. Suppose that B_{X^*} is descriptive. We may fix a finer fragmenting metric d in order to apply Theorem 3.3. Let $\|.\|_K$ be the equivalent W*LUR norm on $C(B_{X^*})^*$ given by Theorem 3.3. Consider the norm $|||.|||$ on X^* given by the formula

$$|||x^*||| = \inf\{\|\mu\|_K : \mu \in C(B_{X^*})^*, \int \mathbb{I} \, d\mu = x^*\}.$$

It is easy to see that $|||.|||$ is an equivalent dual norm. Also notice that the infimum is attained. We claim that $|||.|||$ is a W*LUR norm. Indeed, take

$x^*, x^*_\omega \in X^*$, with $|||x^*||| = |||x^*_\omega||| = 1$ and $\lim_\omega |||x^* + x^*_\omega||| = 2$. We can take measures $\mu, \mu_\omega \in C(B_{X^*})^*$ such that $\int \mathbb{I}\, d\mu = x^*$, $\int \mathbb{I}\, d\mu_\omega = x^*_\omega$ and $\|\mu\|_K = \|\mu_\omega\|_K = 1$. Triangular inequality implies that $\lim_\omega \|\mu + \mu_\omega\|_K = 2$. Since $\|.\|_K$ is W*LUR, we have w^*-$\lim_\omega \mu_\omega = \mu$. The w^*-w^*-continuity of the integration operator implies that w^*-$\lim_\omega x^*_\omega = x^*$, and thus $|||.|||$ is W*LUR.■

Given any bounded subset $M \subset X$, we may define a seminorm $\mathfrak{p}_M$ on X^* by the formula

$$\mathfrak{p}_M(x^*) = \sup\{|x^*(x)| : x \in M\}.$$

If M is an Asplund subset, then (B_{X^*}, w^*) is fragmented by $\mathfrak{p}_M$.

Lemma 4.1 *Let X be a Banach space. Assume that there is a total Asplund set $M \subset X$ and B_{X^*} is a descriptive compactum. Then there is equivalent W*LUR norm $|||.|||$ on X^* verifying that* $\lim_\omega \mathfrak{p}_M(x^*_\omega - x^*) = 0$ *whenever $x^*, x^*_\omega \in X^*$ are such that $|||x^*_\omega||| = |||x^*||| = 1$ and* $\lim_\omega |||x^* + x^*_\omega||| = 2$.

Proof Let $\|.\|_K$ be the equivalent W*LUR norm on $C(B_{X^*})^*$ given by Theorem 3.3 where the fragmenting metric is the one induced by $\mathfrak{p}_M$. Define the norm $|||.|||$ on X^* by the same formula that in the proof of Theorem 1.3 (a). Suppose that $x^*, x^*_\omega \in X^*$ verifies $|||x^*||| = |||x^*_\omega||| = 1$ and $\lim_\omega |||x^* + x^*_\omega||| = 2$. As above, take measures $\mu, \mu_\omega \in C(B_{X^*})^*$ such that $T(\mu) = x^*$, $T(\mu_\omega) = x^*_\omega$ and $\|\mu\|_K = \|\mu_\omega\|_K = 1$. Again as above, we also have $\lim_\omega \|\mu + \mu_\omega\|_K = 2$. Theorem 3.3 and the continuity of the identity map $\mathbb{I} : (B_{X^*}, \mathfrak{p}_M) \to (X^*, \mathfrak{p}_M)$ imply that

$$\lim_\omega \mathfrak{p}_M\Big(\int \mathbb{I}\, d\mu_\omega - \int \mathbb{I}\, d\mu\Big) = 0,$$

what is the same that $\lim_\omega \mathfrak{p}_M(x^*_\omega - x^*) = 0$. ■

Proof of Theorem 1.3 (b) Take $X_0 = \overline{span}^{\|\cdot\|}(M)$ and let $\|.\|_0$ the dual norm on X_0^* given by Lemma 4.1. Let $\|.\|$ be a W*LUR norm on X^*, after the proof of Theorem 1.3 (a). Define an equivalent dual norm $|||.|||$ on X^* by

$$|||x^*|||^2 = \|x^*\|^2 + \|x^*|_{X_0}\|_0^2.$$

A convexity argument shows that $\lim_\omega \mathfrak{p}_M(x^*_\omega - x^*) = 0$ whenever $x^*, x^*_\omega \in X^*$ are such that $\lim_\omega |||x^*_\omega||| = |||x^*|||$ and $\lim_\omega |||x^* + x^*_\omega||| = 2|||x^*|||$. Šmulyan's criterion [3, Theorem I.1.4] (with small modifications) implies that the predual norm on X is M-differentiable. ∎

The application of Lemma 4.1 to an Asplund Banach space gives a dual version of the result of Moltó, Orihuela, Troyanski and Valdivia [16]. See [18] for more characterizations of the existence of equivalent dual LUR norms.

Corollary 4.2 *If X is an Asplund Banach space and X^* has a W*LUR norm, then X^* has an equivalent dual LUR norm.*

Remark 4.3 *The existence of a dual strictly convex norm on X^* does not imply the existence of an equivalent W*LUR norm on X^* even when X is supposed to be Asplund. Haydon [12] has built an Asplund space X with no equivalent Fréchet differentiable norm such that the dual norm on X^* is strictly convex. Corollary 4.2 shows that X^* cannot be W*LUR renormable.*

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