First Borel class sets in Banach spaces and the asymptotic-norming property

M. Raja*

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Abstract

The Radon-Nikodým property in a separable Banach space X is related to the representation of X as a weak* first Borel class subset of some dual Banach space (its bidual X^{**}, for instance) by well known results due to Edgar and Wheeler [8], and Ghoussoub and Maurey [9, 10, 11]. The generalizations of those results depend on a new notion of Borel set of the first class "generated by convex sets" which is more suitable to deal with non separable Banach spaces. The asymptotic-norming property, introduced by James and Ho [12], and the approximation by differences of convex continuous functions are also studied in this context.

1 Introduction

This paper is devoted to show several connections between geometrical properties of a Banach space and certain kinds of descriptive sets. Dealing with non metrizable topologies, as the weak topology of a Banach space, new types of Borel subsets appear, even of the first class, that is, obtained by just one countable operation. We say that a subset A of a topological space V is $(\mathcal{F} \wedge \mathcal{G})_\sigma$ (resp. $(\mathcal{F} \vee \mathcal{G})_\delta$) if there are closed subsets F_n and open subsets G_n of V such that $A = \bigcup_{n=1}^{\infty}(F_n \cap G_n)$ (resp. $A = \bigcap_{n=1}^{\infty}(F_n \cup G_n)$). Clearly, a subset is $(\mathcal{F} \wedge \mathcal{G})_\sigma$ if, and only if, its complement is $(\mathcal{F} \vee \mathcal{G})_\delta$.

We shall consider a Banach space X as a topological subspace of its bidual X^{**} endowed with the weak* topology. Jayne, Namioka and Rogers [13] proved that if a Banach space X is a Borel subset of X^{**}, then X has a certain topological property called σ-fragmentability. It is natural to expect that X will have stronger properties if it is of lower Borel class in X^{**}. Indeed, Edgar and Wheeler showed [8] that X has the point of continuity property (PCP) if X is a

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$(\mathcal{F} \vee \mathcal{G})_\delta$ subset of X^{**} and the converse is true if X is separable. Recall that a map has the **point of continuity property** if its restriction to any nonempty closed subset has a point of continuity and a Banach space X has the point of continuity property (PCP) if the identity map on B_X from the weak to the norm topology has it. If $X^{**} \setminus X = \bigcup_{n=1}^{\infty} K_n$ where every K_n is convex and weak* compact, then X has the Radon-Nikodým property (RNP) [8]. This is far from being a characterization, because if a separable Banach space X is a $\mathcal{G}_\delta$ subset of X^{**}, then X^* is also separable. Ghoussoub and Maurey [9] proved the following: *A separable space X has the RNP if, and only if, there is a separable Banach space Y such that $X \subset Y^*$ (isomorphically) and $Y^* \setminus X = \bigcup_{n=1}^{\infty} K_n$ where every K_n is convex and weak* compact.*

We shall consider a particular subclass of the $(\mathcal{F} \wedge \mathcal{G})_\sigma$ sets in a topological linear space (V, τ). A subset $D \subset V$ is said to be a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ **set with respect to** τ if $D = \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$, where A_n and B_n are convex τ-closed subsets of V. In case τ is the norm topology, we just say $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set. Using the notion of $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set we are able to give the following characterization of the RNP.

Theorem 1.1 *A separable Banach space X has the RNP if, and only if, $X^{**} \setminus X$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set with respect to the weak* topology.*

A well known result of Edgar [7] connecting renorming theory and Borel structure establishes that, in a Banach space with a Kadec norm, every norm open set is an $(\mathcal{F} \wedge \mathcal{G})_\sigma$ set with respect to the weak topology. Recall that the norm $\|.\|$ of X is said to be **locally uniformly rotund** (LUR) if for every $x, x_k \in X$, such that $\lim_k \|x_k\| = \|x\|$ and $\lim_k \|x + x_k\| = 2\|x\|$, then $\lim_k \|x - x_k\| = 0$. A LUR norm is Kadec, see [6] for this and further information.

Theorem 1.2 *Every norm open subset of a Banach space X is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set if, and only if, X has an equivalent locally uniformly rotund norm.*

Let V be a topological space and d a metric on V not necessarily related to the topology of V. We say that V is **fragmented** by d if every nonempty subset of V has a nonempty relatively open subset of arbitrarily small diameter. Weakly compact subsets of a Banach space are norm fragmented [21]. If X is a Banach space and $Y \subset X^*$ is a norming subspace, then there is a canonical embedding of X into Y^*. In this case, the topology $\sigma(X, Y)$ is the restriction of the weak* topology of Y^*. We shall always regard X as a subset of Y^*. Further, after a suitable renorming, we may assume that $B_{Y^*} = \overline{B_X}^{w^*}$. The following results generalize [8, Theorem 4.13] and part of [9, Theorem III.1].

Theorem 1.3 *Let X be a Banach space and $Y \subset X^*$ a norming subspace. Assume that the $\sigma(X, Y)$-compact subsets of X are fragmented by the norm and X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of Y^* in the weak* topology. Then the identity map on B_X from the $\sigma(X, Y)$-topology to the norm has the point of continuity property.*

Theorem 1.4 *Let X be a Banach space and $Y \subset X^*$ a norming subspace. Assume that the $\sigma(X,Y)$-compact subsets of X are fragmented by the norm and $Y^* \setminus X$ is $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* in the weak* topology. Then X has the RNP.*

For a separable Banach space Y the following three conditions are equivalent: (1) Y^* *has the RNP*; (2) Y^* *is separable*; (3) Y^* *has an equivalent dual norm* $\|.\|$ *such that* $w^*\text{-}\lim_n y_n^* = y^*$ *and* $\lim_n \|y_n^*\| = \|y^*\|$ *implies that* $\lim_n \|y_n^* - y^*\| = 0$. Bourgain and Delbaen [2], and McCartney and O'Brien [18], built separable Banach spaces having the RNP and isomorphic to no subspace of a separable dual Banach space. However, separable Banach spaces with RNP could be characterized by a renorming property in the spirit of condition (3) above. In this sense, James and Ho [12] introduced the asymptotic-norming property (ANP) and showed that it implies the RNP. The equivalence between the RNP and the ANP for separable spaces was established by Ghoussoub and Maurey [11]. Although we leave the technical definition of the ANP for next paragraph, what they actually proved read as follows: *A separable Banach space X has the RNP if, and only if, there is a separable Banach space Y such that X is isomorphic to a subspace of Y^* and the following property is verified: for any sequence $(x_n) \subset X$, such that* $w^*\text{-}\lim_n x_n = y^* \in Y^*$ *and* $\lim_n \|x_n\| = \|y^*\|$, *then* $\lim_n \|x_n - y^*\| = 0$. A main argument used in [11] to prove the previous equivalence motivated the authors to introduce the concept of strong $\mathcal{H}_\delta$ subset. We shall take advantage of their idea to define a similar notion. Let $(V, \|.\|)$ be a normed space and let τ be a vector topology on V and denote by d the induced distance between sets. A subset $D \subset V$ is said to be a **strong** $(\mathcal{C} \setminus \mathcal{C})_\sigma$ **set** (with respect to τ) if $D = \bigcup_{n=1}^{\infty} (A_n \setminus B_n)$ where A_n and B_n are convex τ-closed subsets of V such that $d(X \setminus D, A_n \setminus B_n) > 0$ for every $n \in \mathbb{N}$.

We do not need to deal with all the different ANP's introduced in [12], so we shall just use the weakest one along the paper, named ANP-III there. Given a norming subset $\Phi \subset B_{X^*}$, we say that a sequence $(x_n) \subset S_X$ is asymptotically normed by Φ, if for every $\varepsilon > 0$, there is $y \in \Phi$ and $N \in \mathbb{N}$, such that $y(x_n) > 1 - \varepsilon$ for all $n \geq N$. We say that a Banach space X has the Φ-ANP if $\bigcap_{n=1}^{\infty} \overline{conv}^{\|\cdot\|}(\{x_m \colon m \geq n\}) \neq \emptyset$ for every sequence (x_n) asymptotically normed by Φ. We say that X has the ANP if it has the Φ-ANP for some norming subset $\Phi \subset B_{X^*}$. The characterization of the ANP obtained by Hu and Lin [17], which avoids asymptotically normed sequences, is a main tool to prove the following:

Theorem 1.5 *A Banach space X has the ANP with some equivalent norm if, and only if, $X^{**} \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X^{**} with respect to the weak* topology.*

In relation with this result, we shall discuss conditions on a Banach space X implying that $X^{**} \setminus X$ is $(\mathcal{C} \setminus \mathcal{C})_\sigma$ or strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ into (X^{**}, w^*). The separable case is generalized in a quite natural way in terms of Szlenk type indices, see

Theorem 3.5. A different kind of example is provided by Banach spaces isomorphic to dual Banach spaces having a LUR norm, Theorem 4.4.

Measurability with respect to $(\mathcal{C} \setminus \mathcal{C})_\sigma$ sets is related to approximation by differences of convex functions, see Proposition 2.1. We give the following extension of a recent result of Cepedello-Boiso [5]:

Theorem 1.6 *A function $h\colon X \to \mathbb{R}$ defined on a Banach space is a pointwise limit of a sequence of differences of convex continuous functions if the sets $h^{-1}(r, +\infty)$ and $h^{-1}(-\infty, r)$ are strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ for all $r \in \mathbb{R}$.*

The rest of the paper is devoted to prove the results listed in this introduction as a part of a common frame. In section 2 we study $(\mathcal{C} \setminus \mathcal{C})_\sigma$ and strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of a Banach space. In section 3 we deal with Banach spaces representable as first Borel class subsets of a dual space. The Banach spaces which embed as strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of a dual space are studied in section 4 in relation with the ANP. Although the paper deals mainly with non separable Banach spaces, we show in Proposition 4.7 how our techniques apply to provide a new proof of the equivalence between RNP and ANP for a separable Banach space.

2 Differences of convex sets

The first result of the section shows that $(\mathcal{C} \setminus \mathcal{C})_\sigma$ sets appear naturally when one deals with differences of convex functions.

Proposition 2.1 *Let X be a Banach space and $h\colon X \to \mathbb{R}$ a function which is uniform limit of differences of convex functions. Then both $h^{-1}(-\infty, r)$ and $h^{-1}(r, +\infty)$ are $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of X for every $r \in \mathbb{R}$.*

Proof. First assume that $h = f_1 - f_2$ where f_1 and f_2 are convex. Clearly, we can restrict ourselves to examine the set $h^{-1}(0, +\infty)$. We have

$$h^{-1}(0, +\infty) = \bigcup_{r<s} \{x \in X\colon f_1(x) > s, f_2(x) \le r\}$$

$$= \bigcup_{r<s} (\{x \in X\colon f_2(x) \le r\} \setminus \{x \in X\colon f_1(x) \le s\})$$

where the indices r and s are rational. Thus $h^{-1}(0, +\infty)$ is expressed as a countable union of differences of convex closed sets.
Let h be uniform limit of a sequence (h_n) of differences of convex functions. Changing $h_n(x)$ by $h_n(x) - \|h - h_n\|_\infty$ we may assume that $h_n \le h$. In that case, $h^{-1}(r, +\infty) = \bigcup_{n=1}^{\infty} h_n^{-1}(r, +\infty)$ for any $r \in \mathbb{R}$. ■

Remark 2.2 *A similar result does nor hold for pointwise limits even in case of $X = \mathbb{R}$. The Cantor set C is not $(\mathcal{C} \setminus \mathcal{C})_\sigma$ and it is quite easy to put the characteristic function of C as a pointwise limit of differences of convex functions.*

For a subset $A \subset X$, we shall denote A^o the norm interior of A. Given $Z \subset X^*$ a norming subspace and $A \subset X$ convex and $\sigma(X,Z)$-closed, we shall consider the set

$$\mathcal{B}[A,r] = \overline{A + rB_X}^{\sigma(X,Z)}$$

It is easy to see that $\bigcap_{r>0} \mathcal{B}[A,r] = A$. In the following, we shall assume that the norming subspace $Z \subset X^*$ is 1-norming. To do that, just take the norm on X which has as unit ball $\overline{B_X}^{\sigma(X,Z)}$. The set of all $\sigma(X,Z)$-open halfspaces of X will be denoted by $\mathbb{H}(Z)$.

Lemma 2.3 *Let X be a Banach space and $Z \subset X^*$ a norming subspace. For a given subset E of X the following conditions are equivalent:*

i) $X \setminus E$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X with respect to $\sigma(X,Z)$.

ii) There are sequences of convex $\sigma(X,Z)$-closed sets (A_n) and (B_n) with nonempty norm interior such that $X \setminus E = \bigcup_{n=1}^{\infty}(A_n^o \setminus B_n)$.

iii) There is a sequence of convex $\sigma(X,Z)$-closed subsets (A_n) such that for every $x \in X \setminus E$, there is $H \in \mathbb{H}(Z)$ verifying that $x \in A_n \cap H$ and $d(A_n \cap H, E) > 0$.

iv) There is a sequence of convex $\sigma(X,Z)$-closed subsets (A_n) with nonempty norm interior such that for every $x \in X \setminus E$ there is $H \in \mathbb{H}(Z)$ such that $x \in A_n^o \cap H$ and $A_n \cap H \subset X \setminus E$.

Proof. *i)* $\Rightarrow$ *iii)* Follows easily by the Hahn-Banach theorem.
iii) $\Rightarrow$ *iv)* Let A be a $\sigma(X,Z)$-closed convex set, suppose that $x_0 \in A$ and there is $H \in \mathbb{H}(Z)$ such that $d(A \cap H, E) > 0$. If $H = \{x \in X\colon z(x) > a\}$, then take $b > a$ such that the halfspace $H' = \{x \in X\colon z(x) > b\}$ still contains x_0. Take $0 < r < \inf(d(A \cap H, E), d(H', A \setminus H))$. It is easy to see that

$$\mathcal{B}[A,r] \cap H' \subset \mathcal{B}[A \cap H, r] \subset X \setminus E$$

Suppose now that (A_n) is a sequence as in statement *iii)*. The argument above shows that the double sequence $\mathcal{B}[A_n, m^{-1}]$ satisfies statement *iv)*.
iv) $\Rightarrow$ *ii)* Let B_n be the $\sigma(X,Z)$-closed convex set obtained from A_n removing the $\sigma(X,Z)$-open slices disjoint from E, namely

$$B_n = \{x \in A_n\colon \text{such that } A_n \cap H \cap E \neq \emptyset \text{ whenever } x \in H \in \mathbb{H}(Z)\}$$

Then we have $X \setminus E = \bigcup_{n=1}^{\infty}(A_n^o \setminus B_n)$. To ensure nonempty norm interiors replace the sequence (B_n) by the double sequence $\mathcal{B}[B_n, m^{-1}]$.
ii) $\Rightarrow$ *i)* Fix $a_n \in A_n^o$ and take

$$A_{n,m} = a_n + (1 - m^{-1})(A_n - a_n)$$

Then we have $d(A_{n,m}, X \setminus A_n^o) > 0$. Define the sets $B_{n,m} = \mathcal{B}[B_n, m^{-1}]$. Since $E \subset (X \setminus A_n^o) \cup B_n$, we deduce that $d(A_{n,m} \setminus B_{n,m}, X \setminus E) > 0$. On the other hand $X \setminus E = \bigcup_{n,m=1}^{\infty}(A_{n,m} \setminus B_{n,m})$. ■

Theorem 2.4 *Let X be a Banach space, let $Z \subset X^*$ be a norming subspace and let $E \subset X$ be a subset. The following properties are equivalent:*

i) $X \setminus E$ *is a strong* $(\mathcal{C} \setminus \mathcal{C})_\sigma$ *subset of* X *with respect to* $\sigma(X, Z)$.

ii) There is an equivalent $\sigma(X, Z)$*-lower semicontinuous norm* $|||.|||$ *such that for any* $(x_n) \subset E$ *and* $x \in X$ *with* $\lim_n |||x_n||| = |||x|||$ *and* $\lim_n |||x_n + x||| = 2|||x|||$*, then* $x \in E$.

Proof. *i)* $\Rightarrow$ *ii)* By Lemma 2.3, we may assume that $X \setminus E = \bigcup_{n=1}^{\infty} (A_n^o \setminus B_n)$ where the sets A_n and B_n are $\sigma(X, Z)$-closed convex and have nonempty norm interior. Fix interior points $a_n \in A_n$ and $b_n \in B_n$ for every $n \in \mathbb{N}$ and let f_n and g_n be the Minkowski functionals with respect to the points a_n and b_n of the set A_n and B_n, respectively. We define a symmetric convex function F on X by the formula

$$F(x)^2 = \|x\|^2 + \sum_n \alpha_n f_n(x)^2 + \sum_n \beta_n g_n(x)^2$$

$$+ \sum_n \alpha_n f_n(-x)^2 + \sum_n \beta_n g_n(-x)^2$$

where (α_n) and (β_n) are positive constants taken in such a way to guarantee the uniform convergence on bounded subsets of X of the series, so F is uniformly continuous on bounded sets, and that the absolutely convex set $B = \{x \in X\colon\ F(y^*) \leq 1\}$ contains 0 as an interior point. Let $|||.|||$ be the functional of Minkowski of B. Then $|||.|||$ is an equivalent $\sigma(X, Z)$-lower semicontinuous norm on X. It is standard to check that for every sequence $(x_k) \subset B$ with $\lim_k |||x_k||| = 1$, then $\lim_k F(x_k) = 1$.
Suppose that we are given a sequence $(x_n) \subset E$ such that $\lim_n |||x_n||| = |||x|||$ and $\lim_n |||x_n + x||| = 2|||x|||$. We want to show that $x \in E$. Suppose that $x \in X \setminus E$ in order to get a contradiction. Then $x \in A_n^o \setminus B_n$ for some $n \in \mathbb{N}$, and thus $f_n(x) < 1$ and $g_n(x) > 1$. An standard convexity argument [6, Fact 2.3] shows easily that $\lim_k f_n(x_k) = f_n(x)$ and $\lim_k g_n(x_k) = g_n(x)$. For k large enough we should have $f_n(x_k) < 1$ and $g_n(x_k) > 1$, which implies that $x_k \in A_n^o \setminus B_n \subset X \setminus E$, a contradiction.
ii) $\Rightarrow$ *i)* Assume X is endowed with $|||.|||$. First we shall show that for every $x \in X \setminus E$ there is a rational $r > |||x|||$ and $H \in \mathbb{H}(Z)$ such that $x \in \mathcal{B}[0, r] \cap H$ and $\mathcal{B}[0, r] \cap H \subset X \setminus E$. Suppose not and take rational numbers $s_n < |||x||| < r_n$ such that $\lim_n (r_n - s_n) = 0$ and take $H_n \in \mathbb{H}(Z)$ containing x and disjoint from $\mathcal{B}[0, s_n]$. To get a contradiction, take $x_n \in \mathcal{B}[0, r_n] \cap H \cap E$. By the construction, $\lim_n |||x_n||| = |||x|||$ and $\lim_n |||x_n + x||| = 2|||x|||$, thus $x \in E$ which is impossible. If (A_n) is an enumeration of the balls $\mathcal{B}[0, r]$ with $r > 0$ rational, the argument above shows that for every $x \in X \setminus E$, there is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n^o \cap H$ and $A_n \cap H \subset X \setminus E$. Finally apply Lemma 2.3. ■

The following result includes Theorem 1.2.

Theorem 2.5 *For a Banach space X the following conditions are equivalent:*

i) X has an equivalent LUR norm.

ii) Every norm open subset of X is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set.

iii) Every norm open subset of X is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set.

Proof. *iii)* $\Rightarrow$ *ii)* It is trivial.
ii) $\Rightarrow$ *i)* Let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ be a basis for the norm topology such that every $\mathfrak{B}_n$ is discrete [14, p. 236]. Take $V_n = \bigcup \mathfrak{B}_n$. By the hypothesis we can find convex closed sets $(A_{m,n})$ and $(B_{m,n})$ such that $V_n = \bigcup_{m=1}^{\infty}(A_{m,n} \setminus B_{m,n})$. We claim that for every $x \in X$ and $\varepsilon > 0$ there is an open halfspace H and $m, n \in \mathbb{N}$ such that $x \in A_{m,n} \cap H$ and $\operatorname{diam}(A_{m,n} \cap H) < \varepsilon$. Indeed, for some $n \in \mathbb{N}$, there is $W \in \mathfrak{B}_n$ such that $x \in W \subset B(x, \varepsilon)$. Now, for some m we have $x \in A_{m,n} \setminus B_{m,n}$. Let H be an open halfspace containing x and disjoint from $B_{m,n}$. We have $A_{m,n} \cap H$ is convex and it is contained in V_n. By the discreteness of $\mathfrak{B}_n$ we must have $A_{m,n} \cap H \subset W \subset B(x, \varepsilon)$. The existence of an equivalent LUR norm is now a consequence of the theorem of Moltó, Orihuela and Troyanski [19] (see also [23]).
i) $\Rightarrow$ *iii)* An equivalent LUR norm satisfies condition ii) of Theorem 2.4 for any norm closed subset E, and thus, every norm open subset is strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$. ■

Proof of Theorem 1.6. Expressing h as the difference of its positive and negative parts, we may assume that $h \geq 0$. Indeed, if $h = h_+ - h_-$ then $h_+^{-1}(r, +\infty) = X$, $h_+^{-1}(-\infty, r) = \emptyset$ for $r < 0$, and $h_+^{-1}(r, +\infty) = h^{-1}(r, +\infty)$, $h_+^{-1}(-\infty, r) = h^{-1}(-\infty, r)$ for $r \geq 0$ (equalities for h_- are similar). Let (E_n) be an enumeration of the sets of the form $h^{-1}[r, +\infty)$ and $h^{-1}(-\infty, r]$ with r rational. Let $\|.\|_n \leq \|.\|$ be the norm given by Theorem 2.4 for the set E_n. Define an equivalent norm

$$\|x\|^2 = \sum_{n=1}^{\infty} 2^{-n} \|x\|_n^2$$

We claim that the norm $\|.\|$ has the following property: $\lim_k h(x_k) = h(x)$, whenever that $\lim_k \|x_k\| = \|x\|$ and $\lim_k \|x_k + x\| = 2\|x\|$. Take $r > h(x)$ a rational. Clearly, it is enough to show that $h(x_k) < r$ for k large. Suppose not, then $h(x_k) \geq r$ for infinitely many k's. Take n such that $E_n = h^{-1}[r, +\infty)$. By [6, Fact 2.3] we have $\lim_k \|x_k\|_n = \|x\|_n$ and $\lim_k \|x_k + x\|_n = 2\|x\|_n$. This implies $x \in E_n$, and thus $h(x) \geq r$ which is a contradiction.
To express h as a pointwise limit of differences of convex continuous functions we shall use an argument from [5]. Consider

$$h_n(x) = \inf_{y \in X} \{h(y) + n(2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2)\}$$

$$= 2n\|x\|^2 - \sup_{y \in X} \{n\|x + y\|^2 - 2n\|y\|^2 - h(y)\}$$

Notice that $h_n(x) \le h(x)$ just putting $y = x$. The last equality expresses h_n as a difference of two convex continuous functions. We claim that $\lim_n h_n(x) = h(x)$. Indeed, take $x_n \in X$ such that

$$h(x_n) + n(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) < h_n(x) + n^{-1}$$

Thus we have

$$\begin{aligned} 0 &\le 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \\ &< n^{-1}(h_n(x) - h(x_n)) + n^{-2} \\ &\le n^{-1}h(x) + n^{-2} \end{aligned}$$

As the last term goes to 0 when n grows, we deduce that $\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$, and by [6, Fact 2.3] this is equivalent to $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x_n + x\| = 2\|x\|$. Using the first part of the proof, we have $\lim_n h(x_n) = h(x)$. On the other hand

$$h(x_n) - n^{-1} < h_n(x) \le h(x)$$

and thus $\lim_n h_n(x) = h(x)$. This ends the proof. ■

Remark 2.6 *Theorem 1.6 remains true, with minor changes in the proof, if the function h is just defined on a subset of X.*

Corollary 2.7 *Let X be a Banach space and let $h\colon X \to \mathbb{R}$ be a uniformly continuous function such that for every $r \in \mathbb{R}$ the sets $h^{-1}(-\infty, r)$ and $h^{-1}(r, +\infty)$ are both $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subsets of X. Then h is a pointwise limit of differences of convex continuous functions.*

Proof. Given any real number r, the decomposition

$$h^{-1}(-\infty, r) = \bigcup_{n=1}^{\infty} h^{-1}(-\infty, r - n^{-1})$$

shows us how to put $h^{-1}(-\infty, r)$ as a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X (positive distances are ensured by the uniform continuity). For the set $h^{-1}(r, +\infty)$ we can find a similar decomposition. ■

Theorem 1.6 clearly extends the following result due to Cepedello-Boiso [5].

Corollary 2.8 *Let X be a Banach space having an equivalent LUR norm. Then any continuous function is the pointwise limit of a sequence of differences of convex continuous functions.*

3 Banach spaces of the first Borel class

In this section we shall study Banach spaces which are $(\mathcal{F}\vee\mathcal{G})_\delta$ subsets into some dual Banach space endowed with the weak* topology. The important particular class of Banach spaces X such that B_X is a $\mathcal{G}_\delta$-subset of $(B_{X^{**}}, w^*)$ was studied by Edgar and Wheeler in [8]. In that case they proved that $X = X_1 \oplus X_2$ (direct sum) where X_1 and its dual X_1^* are separable and X_2 is reflexive. A trivial but useful observation is the following: if X is a subspace of a dual Banach space Y^*, then $Y^* \setminus X$ is a countable union of $\mathcal{F} \wedge \mathcal{G}$-sets (resp. differences of w^*-compact convex sets, differences of w^*-compact symmetric convex sets) if and only if $B_{Y^*} \setminus B_X$ is a countable union of sets of the same kind.

Lemma 3.1 *If a topological space V is homeomorphic to a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of a compact Hausdorff space, then V is hereditarily Baire.*

Proof. Suppose that $V \subset K$ and $V = \bigcap_{n=1}^{\infty}(F_n \cup G_n)$ where F_n is closed and G_n is open in K for every $n \in \mathbb{N}$. Is enough to show that V is Baire since any closed subset of V is also a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of K. We may assume that V is dense in K. Let F_n^o denote the interior of F_n. It is easy to verify that $F_n^o \cup G_n$ is dense in K. The intersection $\bigcap_{n=1}^{\infty}(F_n^o \cup G_n)$ is a dense Čech-complete subset of V, and thus V is a Baire space. ■

Proof of Theorem 1.3. The space $(B_X, \sigma(X,Y))$ is Čech-analytic (see [13]) since B_X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of (B_{Y^*}, w^*) and thus it is also hereditarily Baire by the former lemma. By [13, Theorem 4.1], a Čech-analytic space with a finer lower semicontinuous metric d such that its compact subsets are fragmentable by d is itself σ-fragmentable (see [13]) by d. A hereditarily Baire topological space which is σ-fragmentable by a lower semicontinuous metric d is fragmentable by d, [13, Corollary 3.1.2]. If any hereditarily Baire space is fragmented by a metric, then the identity map to the space endowed with the metric has the point of continuity property by [21, Lemma 1.1]. ■

Remark 3.2 *The $\sigma(X,Y)$-compact subsets of a Banach space X are fragmented by the norm if X has an equivalent $\sigma(X,Y)$-Kadec norm, that is, on its unit sphere the norm topology and $\sigma(X,Y)$ coincide [13]. If the dual unit ball (B_{X^*}, w^*) is a Corson compact (for instance, if X is WCD), then the $\sigma(X,Y)$-compact subsets of X are fragmented by the norm for any norming subspace $Y \subset X^*$ [4].*

Proof of Theorem 1.4. Let (Ω, Σ, μ) a probability space and let $\nu\colon \Sigma \to X$ be a μ-continuous vector measure with average range in B_X. There exists a w^*-Borel measurable density $f\colon \Omega \to B_{Y^*}$, that is, $\langle \nu(C), y\rangle = \int_C \langle f, y\rangle\, d\mu$ for every $C \in \Sigma$ and $y \in Y$. The proof is based on the theory of liftings. We shall sketch the proof because we found no suitable reference. For any $y \in Y$, the signed measure $\langle \nu, y\rangle$ is μ-continuous, so it has a Radon-Nikodým derivative $f_y \in L^1(\mu)$. Let ρ be a lifting of $L^\infty(\mu)$, see [1]. It is easy to check that the map

$y \to \rho(f_y)(\omega)$ is linear for every $\omega \in \Omega$ and bounded by $\|y\|$, so there is $y^*_\omega \in B_{Y^*}$ such that $y^*_\omega(y) = \rho(f_y)(\omega)$. Clearly, the map defined by $f(\omega) = y^*_\omega$ is w^*-scalarly measurable, so it is w^*-Baire measurable by [7, Theorem 2.3]. We claim that f also is w^*-Borel measurable and the measure image $f(\mu)$ is w^*-Radon. Indeed, if ρ_K is the abstract lifting considered in [1, §2] for the compact space $K = B_{Y^*}$, then

$$h \circ \rho_K(f)(\omega) = \rho(h \circ f)(\omega)$$

for every $\omega \in \Omega$ and every $h \in C(K)$. From the definition of f we get that $\rho_K(f) = f$ just taking as continuous functions h the elements $y \in Y$. The desired properties follow from [1, Theorem 2.1].

Put $B_{Y^*} \setminus B_X = \bigcup_{n=1}^{\infty}(A_n \setminus B_n)$ and observe that B_X is a w^*-Borel subset of B_{Y^*}. We want to show that $f(\omega) \in B_X$ for μ-almost all $\omega \in \Omega$. That is equivalent to show that $f(\mu)(B_X) = 1$. Fix $n \in \mathbb{N}$ and suppose that $f(\mu)(A_n \setminus B_n) > 0$. Using the fact that $f(\mu)$ is w^*-Radon, we can find a w^*-open halfspace H disjoint with B_n such that $f(\mu)(A_n \cap H) > 0$. Without loss of generality, assume that $L \cap X = \emptyset$, where $L = \overline{A_n \cap H}^{w^*}$ is w^*-compact convex. Let $C = f^{-1}(L)$ and $x = \nu(C)/\mu(C) \in X$. Take $y \in Y$ such that $\langle x, y \rangle < \inf_{y^* \in L}\langle y^*, y\rangle$. Thus

$$\langle x, y \rangle < \mu(C)^{-1} \int_C \langle f, y \rangle \, d\mu$$

which is a contradiction.

We have now that $f(\mu)$ is a Radon measure on $(B_X, \sigma(X,Y))$. Since the $\sigma(X,Y)$-compact subsets of X are fragmentable, $f(\mu)$ is concentrated on a norm separable subset of B_X. By [7, Theorem 5.2], there is a Bochner measurable function g: $\Omega \to B_X$, such that for every $y \in Y$, the equality $\langle g(\omega), y\rangle = \langle f(\omega), y \rangle$ holds for μ-almost all $\omega \in \Omega$. This implies $\langle \nu(C), y \rangle = \langle \int_C g \, d\mu, y \rangle$ for every $C \in \Sigma$ and $y \in Y$, and thus $\nu(C) = \int_C g \, d\mu$. This proves the RNP of X. ■

To provide examples of Banach spaces being weak* first Borel class subsets into some dual Banach space we need two definitions which are based on the Szlenk index [15].

Definition 3.3 *We say that X has countable Y-fragmentability index if for every $\varepsilon > 0$ there exists a decreasing transfinite sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of $\sigma(X,Y)$-closed subsets of B_X, where γ_ε is countable, such that $B_X = \bigcup_{\alpha<\gamma_\varepsilon}(C_\alpha \setminus C_{\alpha+1})$ and for every $x \in C_\alpha \setminus C_{\alpha+1}$ there is a $\sigma(X,Y)$-open neighbourhood U of x with diam$(C_\alpha \cap U) < \varepsilon$.*

Clearly, if the Banach space X has countable Y-fragmentability index, then B_X endowed with the $\sigma(X,Y)$-topology is norm fragmentable. The converse is true for X a separable Banach space.

Definition 3.4 *We say that X has countable Y-dentability index if for every $\varepsilon > 0$ there exists a decreasing transfinite sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of $\sigma(X,Y)$-closed* *convex subsets of B_X, where γ_ε is countable, such that $B_X = \bigcup_{\alpha<\gamma_\varepsilon}(C_\alpha \setminus C_{\alpha+1})$ and for every $x \in C_\alpha \setminus C_{\alpha+1}$ there is a $\sigma(X,Y)$-open halfspace H containing x with diam$(C_\alpha \cap H) < \varepsilon$.*

Lancien [15, 16] has studied Banach spaces with countable indices. Among other results, he proved that a dual Banach space X^* has countable X-dentability index if it has countable X-fragmentability index.

Theorem 3.5 *Let X be a Banach space, $Y \subset X^*$ a norming subspace.*

a) If X has countable Y-fragmentability index, then X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of Y^* with respect to the weak topology.*

b) If X has countable Y-dentability index, then $Y^* \setminus X$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak topology.*

Proof. Given $n \in \mathbb{N}$, let $(C^n_\alpha)_{\alpha<\gamma_n}$ be a family of sets like in Definition 3.3 with $\varepsilon = n^{-1}$. For every α we can put

$$C^n_{\alpha+1} = C^n_\alpha \setminus \bigcup_{i \in I^n_\alpha} U^n_i$$

where the family $\{U^n_i \colon i \in I^n_\alpha\}$ consists of w^*-open subsets of Y^* such that diam$(C^n_\alpha \cap U^n_i) < n^{-1}$. The set $U^n_\alpha = \bigcup_{\beta<\alpha} \bigcup_{i \in I^n_\beta} U_i$ is w^*-open. Take

$$A = B_{Y^*} \cap \bigcap_{n \in \mathbb{N}} \bigcap_{\alpha<\gamma_n} (\overline{C^n_\alpha}^{w^*} \cup U^n_\alpha)$$

which is clearly $(\mathcal{F} \vee \mathcal{G})_\delta$ in (Y^*, w^*). Clearly $B_X \subset A$. We claim that $B_X = A$. Indeed, take any $z \in A$ and fix $n \in \mathbb{N}$. Since $(\overline{C^n_\alpha}^{w^*})$ is decreasing in α, we can take α_n such that $z \in \overline{C^n_{\alpha_n}}^{w^*}$ but $z \notin \overline{C^n_{\alpha_n+1}}^{w^*}$. This means that $z \in U^n_{\alpha_n}$ and so $z \in U^n_{i_n}$ for some $i_n \in I^n_{\alpha_n}$. We have now that $z \in \overline{C^n_{\alpha_n}}^{w^*} \cap U^n_{i_n}$. Since $\overline{C^n_{\alpha_n}}^{w^*} \cap U^n_{i_n} \subset \overline{C^n_{\alpha_n} \cap U^n_{i_n}}^{w^*}$ and diam$(\overline{C^n_{\alpha_n} \cap U^n_{i_n}}^{w^*}) \leq n^{-1}$ by the w^*-lower semicontinuity of the norm, so we have $d(z, X) \leq n^{-1}$. Since that happens for every $n \in \mathbb{N}$, then $z \in X$.
Assume now that X has countable Y-dentability index. Then the w^*-open sets U_i above can be taken to be w^*-open halfspaces and it is easy to see that the sets (C^n_α) can be taken to be symmetric. Let us change the notation putting $H_i = U_i$. We have

$$B_X = \bigcap_{n \in \mathbb{N}} \bigcap_{\alpha<\gamma_n} (\overline{C^n_\alpha}^{w^*} \cup \bigcup_{\beta<\alpha} \bigcup_{i \in I^n_\beta} H_i)$$

thus

$$B_{Y^*} \setminus B_X = \bigcup_{n \in \mathbb{N}} \bigcup_{\alpha<\gamma_n} (D_n \setminus \overline{C^n_\alpha}^{w^*})$$

where $D^n_\alpha = \bigcap_{\beta<\alpha} \bigcap_{i\in I^n_\beta} (Y^* \setminus H_i)$ is w^*-compact and convex. ■

If X is a separable Banach space with the PCP (resp. RNP) then X has countable X^*-fragmentability (resp. X^*-dentability) index. The following result due to Edgar and Wheeler [8] follows from Theorems 1.3 and 3.5.

Corollary 3.6 *Let X be a separable Banach space. Then X has the PCP if and only if X is a $(\mathcal{F} \vee \mathcal{G})_\delta$ subset of X^{**} with respect to the weak* topology.*

Proof of Theorem 1.1. Just apply Theorems 1.4 and 3.5. ■

The convex counterpart of $\mathcal{G}_\delta$-sets was introduced in [9, 11] as follows. Let $C \subset D$ be subsets of a dual Banach space Y^*. We say that C is a $\mathcal{H}_\delta$ **subset** (resp. **strong** $\mathcal{H}_\delta$ **subset**) of D if $D \setminus C = \bigcup_{n=1}^\infty K_n$ where the sets K_n are w^*-compact and convex (resp. and $d(C, K_n) > 0$ for every $n \in \mathbb{N}$). Clearly, if C is an $\mathcal{H}_\delta$ set then $Y^* \setminus C$ is a $(\mathcal{C} \setminus \mathcal{C})_\sigma$ set. It is not difficult to show that the converse is true for Y separable. Analogous results holds adding "strong".

4 Strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ remainders and the ANP

In this section we shall discuss some links between the representation of a Banach space X into a dual and renorming properties.

Lemma 4.1 *Let X be a Banach space and let $Y \subset X^*$ be a norming subspace. The following are equivalent:*

i) $Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.

ii) There is a sequence (A_n) of convex w^*-compact subsets with nonempty norm interior such that for every $y^* \in Y^* \setminus X$ there is a w^*-open halfspace H such that $y^* \in A_n \cap H$ and $A_n \cap H \subset Y^* \setminus X$.

iii) There are sequences of convex w^*-compact sets (A_n), (B_n) with nonempty norm interior such that $Y^* \setminus X = \bigcup_{n=1}^\infty (A_n \setminus B_n)$.

Proof. Implications $i) \Rightarrow ii) \Leftrightarrow iii)$ can be deduced from Lemma 2.3 and the ideas of its proof. It is enough to show that statement $ii)$ above implies statement $iii)$ of Lemma 2.3, namely we shall prove that the sequence (A_n) given by $ii)$ verifies that for any $y^* \in Y^* \setminus X$ there is a w^*-open halfspace H such that $y^* \in A_n \cap H$ and $d(A_n \cap H, X) > 0$. Let $A \subset Y^*$ be a w^*-compact convex subset with nonempty norm interior. Suppose that $y^* \in A$ is such that it is contained in a w^*-open halfspace H with $A \cap H \subset Y^* \setminus X$. Take $x \in A^o \cap X$. Since $X = X - x$, we may change A by $A - x$, and without loss of generality assume that 0 is an interior point of A. In that case $y^* \neq 0$. If $H = \{u^* \in Y^* \colon u^*(y) > a\}$, then take

$b > a$ such that the halfspace $H' = \{u^* \in Y^*\colon u^*(y) > b\}$ still contains y^*. Since $A \cap X \subset A \setminus H$, then $d(A \cap H', A \cap X) > 0$. Take $\varepsilon > 0$ small enough to have

$$d(A \cap H', (1+\varepsilon)(A \cap X)) > 0$$

Since 0 is an interior point of A, then

$$d(A \cap H', Y^* \setminus (1+\varepsilon)A) > 0$$

The last two inequalities imply that $d(A \cap H', X) > 0$. ■

Remark 4.2 *Let X be a Banach space with countable Y-dentability index and the index and the sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of Definition 3.4 can be taken with nonempty norm interior. Then the proof of Theorem 3.5 together with Lemma 4.1 gives that $Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.*

Proposition 4.3 *Let X be a Banach space and $Y \subset X$ a norming subspace. Then the following statements are equivalent:*

i) *$Y^* \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of Y^* with respect to the weak* topology.*

ii) *There is an equivalent dual norm $\|.\|$ on Y^* such that for any $(x_n) \subset X$ and $y^* \in Y^*$ with $\lim_n \|x_n\| = \|y^*\|$ and $\lim_n \|x_n + y^*\| = 2\|y^*\|$, then $y^* \in X$.*

iii) *There is an equivalent dual norm $\|.\|$ on Y^* such that $S_{Y^*} \cap X$ is relatively weak* closed in S_{Y^*}.*

Proof. *i)* ⇒ *ii)* Follows from Theorem 2.4.
ii) ⇒ *iii)* Consider Y^* endowed with a norm satisfying *ii)*. It is enough to show that $y^* \in X$, if $\|y^*\| = 1$ and $(x_\omega) \subset S_X$ is a net w^*-converging to y^*. Indeed, the net $(x_\omega + y^*)$ is w^*-converging to $2y^*$. Since the norm is w^*-lower semicontinuous we have $\lim_\omega \|x_\omega + y^*\| = 2$. Thus we can take an increasing sequence (ω_n) such that $\lim_n \|x_{\omega_n} + y^*\| = 2$, and so $y^* \in X$.
iii) ⇒ *i)* First we claim that for any $y^* \in S_{Y^*} \setminus S_X$, there is a w^*-open halfspace H such that $y^* \in H$ and $B_{Y^*} \cap H \subset Y^* \setminus X$. To see that, just remark that $y^* \notin \overline{B_X}^{w^*}$ and the existence of H is a consequence of the Hahn-Banach theorem. Now take any $y^* \in Y^* \setminus X$. By homogeneity and a small perturbation, there is a rational $r \geq \|y^*\|$ and a w^*-open halfspace H containing y^* such that $rB_{Y^*} \cap H \subset Y^* \setminus X$. Let (A_n) be a reenumeration of the sets of the form rB_{Y^*} with $r \geq 0$ rational. We have proved that for any $y^* \in Y^* \setminus X$, there is $n \in \mathbb{N}$ and H a w^*-open halfspace containing y^* such that $A_n \cap H \subset Y^* \setminus X$. Then the conclusion follows from Lemma 4.1, because the sets A_n have nonempty norm interior. ■

The following result implies that $\ell_1(\Gamma)$ is a $(\mathcal{F} \vee \mathcal{G})_\delta$ its bidual, which improves the estimate $(\mathcal{F} \wedge \mathcal{G})_{\sigma\delta}$ given in the proof of [7, Theorem 3.3].

Theorem 4.4 *Assume that X is Banach space which is isomorphic to a dual space with a weak*-Kadec norm. Then $X^{**}\setminus X$ is a strong $(\mathcal{C}\setminus\mathcal{C})_\sigma$ subset of X^{**} with respect to the weak* topology.*

Proof. Let Y be such that $X=Y^*$. Without loss of generality we may assume that the norm of Y^* is w^*-Kadec. Consider in X^* the symmetric convex body $\overline{conv}^{\|\cdot\|}(B_Y\cup 2^{-1}B_{Y^{**}})$ which is the unit ball of some equivalent (but non dual) norm on X^*. Let $|\!|\!|.|\!|\!|$ be is dual norm, defined on X^{**}. It is easy to verify that if we endow X^{**} with $|\!|\!|.|\!|\!|$, then $S_{X^{**}}\cap X=S_X$. We shall prove that S_X is a closed subset of $S_{X^{**}}$ and the result will follow from Proposition 4.3. Let $(x_\omega)\subset S_X$ a net which is w^*-converging to $x^{**}\in S_{X^{**}}$. Since $x^{**}\in B_{Y^{***}}$, its supremum on $2^{-1}B_{Y^{**}}$ is less that 2^{-1}. But $|\!|\!|x^{**}|\!|\!|=1$, so the supremum of x^{**} on B_Y must be 1. Put $x=x^{**}|_Y$ and realize that $x\in S_{Y^*}$. Clearly (x_ω) converges to x in the weak* topology of Y^* and $\|x_\omega\|=\|x\|$. Since the norm is w^*-Kadec, the net (x_ω) is norm converging to x. This implies that $x^{**}=x$ and thus $x^{**}\in S_X$. ■

Recall that ANP along this paper means ANP-III in the terminology of [12]. We shall use a nice characterization from [17] of the ANP to prove the following.

Proposition 4.5 *A Banach space X has the ANP if and only if there is an equivalent norm on X^* such that its dual norm on X^{**} induces the original norm of X, that is $S_X=S_{X^{**}}\cap X$, and S_X relatively w^*-closed in $S_{X^{**}}$.*

Proof. Assume that X has the Φ-ANP for some norming subset $\Phi\subset B_{X^*}$. It is easy to see that if a sequence $(x_n)\subset S_X$ is asymptotically normed by $\overline{conv}^{\|\cdot\|}(\Phi\cup-\Phi\cup 2^{-1}B_{X^*})$, then (x_n) has a subsequence (x_{n_k}) such that (x_{n_k}) or $(-x_{n_k})$ is asymptotically normed by Φ. Without loss of generality we may replace Φ by that convex set, which is the unit ball of some equivalent norm on X^*. Call $|\!|\!|.|\!|\!|$ that norm on X^* and notice that once X^{**} is endowed with the dual norm, then $S_{X^{**}}\cap X=S_X$. In [17] it is proved that if X has the Φ-ANP for any $x^{**}\in X^{**}\setminus X$, then $\sup_{\phi\in\Phi}x^{**}(\phi)<|\!|\!|x^{**}|\!|\!|$. It follows that $S_X=\overline{B_X}^{w^*}\cap S_{X^{**}}$, so S_X is relatively w^*-closed in $S_{X^{**}}$. Reciprocally, if $S_X=S_{X^{**}}\cap X$ and S_X relatively w^*-closed in $S_{X^{**}}$, when X^{**} is endowed with the dual of some norm $|\!|\!|.|\!|\!|$ on X^*. Let $\Phi=B_{X^*}$ and consider a Φ-asymptotically normed sequence (x_n). Clearly, any w^*-cluster point of (x_n) in X^{**} must be in $S_{X^{**}}$, and thus in X by the hypothesis. This implies that (x_n) is a relatively weakly compact subset and so $\bigcap_{n=1}^\infty\overline{conv}^{\|\cdot\|}(\{x_m\colon m\geq n\})\neq\emptyset$. ■

Combining the last result with Proposition 4.3 we obtain the following result which contains Theorem 1.5.

Theorem 4.6 *For a Banach space X the following are equivalent:*

i) X has the ANP with some equivalent norm.

ii) $X^{**} \setminus X$ *is a strong* $(\mathcal{C}\setminus\mathcal{C})_\sigma$ *subset of* X^{**} *with respect to the weak* topology.*

iii) *There is an equivalent dual norm* $\|.\|$ *on* X^{**} *such that for any* $(x_n) \subset X$ *and* $x^{**} \in X^{**}$ *with* $\lim_n \|x_n\| = \|x^{**}\|$ *and* $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$*, then* $x^{**} \in X$*.*

As an application, we shall prove that separable spaces with the RNP can be renormed with a property stronger than the ANP.

Proposition 4.7 *Let* X *be a separable Banach space with the RNP. Then there is an equivalent dual norm* $\|.\|$ *on* X^{**} *such that for any* $(x_n) \subset X$ *and* $x^{**} \in X^{**}$ *with* $\lim_n \|x_n\| = \|x^{**}\|$ *and* $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$*, then* $\lim_n \|x_n - x^{**}\| = 0$*.*

Proof. Since X is separable, there is an equivalent dual norm $\|.\|_1$ on X^{**} such that its restriction to X is LUR. Suppose that $X^{**}\setminus X$ is a strong $(\mathcal{C}\setminus\mathcal{C})_\sigma$ subset of X^{**} and let $\|.\|_2$ be the norm on X^{**} given by *iii)* of Theorem 4.6. We claim that the norm $\|.\|$ defined by $\|.\|^2 = \|.\|_1^2 + \|.\|_2^2$ satisfies the required condition. Indeed, if $(x_n) \subset X$ is a sequence with $\lim_n \|x_n\| = \|x^{**}\|$ and $\lim_n \|x_n + x^{**}\| = 2\|x^{**}\|$, then by [6, Fact 2.3], we have $\lim_n \|x_n\|_i = \|x^{**}\|_i$ and $\lim_n \|x_n + x^{**}\|_i = 2\|x^{**}\|_i$, for $i = 1, 2$. Taking $i = 2$, we deduce that $x^{**} \in X$, and taking $i = 1$, we obtain $\lim_n \|x_n - x^{**}\| = 0$ by the LUR property.
We shall prove that $X^{**} \setminus X$ is a strong $(\mathcal{C} \setminus \mathcal{C})_\sigma$ subset of X^{**}. According to Remark 4.2 we just need to build sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ as in Definition 3.4 with nonempty norm interior. We shall use the following fact: if X is a Banach space with the RNP and $C, D \subset X$ closed convex subsets such that $C \setminus D \neq \emptyset$ and $\varepsilon > 0$, then there is an open halfspace H such that $H \cap C \neq \emptyset$, $D \cap H = \emptyset$ and $\operatorname{diam}(C \cap H) < \varepsilon$. That is a consequence of the norm density in X^* of the elements x^* strongly exposing points of C [3, Theorem 3.5.4]. Fix $\varepsilon > 0$ and define a transfinite sequence of closed convex sets $(C_\alpha)_{\alpha\leq\gamma}$ which is strictly decreasing and such that $C_1 = B_X$, $C_\gamma = 2^{-1}B_X$, $C_\alpha = \bigcap_{\beta<\alpha} C_\beta$ if α is a limit ordinal and $C_{\alpha+1} = C_\alpha \setminus H_\alpha$, where $H_\alpha = \{x \in X\colon x^*_\alpha(x) > s_\alpha\}$ is an open halfspace disjoint from $2^{-1}B_X$ such that $\operatorname{diam}(C_\alpha \cap H_\alpha) < \varepsilon$. The construction is possible because of the fact and γ must be countable since (C_α) is strictly decreasing and X is separable. To obtain a sequence $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ as in Definition 3.4 we have to repeat the process again starting at $2^{-1}B_X$ and finishing at $2^{-2}B_X$, and so on. Clearly, it is enough to iterate that process n times, where n is such that $2^{-n+1} < \varepsilon$, to reach the empty set with some C_α. Then take γ_ε as the first index α such that $C_\alpha = \emptyset$.■

It is possible to arrange the ideas behind Proposition 4.7 in order to give a quite selfcontained proof of the result of Ghoussoub and Maurey [10] on the equivalence of the RNP and the ANP for separable Banach spaces. The key fact, as above, is "to eat" the unit ball in such a way that the remainders have nonempty interior.

We do not know if the ANP implies the existence of an equivalent Kadec norm. A negative answer will provide a Banach space with the RNP and no equivalent LUR norm. Actually, a Banach space with RNP and a Kadec norm has an equivalent LUR norm [20]. As a consequence, if a Banach space has the ANP and an equivalent Kadec norm, then X can be renormed to verify stronger asymptotic-norming properties [17]. Recall that Banach spaces with countable dentability index has an equivalent LUR norm [15]. In fact, if the sets $(C_\alpha)_{\alpha<\gamma_\varepsilon}$ of Definition 3.4 have nonempty norm interior, a convex series of their Minkowski functionals produces a norm which is LUR and has the ANP, see Remark 4.2.

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Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia. SPAIN
E-mail: matias@um.es