Borel properties of linear operators

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Abstract

Given an injective bounded linear operator $T : X \to Y$ between Banach spaces, we study the Borel measurability of the inverse map $T^{-1} : TX \to X$. A remarkable result of Saint-Raymond [17] states that if X is separable, then the Borel class of T^{-1} is α if, and only if, X^* is the α'th iterated sequential weak*-closure of T^*Y^* for some countable ordinal α. We show that Saint-Raymond's result holds with minor changes for arbitrary Banach spaces if we assume that T has certain property named co-σ-discreteness after Hansell [5]. As an application, we show that the Borel class of the inverse of a co-σ-discrete operator T can be estimated by the image of the unit ball or the restrictions of T to separable subspaces of X. Our results apply naturally when X is a WCD Banach space since in this case any injective bounded linear operator defined on X is automatically co-σ-discrete

1 Introduction

Let X^* be the dual of a Banach space X. For any subset $A \subset X^*$, let $A_{(1)}$ denote the set of the limits of all the w^*-converging sequences of elements from A, and define inductively $A_{(\alpha)} = \bigcup_{\beta<\alpha}(A_{(\beta)})_{(1)}$ for any ordinal α. If X is a separable Banach space and Z is a separating subspace of X^*, a classical result of S. Banach states that $Z_{(\alpha)} = X^*$ for some countable ordinal α, see [11]. The least α with that property is called the sequential order (s-order) of Z. Consider an injective bounded linear operator $T : X \to Y$ from the separable

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Banach space X into a normed space Y. A well known result due to Souslin [6, §39.IV] implies that $T^{-1} : TX \to X$ is Borel measurable. Saint-Raymond found in [17] (Corollaries 42 and 45) a nice link between these results, see also [11] and [12].

Theorem (Saint-Raymond [17]) *If $T : X \to Y$ is an injective bounded linear operator between separable Banach spaces, then T^{-1} is of Borel class α if, and only if, $(T^*Y^*)_{(\alpha)} = X^*$.*

For a non separable Banach space X, an analogous result may fail for several reasons that we describe in the following two examples.

Example 1.1 *Let $T : \ell_1(\Gamma) \to \ell_2(\Gamma)$ be the natural embedding for Γ uncountable. It is not very hard to prove that T^{-1} is of first Borel class. Clearly $T^*\ell_2(\Gamma)$ is a subset of $\ell_\infty^c(\Gamma)$, the subspace of $\ell_1(\Gamma)^* = \ell_\infty(\Gamma)$ made up of the elements with countable support. Since $\ell_\infty^c(\Gamma)_{(1)} = \ell_\infty^c(\Gamma)$, the space $\ell_\infty(\Gamma)$ cannot be reached by any sequential closure of $T^*\ell_2(\Gamma)$.*

Example 1.2 *Let X be the dual of the James-Tree space JT. Recall that JT is separable not containing a copy of ℓ_1, and X is not separable [3]. Fix a dense sequence (t_n) in the unit ball of JT and define and operator $T : X \to c_0$ by taking $(Tx)_n = n^{-1}x(t_n)$. It is easy to check that $T^*\ell_1$ is norm dense in JT and applying Rosenthal's theorem $(T^*\ell_1)_{(1)} = X^*$. On the other hand, T^{-1} is not Borel measurable. Indeed, T^{-1} sends TB_X, which is a closed separable subset of c_0 (so Polish) to the non separable metric space B_X.*

The first example suggests us that some modification of the s-order is necessary. An essential fact is that w^*-convergent sequences are bounded.

Definition 1.3 *For any subset A of a dual Banach space X^*, denote by $A_{[1]}$ the set of the limits of all the w^*-converging bounded nets of elements from A, and for any ordinal $\alpha > 1$ define $A_{[\alpha]} = \bigcup_{\beta<\alpha}(A_{[\beta]})_{[1]}$. The bounded-net order (bn-order) of A is the first ordinal α such that $A_{[\alpha]} = A_{[\alpha+1]}$.*

If A is convex and has bn-order α, a straightforward application of the Krein-Smulyan theorem gives that $A_{[\alpha]} = \overline{A}^{w^*}$.

To avoid the pathology showed in Example 1.2 we need some topological notions. Consider a family $\mathcal{H} = \{H_i : i \in I\}$ of subsets of a metric space X.

Recall that $\mathcal{H}$ is said to be discrete if any point $x \in X$ has a neighbourhood V which meets at most one member of $\mathcal{H}$. The family $\mathcal{H}$ is σ-discretely decomposable if there are discrete families $(\{H_i^n : i \in I\})_{n=1}^{\infty}$ such that $H_i = \bigcup_{n=1}^{\infty} H_i^n$ for every $i \in I$. The following definition is due to Hansell [5].

Definition 1.4 *Let $T : X \to Y$ be a injective map between metric spaces. We say that T is co-σ-discrete if for every discrete family $\{H_i : i \in I\}$ in X, the family $\{T(H_i) : i \in I\}$ is σ-discretely decomposable.*

The definition for non injective maps is possible but more technical. If X is separable, then any map $T : X \to Y$ is co-σ-discrete since any discrete family in X must be countable. A great part of the classical descriptive set theory in Polish spaces can be generalized to a non separable setting introducing discreteness conditions, for further information on this topic see the survey "Analytic sets in non-separable metric spaces" by A.H. Stone in [15]. Co-σ-discrete maps are called co-σ-continuous in [8], where this kind of maps are used in renorming theory.

Our extension of Saint-Raymond's result is stated as follows.

Theorem A *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces which is also co-σ-discrete. Then T^{-1} is Borel measurable if, and only if, the bn-order of T^*Y^* is countable, and in such a case, it coincides with the Borel class of T^{-1}.*

We shall show below that if we are looking for Borel properties of T^{-1} the hypothesis of T being co-σ-discrete is not very restrictive, see Corollary 3.8. Moreover, the Borel measurability of T^{-1} implies that T is co-σ-discrete if we assume Fleissner's Axiom [4]: *a family of subsets of a metric space which is point-finite and analytic-additive, i.e. the union of any subfamily is Souslin-$\mathcal{F}$ (see the beginning of the next section), is σ-discretely decomposable.* On the other hand, for the class of weakly countably determined spaces (WCD) every injective bounded linear operator is automatically co-σ-discrete, Corollary 2.11. The construction of co-σ-discrete injective bounded linear operator from certain Banach spaces into $c_0(\Gamma)$ has been studied by Oncina [9], with a different terminology, using projectional resolutions of identity.

Under the assumption that T is co-σ-discrete, we have simple criteria to check the Borel measurability of T^{-1}.

Theorem B *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces which is also co-σ-discrete. Then T^{-1} is Borel measurable of class α provided that one of the following conditions holds:*

(a) There exists a nonempty bounded open set $B \subset X$ such that TB is of additive class α in TX.

(b) For every separable closed subspace $E \subset X$, the restriction $T|_E$ has inverse of Borel class at most α.

Notice that conditions (a) and (b) above are necessary for being T^{-1} of Borel class α. Without the co-σ-discreteness assumption Theorem B is not longer true. The operator $T : \ell_\infty \to c_0$ defined by $T((x_n)) = (n^{-1}x_n)$ satisfies conditions (a) and (b) of Theorem B for $\alpha = 1$, but T^{-1} is not Borel measurable by the same argument that in Example 1.2.

Let us describe the contents of the next sections of the paper. Section 2 provides the tools that we shall need about co-σ-discrete maps for the next section as well as some applicable criteria to check the co-σ-discreteness. Section 3 is devoted to prove Theorem A and some of its consequences. To this aim we shall need some of the ideas and results used by Plichko in [12] to give an alternative proof of Saint-Raymond's result. In section 4 we shall prove that the Borel class of the inverse of an injective bounded linear operator $T : X \to Y$ is separably determined. We believe that the unexplained notions and terminology used in this paper are standard. As general references we have followed [6] for general topology, [6, 15] for descriptive set theory, [3, 1] for Banach spaces and [18] as specific source for the properties of WCD Banach spaces.

2 Co-σ-discrete maps

If τ is a given topology on X, the family of τ-Borel sets of additive class α will be denoted by $\mathcal{A}_\alpha(\tau)$, and $\mathcal{M}_\alpha(\tau)$ will stand for the sets of multiplicative class α. Given $\sigma = (\sigma_k)_{k=1}^\infty \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, let $\sigma|n \in \mathbb{N}^{<\mathbb{N}}$ denote the finite sequence $(\sigma_k)_{k=1}^n$. A subset A of a metric space is said to be a Souslin-$\mathcal{F}$ if it can be written as

$$A = \bigcup_{\sigma \in \mathbb{N}^\mathbb{N}} \bigcap_{n=1}^{\infty} A(\sigma|n)$$

where $A(s)$ is closed for every $s \in \mathbb{N}^{<\mathbb{N}}$. Every Souslin-$\mathcal{F}$ in a complete metric space X is measurable with respect to any Radon measure on X and every Borel subset of X is Souslin-$\mathcal{F}$ [15]. However, in non separable metric spaces the analogous of the Lusin separation theorem does not hold, at least with the usual Borel sets. Next result is due to Hansell.

Theorem 2.1 *Let $T : X \to Y$ be an injective continuous map between complete metric spaces. Then T is co-σ-discrete if, and only if, $T(A)$ is Souslin-$\mathcal{F}$ in Y for every closed $A \subset X$.*

Proof. Put together Proposition 3.14 and Corollary 4.2 of [5]. ■

Remark 2.2 *Assuming Fleissner's Axiom [4], the completeness of Y is not necessary as hypothesis in Theorem 2.1.*

The following property has been used in [13] and [14] in relation with the comparison of Borel subsets for different topologies, and the Borel measurability of maps.

Definition 2.3 *Let X be a set, τ_1 and τ_2 two topologies on X and let $\mathcal{C}$ be a class of subsets of X. We say that X has the property $P(\tau_1, \tau_2)$ (with sets from $\mathcal{C}$) if there exists a sequence (A_n) of subsets of X (belonging to $\mathcal{C}$) such that, for every $x \in X$ and every $V \in \tau_1$ with $x \in V$ there exists $n \in \mathbb{N}$ and $U \in \tau_2$ verifying $x \in A_n \cap U \subset V$.*

The argument that we shall use in the paper to show that two topologies τ_1 and τ_2 on X have the same Borel sets can be sketched as follows. Assume that τ_1 is finer than τ_2. Clearly it is enough to check that every τ_1-open set is τ_2-Borel. By Lemma 2.4 (1) we just need to prove that X has the property $P(\tau_1, \tau_2)$ with τ_2-Borel sets.

Lemma 2.4 *Let X be a set with topologies τ_i with $i = 0, 1, 2, \ldots, \infty$ defined on it and let $\mathcal{C}$ be a class of subsets of X stable by finite intersections.*

(1) If X has $P(\tau_1, \tau_2)$ with sets which are of multiplicative class less than α with respect to τ_2, then every τ_1-open set belongs to $\mathcal{A}_\alpha(\tau_2)$.

(2) If X has $P(\tau_1, \tau_2)$ and $P(\tau_2, \tau_3)$ with sets from $\mathcal{C}$, then X has $P(\tau_1, \tau_3)$ with sets from $\mathcal{C}$.

(3) *If X has $P(\tau_1,\tau_2)$ and τ_1 is metrizable by a τ_2-lower semicontinuous metric, then X has $P(\tau_1,\tau_2)$ with τ_2-closed sets.*

(4) *If X has $P(\tau_n,\tau_0)$ for every $n\in\mathbb{N}$ with sets from $\mathcal{C}$ and τ_∞ is the weaker topology containing $\{\tau_n : n\in\mathbb{N}\}$, then X has $P(\tau_\infty,\tau_0)$ with sets from $\mathcal{C}$.*

Proof. Statements (1), (2) and (3) are essentially contained in [13, Proposition 1] and statement (4) is a particular case of [14, Proposition 2.11]. It is necessary to add the extra information on the class of the sets, which follows by simple inspection of the proofs. ■

Proposition 2.5 *Let X be a set endowed with two metrics d_1 and d_2. The identity map $(X,d_1)\to(X,d_2)$ is co-σ-discrete if and only if X has $P(d_1,d_2)$.*

Proof. Assume that $(X,d_1)\to(X,d_2)$ is co-σ-discrete. Let $\{B_i : i\in I\}$ a basis for the d_1-topology such that $I=\bigcup_{n=1}^\infty I_n$ and every family $\{B_i : i\in I_n\}$ is discrete, see [6, §21.XVII]. Then there is a decomposition $B_i=\bigcup_{m=1}^\infty C_i^m$ such that $\{C_i^m : i\in I_n\}$ is discrete with respect to d_2 for every $n,m\in\mathbb{N}$. Take $A_{n,m}=\bigcup_{i\in I_n} C_i^m$. We claim that X has $P(d_1,d_2)$ with the countable family of sets $(A_{n,m})$. Indeed, given $x\in X$ and a d_1-neighbourhood V of x there is $n\in\mathbb{N}$ and $i\in I_n$ such that $x\in B_i\subset V$. Take $m\in\mathbb{N}$ such that $x\in C_i^m$. There is a d_2-neighbourhood U of x such that $A_{n,m}\cap U\subset C_i^m$, and thus we have $x\in A_{n,m}\cap U\subset V$.
Suppose that X has $P(d_1,d_2)$ with a sequence of sets (A_n). Firstly we shall show that a family $\{H_i : i\in I\}$ is σ-discretely decomposable with respect to d_2 if it satisfies the "ε-separation" condition with respect to d_1: there is $\varepsilon>0$ such that $d_1(x,y)>\varepsilon$, whenever $x\in H_i$ and $y\in H_j$ with $i,j\in I$ and $i\neq j$. Take $V_i=\{x\in X : d_1(x,H_i)<\varepsilon/3\}$ and notice that $\{V_i : i\in I\}$ is a disjoint family of d_1-open sets. Consider the following sets

$$W_i^{n,m}=\{x\in A_n\cap V_i : d_2(x,A_n\cap\bigcup_{j\in I,j\neq i}V_j)>m^{-1}\}$$

We claim that $V_i=\bigcup_{n=1}^\infty\bigcup_{m=1}^\infty W_i^{n,m}$. Indeed, for every $x\in V_i$ there is $n\in\mathbb{N}$ and a d_2-open U such that $x\in A_n\cap U\subset V_i$. Since $A_n\cap\bigcup_{j\in I,j\neq i}V_j\subset X\setminus U$ we have $d_2(x,A_n\cap\bigcup_{j\in I,j\neq i}V_j)>0$ and thus $x\in W_i^{n,m}$ for $m\in\mathbb{N}$ large enough. Clearly, the family $\{W_i^{n,m} : i\in I\}$ is d_2-discrete for every $n,m\in\mathbb{N}$, and this gives a σ-discrete decomposition, with respect to d_2, of $\{V_i : i\in I\}$. It follows that the family $\{H_i : i \in I\}$ is σ-discretely decomposable with respect to d_2. Let $\{H_i : i \in I\}$ an arbitrary d_1-discrete family. Define

$$H_i^n = \{x \in H_i : d_1(x, \bigcup_{j \in I, j \neq i} H_j) > n^{-1}\}$$

The d_1-discreteness of $\{H_i : i \in I\}$ gives that $H_i = \bigcup_{n=1}^{\infty} H_i^n$ for every $i \in I$. The family $\{H_i^n : i \in I\}$ satisfies the ε-separation condition with respect to d_1 for $\varepsilon = n^{-1}$. Every family $\{H_i^n : i \in I\}$ is σ-discretely decomposable with respect to d_2, and it is easy to see that $\{H_i : i \in I\}$ has the same property. ■

Remark 2.6 *We say that a topological space (X, τ) has the property d-SLD (from "countable cover by sets of small local diameter") if for every $\varepsilon > 0$ there is a decomposition $X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ such that any point $x \in X_n^{\varepsilon}$ has a relative τ-neighbourhood in X_n^{ε} of d-diameter less than ε. Oncina proved [9, Proposition 2.2] that the identity map $(X, d_1) \to (X, d_2)$ is co-σ-discrete if, and only if, (X, d_2) has d_1-SLD. Notice that co-σ-discrete maps are called SLD maps there [9, Definition 2.1]. Proposition 2.5 can be obtained form Oncina's result together with [13, Proposition 2].*

From this moment on we shall deal only with linear maps. Next result is a useful criterion for co-σ-discreteness in the linear case due to Moltó, Orihuela and Troyanski.

Theorem 2.7 *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces. The following conditions are equivalent:*

i) T is co-σ-discrete.

ii) For every $x \in X$ there is a separable subset $S(x) \subset X$ in such a way that

$$x \in \overline{span}(\bigcup_{n=1}^{\infty} S(x_n))$$

for any bounded sequence $(x_n) \subset X$ such that (Tx_n) converges to Tx.

Proof. In [7, Lemma 11] it is proved the implication $ii) \Rightarrow i)$ using the terminology mentioned in the proof Proposition 2.5. The complete proof appears in [10, Theorem A]. ■

The operator T considered in Example 1.1 is co-σ-discrete using the previous result with suitable sets $S(x)$ that the reader will find easily. Theorem 2.7 has been used by Oncina [9] to build injective bounded co-σ-discrete linear operators from certain classes of spaces into $c_0(\Gamma)$. The adaptation of [7, Corollary 9] gives the following.

Corollary 2.8 *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces. If T^*Y^* is norm dense in X^*, then T is co-σ-discrete and T^{-1} is of first Borel class.*

Proof. If $x \in X$ and $(x_n) \subset X$ is a bounded sequence such that (Tx_n) converges to Tx, then it is not hard to verify that (x_n) converges weakly to x. Therefore $x \in \overline{conv}(\{x_n : n \in \mathbb{N}\})$. From this we can deduce two consequences. The first one is that condition ii) of Theorem 2.7 is satisfied for $S(x) = \{x\}$. Proposition 2.5 implies that X has $P(\|.\|, |||.|||)$ where $|||.|||$ is the seminorm defined by $|||x||| = \|Tx\|$. The second consequence is that $\|.\|$ is lower semicontinuous with respect to $|||.|||$. Then X has $P(\|.\|, |||.|||)$ with $|||.|||$-closed sets by Lemma 2.4 (3). By Lemma 2.4 (1) every $\|.\|$-open is of first additive class with respect to $|||.|||$, that is, T^{-1} is of first Borel class. ■

Remark 2.9 *The fact that T^{-1} is of the first Borel class in Corollary 2.8 also follows from a result of Srivatsa [16].*

Recall that Banach space X is said to be weakly countably determined (WCD) if there exists a sequence (K_n) of w^*-compact sets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. The class of the WCD Banach spaces includes the weakly compact generated Banach spaces (WCG).

Theorem 2.10 ([13]) *If X is a WCD Banach space, then X has $P(\|.\|, \tau)$ for any Hausdorff vector topology τ weaker than the weak topology of X.*

Corollary 2.11 *If X is a WCD Banach space, then any injective bounded linear operator $T : X \to Y$ to another Banach space Y is co-σ-discrete.*

Proof. Take $Z = T^*Y^*$. By Theorem 2.10, X has $P(\|.\|, \sigma(X, Z))$. Consider the norm $\|.\|_{-1}$ on X defined by $\|x\|_{-1} = \|Tx\|$. Since $\|.\|_{-1}$ generates an intermediate topology between $\sigma(X, Z)$ and $\|.\|$, we deduce that X has $P(\|.\|, \|.\|_{-1})$ which is equivalent to being T co-σ-discrete by Proposition 2.5. ■

3 Main arguments

Saint-Raymond obtained the result mentioned in the introduction within the context of his theory of “espaces a modèle séparable”. Plichko [12] gave a more direct proof of Saint-Raymond’s theorem in the separable case. Some of the arguments used by Plichko in [12] are valid also for non separable Banach spaces, but this fact is not stated explicitly in his paper. The most relevant of them for our development is Lemma 3.3 below.

Let Z be a separating subspace of X^*. The formula

$$\|x\|_Z = \sup\{x^*(x) : x^* \in B_{X^*} \cap Z\}$$

defines a coarser norm on X. If the norm $\|.\|_Z$ is equivalent to the original norm of X, that we shall subsequently denote by $\|.\|$, then Z is said to be a norming subspace. It is well known that Z is norming if and only if $\overline{B_{X^*} \cap Z}^{w^*}$ contains a neighbourhood of the origin. For any subspace $Z \subset X^*$ we have $Z_{[1]} = \bigcup_{n=1}^{\infty} n\overline{B_{X^*} \cap Z}^{w^*}$. Applying Baire’s category theorem we deduce that Z is norming if and only if $Z_{[1]} = X^*$. Next result is well known (the proof is included because we have not found suitable reference).

Lemma 3.1 *If $Z \subset X^*$ is a separating subspace, then $(X, \|.\|_Z)^* = Z_{[1]}$.*

Proof. The dual space $(X, \|.\|_Z)^*$ is a subspace of X^* since the norm $\|.\|_Z$ is coarser than $\|.\|$. Its dual ball is the polar of the set $\{x \in X : \|x\|_Z \leq 1\}$. By the bipolar theorem, $\overline{B_{X^*} \cap Z}^{w^*}$ is the unit ball of $(X, \|.\|_Z)^*$. ■

In all what follows an injective bounded linear operator $T : X \to Y$ is fixed. We shall use Plichko’s notation $\|.\|_{-1}$ for the norm of Y lifted to X, that is $\|x\|_{-1} = \|Tx\|$. We also fix the separating subspace $Z = T^*Y^*$ and define $\|.\|_\alpha = \|.\|_{Z_{[\alpha]}}$ for $\alpha \geq 0$, taking $Z_{[0]} = Z$. Notice that the sequence of norms stabilizes at $\|.\|_\gamma$ if, and only if, $\|.\|_\gamma = \|.\|$. In that case the former lemma implies that $\{\|.\|_\alpha : -1 \leq \alpha < \gamma\}$ are pairwise non-equivalent norms on X coarser than $\|.\|$. The exceptional case, the norm $\|.\|_{-1}$ is equivalent to $\|.\|$, happens if, and only if, T is an isomorphism onto its image.

Lemma 3.2 *The transfinite sequence of norms $\{\|.\|_\alpha : \alpha \geq -1\}$ defined above has the following properties:*

(1) For any two ordinals $\beta < \alpha$ the norm $\|.\|_\alpha$ is finer than $\|.\|_\beta$.

(2) $\|.\|_{\alpha+1}$ *is lower semicontinuous with respect to* $\|.\|_{\alpha}$.

(3) For any ordinal α

$$\bigcap_{-1\leq\beta<\alpha} \overline{B_X}^{\|.\|_\beta} = \{x \in X : \|x\|_\alpha \leq 1\}$$

Proof. Property (1) is easy to check. Property (2) can be deduced from property (3), that appears as Lemma 1 in [12]. The proof of property (3) is consequence of the following chain of identities

$$\bigcap_{-1\leq\beta<\alpha} \overline{B_X}^{\|.\|_\beta} = \bigcap_{-1\leq\beta<\alpha} \overline{B_X}^{\sigma(X,Z_{[\beta+1]})} = \overline{B_X}^{\sigma(X,\ \bigcup_{-1\leq\beta<\alpha} Z_{[\beta+1]})}$$

$$= \overline{B_X}^{\sigma(X,Z_{[\alpha]})} = \{x \in X : \|x\|_\alpha \leq 1\}$$

where we have used the theorems of Mazur (for $(X, \|.\|_\beta)$) and bipolar (with respect to the dual pair $\langle X, Z_\alpha\rangle$). ■

In order to state Plichko's lemma we have to introduce some notation. The following concepts are relevant to the $\|.\|$-topology of X. If $A \subset X$ is a Borel subset, then there exists an open subset U such that the symmetric difference $A\Delta U$ is of first category [6, §11.III]. The subset $D(A)$ of points of X where A is not of the first category coincides with $\overline{U}$ [6, §10.V].

Lemma 3.3 *For any* $A \in \mathcal{A}_\alpha(\|.\|_{-1})$ *verifying that* $D(A) \neq \emptyset$*, there exists* $\beta \in [-1, \alpha)$ *such that* $D(A)$ *contains some* $\|.\|_\beta$*-ball.*

Proof. Notice that for $\alpha = 0$ it is trivial. The case $\alpha > 0$ follows from Lemma 3 and Lemma 1 of [12]. ■

Proposition 3.4 *Let* $T : X \to Y$ *be an injective bounded linear operator between Banach spaces. If there is a nonempty bounded open set* $B \subset X$ *such that* TB *is of additive class* α *in* TX*, then the bn-order of* T^*Y^* *is at most* α.

Proof. We have $B \in \mathcal{A}_\alpha(\|.\|_{-1})$ and then $D(B) = \overline{B}$ contains a $\|.\|_\beta$-ball for some $\beta < \alpha$ by Lemma 3.3. But this implies that the $\|.\|_\beta$-ball is bounded and thus Z_β is a norming subspace. As $\beta + 1 \leq \alpha$, then $Z_{[\alpha]} = X^*$. ■

Lemma 3.5 *Suppose that* X *has* $P(\|.\|, \|.\|_{-1})$*. Then* X *has* $P(\|.\|_\alpha, \|.\|_{-1})$ *with sets from* $\mathcal{M}_\alpha(\|.\|_{-1})$ *for* α *any countable ordinal.*

Proof. We shall prove the lemma by induction. First observe that if X has $P(\tau_1, \tau_2)$ for any pair of intermediate topologies $\|.\|_{-1} \subset \tau_2 \subset \tau_1 \subset \|.\|$.
For $\alpha = 0$, it follows from Lemma 2.4 (3) because $\|.\|_0$ is $\|.\|_{-1}$-lower semicontinuous after Lemma 3.2 (2).
If α is a non limit ordinal, put $\alpha = \beta + 1$. By the induction hypothesis X has $P(\|.\|_\beta, \|.\|_{-1})$ with sets from $\mathcal{M}_\beta(\|.\|_{-1})$. This implies that the $\|.\|_\beta$-open sets belong to $\mathcal{A}_\alpha(\|.\|_{-1})$ by Lemma 2.4 (1), and so the $\|.\|_\beta$-closed sets belong to $\mathcal{M}_\alpha(\|.\|_{-1})$. Reasoning as in the first step, we deduce that X has $P(\|.\|_\alpha, \|.\|_\beta)$ with $\|.\|_\beta$-closed sets, which are in $\mathcal{M}_\alpha(\|.\|_{-1})$. Lemma 2.4 (2) gives that X has $P(\|.\|_\alpha, \|.\|_{-1})$ with sets from $\mathcal{M}_\alpha(\|.\|_{-1})$.
If α is a limit ordinal define an auxiliary metric

$$d(x, y) = \sum_{\beta<\alpha} \lambda_\beta \max\{\|x - y\|_\beta, 1\}$$

where the coefficients $\lambda_\beta > 0$ are such that $\sum_{\beta<\alpha} \lambda_\beta < +\infty$. The d-topology is generated by the union of the $\|.\|_\beta$-topologies for $\beta < \alpha$. By Lemma 2.4 (4), X has $P(d, \|.\|_{-1})$ with sets of multiplicative class strictly less than α with respect to $\|.\|_{-1}$. Since α is a limit ordinal $Z_{[\alpha]} = \bigcup_{\beta<\alpha} Z_{[\beta]}$. Consequently, the functionals of $Z_{[\alpha]}$ are d-continuous. Arguing like in the proof of Lemma 3.2, the norm $\|.\|_\alpha$ is d-lower semicontinuous, and so X has $P(d, \|.\|_\alpha)$ with d-closed sets, which belong to $\mathcal{M}_\alpha(\|.\|_{-1})$. Lemma 2.4 (2) implies that X has $P(\|.\|_\alpha, \|.\|_{-1})$ with sets from $\mathcal{M}_\alpha(\|.\|_{-1})$. ■

Proof of Theorem A. Assume that the bn-order α of T^*Y^* is countable. First we shall show that α cannot be a limit ordinal. Indeed, if we suppose that α is limit, then $Z_{[\beta]}$ with $\beta < \alpha$ is strictly increasing and $\bigcup_{\beta<\alpha} Z_{[\beta]} = X^*$. The norm closure $\overline{Z_{[\beta]}}$ is contained in $Z_{[\beta+1]}$. The Baire category theorem implies that some $Z_{[\beta]}$ with $\beta < \alpha$ must contain an open ball which is a contradiction. Consequently, we can put $\alpha = \beta + 1$, thus $Z_{[\beta]}$ is norming and $\|.\|_\beta$ is an equivalent norm on X. Since T is co-σ-discrete, X has $P(\|.\|, \|.\|_{-1})$ by Proposition 2.5. In order to show that T^{-1} is Borel of class α it is enough to show that X has $P(\|.\|_\beta, \|.\|_{-1})$ with sets from $\mathcal{M}_\beta(\|.\|_{-1})$, but this follows from Lemma 3.5.
For the converse, assume that T^{-1} is Borel measurable of class α. If B is a $\|.\|$-open ball of X, then TB is of additive class α in TX, which implies that $B \in \mathcal{A}_\alpha(\|.\|_{-1})$. Then apply Proposition 3.4 to get $Z_{[\alpha]} = X^*$. ■

Remark 3.6 *Plichko [12] has proved that if X is a Banach space which is the direct sum of separable and reflexive subspaces, then any injective bounded* *linear operators has Borel measurable inverse. He also proved in [12] that the converse is consistent with ZFC.*

Corollary 3.7 *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces. Assume that T is co-σ-discrete and T^*Y^* has bn-order α. Then TX is a Borel subset of Y of multiplicative class $\alpha + 1$ at most. Moreover, TA is Borel in Y for every $A \subset X$ Borel subset.*

Proof. After the proof of Theorem A, we know that X has $P(\|.\|, \|.\|_{-1})$ with sets from $\mathcal{M}_\beta(\|.\|_{-1})$ where $\beta + 1 = \alpha$. The result follows by application of [14, Theorem 4.6]. ■

Corollary 3.8 *For an injective bounded linear operator $T : X \to Y$ between Banach spaces the following conditions are equivalent:*

i) T^{-1} is Borel measurable and TX is a Souslin-$\mathcal{F}$ subset of Y.

ii) T is co-σ-discrete and the bn-order of T^*Y^* is countable.

Proof. *i)* $\Rightarrow$ *ii)* If T^{-1} is Borel measurable, then the bn-order of T^*Y^* is countable by Proposition 3.4. Notice that relative Borel subsets of a Souslin-$\mathcal{F}$ set in Y are also Souslin-$\mathcal{F}$ in Y. Theorem 2.1 implies that T is co-σ-discrete.
ii) $\Rightarrow$ *i)* The Borel measurability of T^{-1} follows from Theorem A and the fact that TX is a Borel subset of Y by the former corollary, so it is Souslin-$\mathcal{F}$. ■

We shall finish this section with some applications to WCD Banach spaces. Notice that if X is a WCD Banach space, then $\overline{B}^{w^*} = B_{(1)}$ for any $B \subset X^*$ bounded [18, Théorème 6.4]. Therefore $A_{(1)} = A_{[1]}$ for any subset $A \in X^*$, and thus the bn-order coincides with the s-order.

Corollary 3.9 *Let X be a WCD Banach space and let $T : X \to Y$ be an injective bounded linear operator to another Banach space Y. Then T^{-1} is Borel measurable if, and only if, the s-order of T^*Y^* is countable, and in such a case, it coincides with the Borel class of T^{-1}.*

If X is a WCD Banach space and $Z \subset X^*$ is norming, then the Borel sets for $\sigma(X, Z)$ and the norm topology coincide [13]. We have the following improvement.

Corollary 3.10 *Let X be a WCD Banach space and let $Z \subset X^*$ be a separating subspace. Then the equality of Borel families*

$$Borel(X, \|.\|) = Borel(X, \sigma(X, Z))$$

holds if, and only if, Z has countable s-order.

Proof. By Theorem 2.10, X has $P(\|.\|, \sigma(X, Z))$. We deduce that X has $P(\|.\|_Z, \sigma(X, Z))$. Since the norm $\|.\|_Z$ is $\sigma(X, Z)$-lower semicontinuous, the Borel sets for $\sigma(X, Z)$ and the $\|.\|_Z$-topology coincide by Lemma 2.4 (3). Let $(Y, |||.|||)$ be the completion of $(X, \|.\|_Z)$ and T the natural embedding of $(X, \|.\|)$ into $(Y, |||.|||)$. Notice that $T^*Y^* = Z_{(1)}$. The result follows from Corollary 3.9 since the equality of $Borel(X, \|.\|)$ and $Borel(X, \|.\|_Z)$ is equivalent to the Borel measurability of T^{-1}. ■

Corollary 3.11 *Let X be a WCD Banach space and let $T : X \to Y$ be a bounded linear operator into another Banach space Y. If there is a nonempty bounded open set $B \subset X$ such that TB is Borel in TX, then TU is Borel in Y for any open $U \subset X$.*

Proof. Reduce to the injective case of Corollary 3.7 by means of the quotient space $X/\mathrm{Ker}T$ and combine with Banach open mapping theorem and Proposition 3.4. ■

4 Separable reduction

We prove in this section that the Borel class of the inverse of an injective bounded linear operator $T : X \to Y$ is determined by its restrictions to separable subspaces of X.

Lemma 4.1 *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces. Assume that the bn-order of T^*Y^* is strictly greater than some countable ordinal $\gamma + 1$. Then there is a separable subspace $E \subset X$ such that bn-order of $(T|_E)^*Y^*$ into E^* is strictly greater than $\gamma + 1$.*

Proof. Let $\{\|.\|_\alpha : -1 \leq \alpha \leq \gamma+1\}$ the transfinite sequence of norms on X and let $\{Z_{[\alpha]} : 0 \leq \alpha \leq \gamma + 1\}$ be the transfinite sequence of separating subspaces of X^* introduced in the former section. The norm $\|.\|_\gamma$ is not equivalent to $\|.\|$ because $Z_{[\gamma+1]} \neq X^*$. Take a $\|.\|$-separable subspace $E_1 \subset X$ such that the

norms $\|.\|_\gamma$ and $\|.\|$ are not equivalent on E_1.
We shall construct an increasing sequence of $\|.\|$-separable subspaces $E_n \subset X$ as follows. Assume that E_n is already built, then E_{n+1} is taken such that $E_n \subset E_{n+1}$ and such that for every $x \in E_n$ there exists a sequence $(x_k) \subset E_{n+1}$ with $\|x_k\| \leq \|x\|_\alpha$ which is $\|.\|_\beta$-converging to x for each $\beta < \alpha$. To do that, first observe that for a given $x \in X$ with $\|x\|_\alpha \leq 1$ there is a sequence $(x_k) \subset B_X$ which is $\|.\|_\beta$-converging to x for each $\beta < \alpha$ with $\|x_k\| \leq \|x\|_\alpha$ because

$$\{x \in X : \|x\|_\alpha \leq 1\} = \bigcap_{-1 \leq \beta < \alpha} \overline{B_X}^{\|\cdot\|_\beta}$$

after Lemma 3.2. Fix $D(n, \alpha)$ a countable $\|.\|$-dense subset of $\{x \in E_n : \|x\|_\alpha \leq 1\}$ and for every $-1 \leq \alpha \leq \gamma$ and every $x \in D(n, \alpha)$ pick a sequence $(x_k) = S(x, \alpha) \subset B_X$ which is $\|.\|_\beta$-converging to x for each $\beta < \alpha$. The space E_{n+1} is constructed as follows

$$E_{n+1} = \overline{span}^{\|\cdot\|} \left(E_n \cup \bigcup_{\alpha \leq \gamma,\ x \in D(n,\alpha)} S(x, \alpha) \right)$$

Let $E \subset X$ be the $\|.\|$-separable closed subspace defined by $E = \overline{\bigcup_{n=1}^{\infty} E_n}^{\|\cdot\|}$. From this moment on the norms $\|.\|_\alpha$ are restricted to E. Observe that by construction E satisfies the equality

$$\bigcap_{-1 \leq \beta < \alpha} \overline{B_E}^{\|\cdot\|_\beta} = \{x \in E : \|x\|_\alpha \leq 1\}$$

for every $-1 \leq \alpha \leq \gamma$. Let $F \subset E^*$ the separating subspace defined by $F = (T|_E)^* Y^*$. We shall consider the transfinite sequence $\{F_{[\alpha]} : 0 \leq \alpha \leq \gamma\}$ of separating subspaces of E^* and the transfinite sequence of norms $\{|\!|\!|.|\!|\!|_\alpha : 0 \leq \alpha \leq \gamma\}$ on E where $|\!|\!|.|\!|\!|_\alpha = \|.\|_{F_{[\alpha]}}$. Put $|\!|\!|.|\!|\!|_{-1} = \|.\|_{-1}$ on E. We claim that $|\!|\!|x|\!|\!|_\alpha = \|x\|_\alpha$ for every $-1 \leq \alpha \leq \gamma$. The claim will be proved using induction on α. For $\alpha = -1$ it is trivial. Assume that the induction hypothesis is proved for every $\beta < \alpha$ where $0 \leq \alpha \leq \gamma$. We have, by construction of E and the induction hypothesis,

$$\{x \in E : \|x\|_\alpha \leq 1\} = \bigcap_{-1 \leq \beta < \alpha} \overline{B_E}^{\|\cdot\|_\beta}$$

$$= \bigcap_{-1 \leq \beta < \alpha} \overline{B_E}^{|\!|\!|\cdot|\!|\!|_\beta} = \{x \in E : |\!|\!|x|\!|\!|_\alpha \leq 1\}$$

where the last equality follows from Lemma 3.2 applied to E. That completes the proof of the claim. To finish the proof of the lemma, observe that $\|.\|_\gamma = \|.\|_{F_{[\gamma]}}$ is not equivalent to $\|.\|$ on E because $E_1 \subset E$. Therefore $F_{[\gamma]}$ is not norming and thus the bn-order of F is strictly greater than $\gamma + 1$. ■

Theorem 4.2 *Let $T : X \to Y$ be an injective bounded linear operator between Banach spaces. Then:*

(a) If the bn-order of T^*Y^* is countable, then it coincides with the s-order of $(T|_E)^*Y^*$ into E^* for some separable closed subspace $E \subset X$.

(b) If the bn-order of T^*Y^* is uncountable, then any countable ordinal is less than the s-order of $(T|_E)^*Y^*$ into E^* for some separable closed subspace $E \subset X$.

Proof. Assume that the bn-order of T^*Y^* is countable and, by the arguments in the proof of Theorem A, we may assume that is of the form $\gamma+1$. The former lemma gives for every $\beta < \gamma$ a separable subspace E_β such that the s-order of $T|^*_{E_\beta}Y^*$ into E^* is strictly bigger than $\beta+1$. If γ is non-limit ordinal the proof is finished. Suppose that γ is a limit ordinal and take $E = \overline{span}^{\|\cdot\|}(\bigcup_{\beta<\gamma} E_\beta)$. The space E is $\|.\|$-separable and by construction, the s-order α of $T|^*_E Y^*$ is not less than γ. On the other hand α cannot be a limit ordinal and is not greater than $\gamma + 1$. Therefore $\alpha = \gamma + 1$. In case that the bn-order of T^*Y^* is uncountable the proof follows directly from the lemma. ■

Remark 4.3 *Small modifications of Lemma 4.1 can be used to prove that the bn-order of a linear subspace $Z \subset X^*$ is determined by the s-orders of $Z|_E$ into E^* where E runs on the separable closed subspaces of X in the sense of Theorem 4.2.*

Proof of Theorem B. It follows from Proposition 3.4 in case (a). For case (b) apply Theorem 4.2. ■

Remark 4.4 *An injective bounded linear operator $T : X \to Y$ between Banach spaces is called a $\mathcal{G}_\delta$-embedding [2] if TA is a $\mathcal{G}_\delta$-set for every bounded separable closed set $A \subset X$. As a consequence of Theorem B (b), a co-σ-discrete $\mathcal{G}_\delta$-embedding has inverse of the second Borel class, at most.*

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