

# Embedding $\ell_1$ as Lipschitz functions

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## Abstract

Let  $K$  be a compact Hausdorff space and  $d$  a lower semicontinuous metric on it. We prove that if  $K$  is fragmented by  $d$  if, and only if,  $C(K)$  contains no copy of  $\ell_1$  made up of Lipschitz functions with respect to  $d$ . As applications we obtain characterizations of Asplund Banach spaces and Radon-Nikodým compacta.

## 1 Introduction

R. C. James [5] and independently J. Lindenstrauss and C. Stegall [7] solved negatively a classical problem of S. Banach: *Let  $X$  be a Banach space such that its dual  $X^*$  is separable, does  $X$  contain a subspace isomorphic to  $\ell_1$ ?* These counterexamples seems to be the split between two branches of Banach space theory that have had a great development in the last decades: Asplund spaces and Banach spaces not containing copy of  $\ell_1$ .

Our aim is to present a link between the Asplund property and the existence of subspaces isomorphic to  $\ell_1$ . We shall show that if  $X$  is separable and  $X^*$  is not, then it is possible to find a copy of  $\ell_1$  in certain linear space where  $X$

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imbeds in a very natural way. Consider the unit ball of the dual  $B_{X^*}$  endowed with the weak\* topology. Then  $X$  embeds isometrically into  $C(B_{X^*})$  as a subset of norm Lipschitz functions. Of course,  $C(B_{X^*})$  always contains copies of  $\ell_1$ , but there is one made up of norm Lipschitz functions if, and only if,  $X^*$  is not separable. More generally we have the following:

**Theorem 1.1** *For a Banach space  $X$  the following are equivalent:*

- i)  $X$  is not Asplund.*
- ii)  $C(B_{X^*})$  contains a copy of  $\ell_1$  made up of norm Lipschitz functions.*

The linear subspace  $\mathcal{L}$  of the norm Lipschitz functions of  $C(B_{X^*})$  become a Banach space when endowed with the norm  $\|\cdot\| = \|\cdot\|_\infty + \|\cdot\|_{Lip}$  where  $\|\cdot\|_\infty$  is the supremum norm and  $\|f\|_{Lip} = \sup\{\frac{|f(x)-f(y)|}{\|x-y\|} : x, y \in B_{X^*}, x \neq y\}$ , but we do not use in Theorem 1.1 this norm. Indeed, it is easy to see that  $\ell_1$  embeds isomorphically into  $(\mathcal{L}, \|\cdot\|)$  for every non trivial  $X$ . The embedding in Theorem 1.1 *ii)* is with respect to the norm  $\|\cdot\|_\infty$  in  $C(B_{X^*})$ .

Theorem 1.1 will be a consequence of a more general result, Theorem 2.1, which is formulated in terms of a  $C(K)$  space and fragmentability. In the last section of the paper we shall study the subsets of  $C(K)$  which are made up of all the Lipschitz functions with respect to some metric on  $K$ . We shall give a characterization for this sets without explicit mention of the metric, obtaining as an application, a new characterization of Radon-Nikodým compacta.

## 2 Fragmentability and copies of $\ell_1$

We shall deal with continuous functions on a compact space in order to describe a more general result. Recall that a topological space  $K$  is fragmentable by a metric defined on it if every nonempty subset of  $K$  has a nonempty relatively open subset of arbitrarily small diameter with respect to  $d$ . Our main result is the following:

**Theorem 2.1** *Let  $K$  be a compact Hausdorff space and let  $d$  be a lower semi-continuous metric defined on it. Then the following are equivalent:*

- i)  $K$  is not fragmentable by  $d$ .*
- ii)  $C(K)$  contains an isomorphic copy of  $\ell_1$  made up of functions which are Lipschitz with respect to  $d$ .*
- iii) There is a bounded sequence  $(f_n) \subset C(K)$  equivalent to the canonical basis of  $\ell_1$  which is equicontinuous with respect to  $d$ .*

**Proof.** Lipschitz property is understood referred to the metric  $d$ , thus  $\eta$ -Lipschitz for  $\eta \in \mathbb{R}$  will mean that  $|f(x) - f(y)| \leq \eta d(x, y)$  for all  $x, y \in K$ .

*i)  $\Rightarrow$  ii)* If  $K$  is not  $d$ -fragmentable, a Cantor-type construction [6] yields an  $\varepsilon > 0$ , a closed subset  $H \subset K$  and a surjective map  $h : H \rightarrow \{-1, 1\}^{\mathbb{N}}$  such that  $d(h^{-1}(r), h^{-1}(s)) \geq \varepsilon$  for every  $r, s \in \{-1, 1\}^{\mathbb{N}}$  with  $r \neq s$ . Let  $p_n$  the projection on the  $n$ 'th coordinate of  $\{-1, 1\}^{\mathbb{N}}$  and consider the function  $p_n \circ h$  defined on  $H$  which is continuous and  $\varepsilon^{-1}$ -Lipschitz. By a result of [8], see also [1], there is a continuous extension  $f_n$  of  $p_n \circ h$  to  $K$  with the same Lipschitz bound  $\varepsilon^{-1}$ . The sequence  $(f_n)$  is equivalent to the canonical basis of  $\ell_1$ . Indeed, given numbers real numbers  $(a_n)$  for  $i = 1, \dots, m$  there is  $x \in H$  such that  $f_n(x) = \text{sign}(a_n)$  and thus

$$\left\| \sum_{n=1}^m a_n f_n \right\| = \sum_{n=1}^m |a_n|$$

which means that  $E = \overline{\text{span}}^{\|\cdot\|_{\infty}} \{f_n : n \in \mathbb{N}\}$  is even isometric to  $\ell_1$ . An easy computation shows that if  $f \in E$ , then  $f$  is  $\varepsilon^{-1}\|f\|$ -Lipschitz.

*ii)  $\Rightarrow$  iii)* Let  $E \subset C(K)$  be an isomorphic copy of  $\ell_1$  made up of Lipschitz functions with respect to  $d$  and take  $E_n \subset E$  the subset of  $n$ -Lipschitz functions. Clearly  $E_n$  is norm closed, thus by Baire's category theorem some  $E_n$  has nonempty interior. By translation we deduce that the functions of the unit ball of  $E$  have a common Lipschitz bound, and thus there is a bounded sequence

of  $C(K)$  which is equivalent to the canonical basis of  $\ell_1$  and equicontinuous with respect to  $d$ .

*iii)  $\Rightarrow$  i)* Let  $(f_n)$  be a bounded  $\ell_1$ -sequence of  $C(K)$  which is equicontinuous with respect to  $d$ . By a standard reduction argument, there is a metrizable compact  $L$ , a continuous surjection  $q : K \rightarrow L$  and functions  $g_n$  defined on  $L$  such that  $f_n = g_n \circ q$  for every  $n \in \mathbb{N}$ . The sequence  $(g_n)$  is equivalent in  $C(L)$  to the basis of  $\ell_1$ , and thus  $(g_n)$  contains no pointwise convergent subsequence. By a result of Rosenthal [11], see also [12, p.20], there is a subsequence of  $(g_n)$ , denoted again  $(g_n)$  without loss of generality, and there are real numbers  $a < b$ , such that  $\bigcap_{n=1}^m A_n \neq \emptyset$  for every  $m \in \mathbb{N}$  and every choice of  $A_n \in \{g_n^{-1}(-\infty, a], g_n^{-1}[b, +\infty)\}$ . Using compactness and composition with  $q^{-1}$  we get that  $\bigcap_{n=1}^\infty A_n \neq \emptyset$  for every every choice of  $A_n \in \{f_n^{-1}(-\infty, a], f_n^{-1}[b, +\infty)\}$ . In fact, we have used the reduction to the separable case just to apply Rosenthal's theorem as appearing in quoted references. Consider the closed subset of  $K$  given by

$$H = \bigcap_{n=1}^{\infty} (f_n^{-1}(-\infty, a] \cup f_n^{-1}[b, +\infty))$$

and define a map  $h : H \rightarrow \{-1, 1\}^{\mathbb{N}}$  such that  $n$ 'th coordinate of  $h(x)$  is  $-1$  if  $f_n(x) \leq a$ , and  $1$  if  $f_n(x) \geq b$ . The construction ensures that  $h$  is continuous and surjective. Take a closed subset  $C \subset H$  minimal with respect to the property that  $h(C) = \{-1, 1\}^{\mathbb{N}}$ . It is easy to see that any relatively open subset of  $C$  contains two points  $x, y$  such that  $f_n(x) \leq a$  and  $f_n(y) \geq b$  for some  $n$ . We claim that  $C$  is not  $d$ -fragmentable. Suppose that  $C$  is  $d$ -fragmentable, then there is point  $z \in C$  of continuity from the inherited topology of  $C$  to the metric  $d$  [10]. Take  $r > 0$  such that the oscillation of  $f_n$  in  $B(z, r)$  is less than  $b - a$  for all  $n \in \mathbb{N}$ . Take  $U$  a neighbourhood of  $z$  such that  $\text{diam}(C \cap U) < r$ . There are points  $x, y \in C \cap U$ , so  $x, y \in B(z, r)$ , and  $n \in \mathbb{N}$  such that  $f_n(y) - f_n(x) \geq b - a$  which is a contradiction. That proves the claim and thus  $K$  is not  $d$ -fragmentable.  $\blacksquare$

Recall that a Banach space  $X$  is said to be Asplund if every separable subspace has separable dual.

**Proof of Theorem 1.1.** A Banach space  $X$  is Asplund if, and only if, the dual ball  $B_{X^*}$  is norm fragmentable, see [3], then apply Theorem 2.1. ■

**Remark 2.2** *The behaviour of Banach spaces is different with respect of Lipschitz copies of  $c_0$ . Indeed, if  $H$  is an infinite dimensional Hilbert space it is not difficult to find an isometric copy of  $c_0$  inside  $C(B_H)$  made up of norm Lipschitz functions.*

Statement *iii*) of Theorem 2.1 suggest to study the sequential properties of bounded subsets of  $C(K)$  which are equicontinuous with respect to a fragmenting metric. However, it is shown in [2] that certain subsets of functions on a fragmentable compact space verifies stronger sequential properties than the given by Rosenthal's  $\ell_1$ -theorem (see in particular [2, Corollary 3.5]). We shall include the following result which is relevant to Theorem 2.1.

**Proposition 2.3** *Let  $(f_n) \subset C(K)$  be a sequence of functions which are equicontinuous with respect to a metric  $d$  fragmenting  $K$ . Then the pointwise closure of  $(f_n)$  in  $\overline{\mathbb{R}}^K$  is a metrizable compact subset.*

**Proof.** Notice that we do not ask the metric  $d$  be lower semicontinuous. Without loss of generality, the sequence  $(f_n)$  may be supposed bounded. Indeed, compose with a homeomorphism from  $\overline{\mathbb{R}}$  to  $[-1, 1]$  and both continuity and equicontinuity with respect to  $d$  are preserved. Consider the pseudometric  $\rho(x, y) = \sup\{|f_n(x) - f_n(y)| : n \in \mathbb{N}\}$  which is continuous with respect to  $d$  by the equicontinuity of  $(f_n)$ . As a consequence,  $K$  is fragmented by  $\rho$ . Passing to a quotient, we may assume that  $K$  is a metrizable compactum, and thus we get that  $K$  is  $\rho$ -separable using the fragmentability property. Let  $H \subset [-1, 1]^K$  the pointwise closure of  $(f_n)$ . Clearly,  $H$  is a pointwise compact set made up of  $\rho$ -continuous functions. Since a  $\rho$ -dense countable subset of  $K$  separates

points of  $H$ , we deduce that  $H$  is metrizable. ■

We shall finish the section applying the results to scattered compact spaces. Notice that a compact space  $K$  is scattered if, and only if, it is fragmentable by the discrete metric. Since the discrete metric make Lipschitz any bounded function and it is lower semicontinuous with respect to any topology on  $K$  we get the following well known result.

**Corollary 2.4** *A compact Hausdorff space  $K$  is scattered if, and only if,  $C(K)$  does not contain copy of  $\ell_1$ .*

Proposition 2.3 implies the following result of Meyer [9].

**Corollary 2.5** *Let  $K$  be a scattered compact Hausdorff space. Then any Baire function on  $K$  is of the first Baire class.*

### 3 The lattice of Lipschitz functions

It is easy to see that the set of  $\eta$ -Lipschitz functions is lattice closed, that is, the functions  $\max\{f, g\}$  and  $\min\{f, g\}$  are  $\eta$ -Lipschitz whenever the functions  $f$  and  $g$  are  $\eta$ -Lipschitz. Notice that the subset of  $C(K)$  made up of functions which are Lipschitz with respect to some metric  $d$  on  $K$  can be given structure of Banach lattice with the norm  $\|\cdot\|$  defined at the introduction.

**Lemma 3.1** *Let  $K$  be a compact Hausdorff space and let  $B \subset C(K)$  be bounded closed symmetric convex which separates points of  $K$ , is lattice closed and contains a non trivial constant function. Define a lower semicontinuous metric  $d$  on  $K$  by the formula*

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in B\}$$

*Then  $L = \bigcup_{n=1}^{\infty} nB$  is the set of continuous functions on  $K$  which are Lipschitz with respect to  $d$ .*

**Proof.** We may assume without loss of generality that  $B \subset B_{C(K)}$ . Let  $\delta > 0$  such that  $t\mathbf{1} \in B$  for all  $t \in [-\delta, \delta]$ . It is clear that  $L = \bigcup_{n=1}^{\infty} nB$  is the linear space spanned by  $B$  and  $L$  is a lattice. Given  $f \in C(K)$  which is Lipschitz with respect to  $d$  we shall prove that  $f \in L$ . We may assume without loss of generality that  $f \in B_{C(K)}$  and  $f$  is 1-Lipschitz, thus  $|f(x) - f(y)| \leq d(x, y)$  for every  $x, y \in K$ . Fix a pair of points  $x, y \in K$  with  $x \neq y$ . There is  $g \in B$  such that  $d(x, y) \leq 2(g(x) - g(y))$ . We may take  $\lambda \in [-2, 2]$  such that  $f(x) - f(y) = \lambda(g(x) - g(y))$ . Take  $\eta = f(x) - \lambda g(x) = f(y) - \lambda g(y)$ . We have  $|\eta| \leq 3$ , and so the function  $p = \lambda g + \eta\mathbf{1}$  belongs to  $D = (2 + 3\delta^{-1})B$ , and verifies that  $p(x) = f(x)$  and  $p(y) = f(y)$ . Applying the method of [4, p.146] to prove the Stone-Weierstrass theorem, for every  $\varepsilon > 0$  there is  $h \in D$  such that  $\|f - h\| < \varepsilon$  because  $D$  is lattice closed, and thus  $f \in D$  because it is closed. ■

**Theorem 3.2** *Let  $L \subset C(K)$  be a dense linear lattice which contains the constant functions. Then  $L$  is the subset of functions which are Lipschitz with respect to some (lower semicontinuous) metric  $d$  on  $K$  if, and only if,  $L$  is an  $\mathcal{F}_\sigma$  and  $L$  is complete for a finer lattice norm. In that case, any two complete finer lattice norms on  $L$  are equivalent.*

**Proof.** If  $L$  is the lattice of the functions which are Lipschitz with respect to some metric  $d$  on  $K$ , then it is complete with the lattice norm  $\|\cdot\|$  defined at the introduction, and  $L = \bigcup_{n=1}^{\infty} F_n$  where  $F_n = \{f \in L : f \text{ is } n\text{-Lipschitz}\}$ . For the converse, let  $\|\cdot\|$  be a complete lattice norm on  $L$  finer than  $\|\cdot\|_\infty$ . Put  $L \cap B_{C(K)} = \bigcup_{n=1}^{\infty} F_n$  where each  $F_n$  is  $\|\cdot\|_\infty$ -closed. Since  $\|\cdot\|$  is finer than the restriction of  $\|\cdot\|_\infty$  to  $L$ , we get that  $L \cap B_{C(K)}$  is  $\|\cdot\|$ -closed and every  $F_n$  is  $\|\cdot\|$ -closed. Baire's Theorem gives that some  $F_n$  has nonempty  $\|\cdot\|$ -interior, say  $C = \{f \in L : \|f - g\| < r\} \subset L$  for some  $g \in L$  and  $r > 0$ . As  $F_n$  is  $\|\cdot\|_\infty$ -closed, we have  $\overline{C}^{\|\cdot\|_\infty} \subset F_n \subset L$ . By translation we deduce that

$$B = \overline{\{f \in L : \|f\| \leq 1\}}^{\|\cdot\|_\infty}$$

is a subset of  $L$  which also is  $\|\cdot\|_\infty$ -bounded,  $\|\cdot\|_\infty$ -closed, lattice closed, symmetric, convex and contains a non trivial constant function. As  $L = \bigcup_{n=1}^{\infty} nB$ ,

then  $B$  also separates points of  $K$ . By the previous lemma,  $L$  is a lattice of Lipschitz functions. Observe that the norm given by the lemma and  $\|\cdot\|$  are comparable, thus the Open Mapping theorem gives that they are equivalent. ■

Recall that a compact Hausdorff space is said Radon-Nikodým if there is a lower semicontinuous metric  $d$  which fragments  $K$ , see [10]. We have the following characterization.

**Corollary 3.3** *A compact Hausdorff space  $K$  is Radon-Nikodým if, and only if, there exists a dense  $\mathcal{F}_\sigma$  lattice  $L \subset C(K)$  containing the constant functions which is complete for a finer lattice norm and such that every bounded sequence in  $L$  has a pointwise Cauchy subsequence.*

**Proof.** A sequence in  $L$  is bounded if, and only if, it is  $\|\cdot\|_\infty$ -bounded and equi-Lipschitz with respect to some lower semicontinuous metric  $d$  given by the former theorem. The result follows from Theorem 2.1 and Rosenthal's  $\ell_1$ -theorem. ■

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