On the dentability of weak*-$\mathcal{H}_\delta$ sets

M. Raja*

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Abstract

We state several conditions equivalent to the weak*-dentability for certain subsets of a dual Banach space which includes both the compact case and a result of Ghoussoub and Maurey.

Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Givat Ram, 91904
Jerusalem, ISRAEL

and

Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, SPAIN
E-mail: matias@um.es

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1 Introduction

It is well known that the Radon-Nikodým property (RNP) of convex subsets of a Banach space is characterized by means of the geometrical notion of dentability, see [2]. This note is concerned with weak*-dentability in dual Banach spaces. Recall that a subset C of a dual Banach space X^* is weak*-dentable if it has weak*-slices of arbitrarily small diameter, and a weak*-slice of C is the nonempty intersection of C with a weak*-open halfspace. We say that C is hereditarily weak*-dentable if every nonempty subset of C is weak*-dentable.

Convex weak*-compact subsets of X^* having the RNP are weak*-dentable, see [2]. For non compact sets the situation becomes more complicate. Bourgain proved in [1] that the bounded convex closed subsets with the RNP of a dual Banach space X^* are weak*-dentable provided that X does not contain a copy of ℓ_1. For a negative result see Example 2.6 below.

The search of intrinsic conditions on a subset $C \subset X^*$ for its weak*-dentability, or at least, in terms of what kind of subset C must be in X^* with respect to the weak* topology, leads to the following notion introduced by Ghoussoub and Maurey [4, 5]:

Definition 1.1 *A bounded subset C of X^* is said to be a weak*-$\mathcal{H}_\delta$ set if $\overline{C}^{w^*} \setminus C$ is a countable union of convex weak*-compact sets.*

The properties of $\mathcal{H}_\delta$ sets and their application to non-compact optimization are extensively studied in the memoir [5]. Ghoussoub and Maurey proved, among other things, that *a closed convex bounded set in the dual of a separable Banach space is hereditarily weak*-dentable if, and only if, it is a separable weak*-$\mathcal{H}_\delta$ set.* Part of the proof, as presented in [5], depends strongly on construction of martingales and separability arguments in order to reduce the problem to subsets of ℓ_2. Our aim is to give a more direct proof of the dentability of the separable weak*-$\mathcal{H}_\delta$ sets of dual Banach space, and by the way, to remove separability assumptions. For this purpose, the optimal hypothesis seems to be the fragmentability of weak*-compact subsets. Recall that a set $A \subset X^*$ is weak*-fragmentable (by the norm) if every nonempty subset of it has nonempty relatively weak*-open subsets of arbitrarily small diameter. It is an easy application of Baire's theorem that norm separable weak*-compact subsets are weak*-fragmentable.

In this note we shall prove the following:

Theorem 1.2 *Let $C \subset X^*$ be a norm closed convex weak*-$\mathcal{H}_\delta$ set. Then the following properties are equivalent:*

i) C is hereditarily weak*-dentable.

ii) Weak*-compact subsets of C are weak*-fragmentable.

iii) C contains no dyadic tree.

iv) C has the Radon-Nikodým property.

The former result is essentially [5, Theorem I.8] without separability assumptions. Recall that a dyadic tree is a set $\{z_s : s \in \{0,1\}^{<\mathbb{N}}\}$, where $\{0,1\}^{<\mathbb{N}}$ denotes all the finite sequences of 0's and 1's, satisfying $z_s = \frac{1}{2}(z_{s\frown 0} + z_{s\frown 1})$ and $\|z_{s\frown 0} - z_{s\frown 1}\| > \varepsilon$ for some $\varepsilon > 0$ and for every $s \in \{0,1\}^{<\mathbb{N}}$. The presence of a bounded dyadic tree is a strong form of failing the RNP.

Since weak*-compact sets are trivially weak*-$\mathcal{H}_\delta$, Theorem 1.2 also includes the well known results for the compact case, see [2] and the references therein. We also get as a consequence that the norm closed convex weak*-$\mathcal{H}_\delta$ set is contained in the weak*-closed convex hull of its weak*-denting points in case one of the above equivalent properties holds. Recall that a point of C is said to be weak*-denting, if it is contained in arbitrarily small weak*-slices of C.

2 Proof and auxiliary results

The first part of the following lemma can be obtained from a result of [6]. Notice that in our particular case, the proof is much more simple. The second part of the lemma applies ideas of van Dulst and Namioka [3].

Lemma 2.1 *Let C be a non-weak*-fragmentable bounded subset of X^* which is weak*-$\mathcal{G}_\delta$ set in $\overline{C}^{w^*}$. Then C contains a non-weak*-fragmentable weak*-compact subset. If C is moreover convex, then C contains a dyadic tree.*

Proof. We will work with the relative topology on $\overline{C}^{w^*}$. There is a decreasing sequence G_n of weak*-open subsets such that $C = \bigcap_{n=1}^{\infty} G_n$. Since C is not weak*-fragmentable, there is $\varepsilon > 0$ and $D \subset C$, such that every weak*-open subset meeting D has diameter bigger than ε.
For any $s \in \{0,1\}^{<\mathbb{N}}$ we may define inductively weak*-open subset U_s such that $U_s \cap D \neq \emptyset$ inductively as follows. The first one $U_\emptyset$ is arbitrary. Suppose U_s already built for all the sequences s of length $n-1$. A sequence of length n is of the form $s\frown i$ with s of length $|s| = n-1$ and $i \in \{0,1\}$. Pick two points $x_0, x_1 \in U_s \cap D$ with $\|x_0 - x_1\| > \varepsilon$. By the weak*-lower semicontinuity of the norm, there are relatively weak*-neighborhoods $U_{s\frown 0}, U_{s\frown 1}$ of x_0, x_1 respectively such that $d(U_{s\frown 0}, U_{s\frown 1}) > \varepsilon$. Using the regularity, we may suppose that $\overline{U_{s\frown i}}^{w^*} \subset U_s \cap G_n$ and $d(\overline{U_{s\frown 0}}^{w^*}, \overline{U_{s\frown 1}}^{w^*}) > \varepsilon$. The set $H = \bigcap_{n=1}^{\infty} \bigcup_{|s|=n} \overline{U_s}^{w^*}$ is a weak*-compact subset of C, and there is a surjective continuous map $f : H \to \{0,1\}^{\mathbb{N}}$ defined in an obvious way, namely $f(x) = \sigma \in \{0,1\}^{\mathbb{N}}$ where $x \in U_s$ for all initial segments s of σ. Take $K \subset H$ a compact subset minimal with respect to the property that $f(K) = \{0,1\}^{\mathbb{N}}$. It is not difficult to see that every nonempty relatively weak*-open subset of K has diameter bigger than ε, so K is not weak*-fragmentable.
We shall assume now that C is convex. Then $\overline{C}^{w^*}$ is also convex and the sets U_s above can be taken to be convex too. In particular, we have

$$\frac{1}{2}(\overline{U_{s\frown 0}}^{w^*} + \overline{U_{s\frown 1}}^{w^*}) \subset U_s$$

for every $s \in \{0,1\}^{<\mathbb{N}}$. Using the continuity of the sum, the sets U_s can be also taken to satisfy the following additional property

$$\frac{1}{2^k} \sum_{|s_2|=k} \overline{U_{s_1 \frown s_2}}^{w^*} \subset G_n$$

for every $n \in \mathbb{N}$, $0 < k < n$ and $s_1 \in \{0,1\}^{<\mathbb{N}}$ with $|s_1| = n - k$. A simple compactness argument gives that there exists points $z_s \in \overline{U_s}^{w^*}$ such that

$$z_s = \frac{1}{2}(z_{s\frown 0} + z_{s\frown 1})$$

for every $s \in \{0,1\}^{<\mathbb{N}}$. Clearly we have $\|z_{s\frown 0} - z_{s\frown 1}\| > \varepsilon$. The property above ensures that $z_s \in G_n$ for every $n \geq |s|$, and thus $z_s \in C$. Therefore, the set $\{z_s : s \in \{0,1\}^{<\mathbb{N}}\}$ is a dyadic tree contained in C. ∎

Definition 2.2 *Let C be a subset of a dual Banach space. We say that $x \in C$ is a weak*-continuity point of C if the identity map from (C, w^*) to the norm topology is continuous at x.*

Lemma 2.3 *Let C be a bounded subset of X^* which is weak*-$\mathcal{G}_\delta$ set in $\overline{C}^{w^*}$. If C is weak*-fragmentable, then the set of weak*-continuity points of C is weak*-dense.*

Proof. It is not difficult to prove that a hereditarily Baire weak*-fragmentable subset of X^* has a weak*-dense set of weak*-continuity points, see [8]. ∎

Lemma 2.4 *Let $C \subset X^*$ be a norm closed and convex weak*-$\mathcal{H}_\delta$ set. Then any weak*-slice of C containing a weak*-continuity point also contains a point which is extreme in $\overline{C}^{w^*}$.*

Proof. First we shall show that the existence of a weak*-continuity point implies the existence in C of an extreme point of $\overline{C}^{w^*}$. Suppose that all the extreme points of $\overline{C}^{w^*}$ belong to $\overline{C}^{w^*} \setminus C = \bigcup_{n=1}^{\infty} K_n$, where each K_n is convex and weak*-compact. Let $x \in C$ be a continuity point. By the Choquet-Bishop-de Leeuw theorem [9], there is a Borel probability measure μ on $\overline{C}^{w^*}$

with barycenter x such that $\mu(D) = 1$ for every $\mathcal{F}_\sigma$ subset D containing the extreme points. In particular, we have $\mu(\bigcup_{n=1}^{\infty} K_n) = 1$. For every $n \in \mathbb{N}$, if the number

$$\lambda_n = \mu(K_n \setminus \bigcup_{j=1}^{n-1} K_j)$$

is not zero, let $x_n \in K_n$ be the barycenter of μ_n where

$$\mu_n(A) = \lambda_n^{-1} \mu(A \cap (K_n \setminus \bigcup_{j=1}^{n-1} K_j))$$

for every $A \subset \overline{C}^{w^*}$ Borel. Take $x_n = 0$ if $\lambda_n = 0$. We have the convex combination

$$x = \sum_{n=1}^{\infty} \lambda_n x_n$$

Now we shall apply an idea of Lin, Lin and Troyanski [7]. Fix $\varepsilon > 0$ and let U be a weak*-open neighbourhood of x such that $\mathrm{diam}(\overline{C}^{w^*} \cap U) < \varepsilon$. Take $n \in \mathbb{N}$ with $\lambda_n \neq 0$. By continuity of the sum, we can find a weak*-open neighbourhood V of x_n such that

$$\lambda_n V + \sum_{m \neq n} \lambda_m x_m \subset U$$

An easy computation gives that $\mathrm{diam}(\overline{C}^{w^*} \cap V)$ is at most $\lambda_n^{-1}\varepsilon$. Since $\varepsilon > 0$ is arbitrary, that would imply that $x_n \in C$, which is impossible because $x_n \in K_n$. We shall show now how to localize the extreme point in a given weak*-slice of C. Assume that the slice is given by a weak*-open halfspace $H = \{y \in X^* : y(t) > a\}$ and let $x \in C \cap H$ be a weak*-continuity point. Define weak*-compact convex sets $K_{n,m} = K_n \cap \{y \in X^* : y(t) \geq a + m^{-1}\}$. If there is no extreme point of $\overline{C}^{w^*}$ in $C \cap H$, then the extreme points of $\overline{C}^{w^*}$ are covered by

$$(\overline{C}^{w^*} \setminus H) \cup (\bigcup_{n,m} K_{n,m})$$

Applying the Choquet-Bishop-de Leeuw theorem as above we shall obtain a convex combination

$$x = \lambda_0 x_0 + \sum_{n,m} \lambda_{n,m} x_{n,m}$$

where $x_0 \in \overline{C}^{w^*} \setminus H$ and $x_{n,m} \in K_{n,m}$. Since $x \in H$, there is some $\lambda_{n,m} \neq 0$. As above, that gives a contradiction. ■

Proof of Theorem 1.2. Clearly $i) \Rightarrow ii)$, $i) \Rightarrow iv)$ and $iv) \Rightarrow iii)$. The proof will be complete if we show that $ii) \Rightarrow i)$ and $iii) \Rightarrow i)$.
To prove that C is hereditarily weak*-dentable, it is enough to find small weak*-open slices of nonempty relatively weak*-closed convex subsets of C. Since relatively weak*-closed convex subsets of C are also weak*-$\mathcal{H}_\delta$ sets, we just need to prove that C has small weak*-open slices. If we assume either $ii)$ or $iii)$, then C is weak*-fragmentable by Lemma 2.1. Let E be the set of extreme points of $\overline{C}^{w^*}$. Consider the set $D = C \cap \overline{C \cap E}^{w^*}$. We know that C has a weak*-dense set of weak*-continuity points by Lemma 2.3, and then the weak*-closed convex hull of D is $\overline{C}^{w^*}$ after Lemma 2.4 and the Hahn-Banach theorem. By the Bourgain-Namioka geometrical lemma [2, p.52], it is enough to show that D has weak*-slices of arbitrarily small diameter. Clearly D is a weak*-fragmentable weak*-$\mathcal{G}_\delta$ subset of $\overline{D}^{w^*}$, so by Lemma 2.3 has a weak*-continuity point, namely $x \in D$. Given $\varepsilon > 0$, we may take a weak*-neighbourhood of x such that $\mathrm{diam}(D \cap U) < \varepsilon$. Take $y \in E \cap C \cap U$. By Choquet's lemma, there is a weak*-open half space H containing y such that $\overline{C}^{w^*} \cap H \subset \overline{C}^{w^*} \cap U$. Then $y \in D \cap H \subset D \cap U$ and thus $\mathrm{diam}(D \cap H) < \varepsilon$.■

Having in mind the ideas urderlying in the Bourgain-Namioka geometrical lemma [2, p.52], the proof above actually gives that every weak*-slice S of the set C, contains another weak*-slice of diameter less than ε. Iterating this argument with $\varepsilon = n^{-1}$, we shall obtain a nested sequence of weak*-slices converging to a weak*-denting point $x \in S$. As a consequence we get that $\overline{C}^{w^*}$ is the weak*-closed convex hull of the weak*-denting points of C.

Remark 2.5 *From the proof of Lemma 2.1 we obtain that a weak*-$\mathcal{H}_\delta$ set C which fails the RNP contains the weak*-closure of a dyadic tree.*

The following example shows that, in general, the RNP does not imply the weak* dentability.

Example 2.6 *A non weak*-dentable separable closed convex set with the RNP.*

Proof. Consider in $\ell_\infty = {\ell_1}^*$ the linear subspace L spanned by the characteristic functions $\{\chi_{A(n,k)} : n, k \in \mathbb{N} : 1 \le k \le 2^n\}$, where

$$A(n,k) = \{k + 2^n(i-1) : i \in \mathbb{N}\} \subset \mathbb{N}$$

Let X be the norm closure of L. It is easy to see that X is 1-norming, and therefore the evaluation map restricted to X gives an isometric embedding of ℓ_1 into X^*. We claim that the set $C = B_{\ell_1}$ is not weak*-dentable as a subset of X^*. It is enough to show that any nonempty slice of C determined by an element of L has diameter bigger than 1. Indeed, if $f \in L$ then there is $n \in \mathbb{N}$ such that f is constant on the sets $A(n,k)$ with $1 \le k \le 2^n$. Suppose that $S = \{x \in C : f(x) > a\}$ is a nonempty slice of C. Take $x = (x_i) \in S$ with $\|x\| = 1$. Since the sets $A(n,k)$ are infinite, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(A(n,k)) = A(n,k)$ for every $1 \le k \le 2^n$ and $\sum_{i=1}^{\infty} |x_i - x_{\phi(i)}| > 1$. If $y = (x_{\phi(i)})$, then $y \in C$ and $f(y) = f(x) > a$, and thus $y \in S$. On the other hand $\|x - y\| > 1$, and thus $\mathrm{diam}(S) > 1$. ■

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