DISTANCE TO SPACES OF CONTINUOUS FUNCTIONS

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ABSTRACT. We link here distances between iterated limits, oscillations, and distances to spaces of continuous functions. For a compact space K, a uniformly bounded set H of the space of real-valued continuous functions $C(K)$, and $\varepsilon \geq 0$, we say that H ε-interchanges limits with K, if the inequality

$$|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| \leq \varepsilon$$

holds for any two sequences (x_n) in K and (f_m) in H, provided the iterated limits exist. We prove that H ε-interchanges limits with K if, and only if, the inequality for the oscillations

$$\operatorname{osc}^*(f) = \sup_{x\in K} \operatorname{osc}^*(f,x) = \sup_{x\in K} \inf\{\sup_{y\in U}|f(y) - f(x)| : U \text{neighb. of } x\} \leq \varepsilon,$$

holds for every f in the closure $\operatorname{cl}_{\mathbb{R}^K}(H)$ of H in $\mathbb{R}^K$. Since oscillations actually measure distances to spaces of continuous functions, we get that if H ε-interchanges limits with K, then

$$\hat{d}(\operatorname{cl}_{\mathbb{R}^K}(H), C(K)) := \sup_{f\in \operatorname{cl}_{\mathbb{R}^K}(H)} d(f, C(K)) \leq \varepsilon.$$

Conversely, if $\hat{d}(\operatorname{cl}_{\mathbb{R}^K}(H), C(K)) \leq \varepsilon$, then H 2ε-interchanges limits with K. We also prove that H ε-interchanges limits with K if, and only if, its convex hull $\operatorname{conv}(H)$ does. As a consequence we obtain that for each uniformly bounded pointwise compact subset H of $\mathbb{R}^K$ we have

$$\hat{d}(\operatorname{cl}_{\mathbb{R}^K}(\operatorname{conv}(H)), C(K)) \leq 5\hat{d}(H, C(K)).$$

The above estimates can be applied to measure distances from elements of the bidual E^{**} to the Banach space E: for a w*-compact subset H of E^{**}, we have $\hat{d}(\text{w}^*\text{-}\operatorname{cl}(\operatorname{conv}(H)), E) \leq 5\hat{d}(H, E)$. These results are *quantitative* versions of the classical Eberlein-Grothendieck and Krein-Smulyan theorems. In the case of Banach spaces these quantitative generalizations have been recently studied by M. Fabian, A. S. Granero, P. Hajék, V. Montesinos, and V. Zizler. Our topological approach allows us to go further: most of the above statements remain true for spaces $C(X, Z)$ for a paracompact (in some cases, a normal countably compact) space X and a convex compact subset Z of a Banach space.

1. INTRODUCTION

Our notation and terminology is standard and explained at the end of this introduction. The following notion was introduced by Grothendieck in [7], for $\varepsilon = 0$, and it has been considered in Banach spaces, for $\varepsilon \geq 0$, in [4]:

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Definition 1. *Let (Z,d) be a metric space, X be a set, H be a subset of functions from X into Z and $\varepsilon \geq 0$. We say that H ε-interchanges limits with a subset A of X if for any two sequences (x_n) in A and (f_m) in H*

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) \leq \varepsilon$$

whenever all limits involved do exist. When $\varepsilon = 0$ we simply say that H interchanges limits with A.

A relationship between the two properties below for a bounded set H of a Banach space E has been investigated in [4]:

(i) *H ε-interchanges limits with the dual unit ball B_{E^*};*
(ii) *the weak* closure of H in the bidual E^{**} satisfies $w^*\text{-cl}(H) \subset E + \varepsilon B_{E^{**}}$.*

Property (i) implies (ii) and (ii) implies that H 2ε-interchanges limits with B_{E^*}, [4]. These results together with Ptak's combinatorial lemma were used in [4] to prove a very interesting quantitative version of the Krein-Smulyan theorem. In [6] this quantitative generalization of the Krein-Smulyan theorem is improved a bit further and formulated as follows: *if H is a w*-compact subset of E^{**} then*

$$\sup\{d(h,E) : h \in \mathrm{w}^*\text{-}\mathrm{cl}(\mathrm{conv}(H))\} \leq 5 \sup\{d(h,E) : h \in H\}. \qquad (1.1)$$

Both papers [4] and [6] use beautiful Banach spaces tricks in their proofs, but their approaches are quite different. In this paper we stress that the above matters are of topological nature and can be considered in a more general framework of spaces of continuous functions embedded in spaces of bounded functions.

Our starting point are the following result and remark:

Result 1.1 (Benyamini and Lindenstrauss, [1, Proposition 1.18])**.** *Let X be a normal space. If $f \in \mathbb{R}^X$ is bounded, then we have*

$$d(f, C^*(X)) = \frac{1}{2}\,\mathrm{osc}(f)\,. \qquad (1.2)$$

Remark 1. Let (Z,d) be a metric space and X a topological space. For a bounded $f \in Z^X$ we have

$$d(f, C^*(X,Z)) \geq \frac{1}{2}\,\mathrm{osc}(f)\,. \qquad (1.3)$$

Actually, in [1] result 1.1 was formulated for paracompact spaces X. However, as was pointed out there, it holds true for normal spaces X. The reader can easily verify that the argument from the proof of [1, Proposition 1.18] combined with [3, 1.7.15.(b)] also works in this case. The remark is a simple observation that holds in general.

In section 2 we link oscillations of functions, the ε-interchanging limit property of sets, and distances to spaces of continuous functions; to take our results to the case of subsets of spaces of vector valued functions we replace equality (1.2) by an estimate for spaces $C^*(X,Z)$ (Z a convex set of a normed space) that completes inequality 1.3, see Lemma 2.7. We include examples showing that our estimates are sharp.

Section 3 is devoted to proving that if Z is a compact convex subset of a normed space E, K a set, and H a subset of the product Z^K that ε-interchanges limits with K, then $\mathrm{conv}(H)$ ε-interchanges limits with K, see Theorem 3.3. As a consequence, for K normal and countably compact and H a uniformly bounded pointwise compact subset of $\mathbb{R}^K$, we have

$$\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H)), C(K)) \leq 5\hat{d}(H, C(K)), \tag{1.4}$$

see Corollary 3.5.

Section 4 deals with the problem of estimating distances to spaces of affine continuous functions. We start by proving that given a compact convex set K of a locally convex space and a bounded affine function f defined on K, the distance of f to the space of continuous functions on K is the same than the distance of f to the space of continuous and affine functions on K, see Proposition 4.1. Once we know this, inequality (1.4) implies inequality (1.1); also the results of [4] presented at the beginning of the introduction follow straightforwardly from our results in section 2.

In the last section of the paper we study the sequential approximation of points in the closure of sets enjoying the ε-interchanging limit property.

Notation and terminology.

X denotes here a set or a completely regular topological space, (Z, d) a metric space (Z if d is implicitly assumed) and $(E, \|\cdot\|)$ a normed space (E if $\|\cdot\|$ is implicitly assumed). The space Z^X is equipped with the product topology τ_p; if $H \subset Z^X$ we write $\mathrm{cl}_{Z^K}(H)$ for the closure of H in (Z^X, τ_p). In the subspace of Z^X consisting of bounded functions we also consider the standard supremum metric, which we usually also denote by d, i.e., $d(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$ for bounded functions $f, g : X \to Z$. $C(X, Z)$ is the space of continuous maps from X into Z, and $C^*(X, Z)$ stands for the maps in $C(X, Z)$ which are bounded ($C(X)$ and $C^*(X)$ are the corresponding spaces of real-valued continuous functions; $\ell_\infty(X)$ is the Banach space of bounded functions on X endowed with the supremum norm $\|\cdot\|_\infty$).

For A and B are nonempty subsets of a metric space (Z, d), we consider the Hausdorff non-symmetrized distance from A to B:

$$\hat{d}(A, B) = \sup\{d(x, B) : x \in A\}.$$

The *oscillation* –$\mathrm{osc}(f, x)$– and *semi-oscillation* –$\mathrm{osc}^*(f, x)$– of a bounded function $f \in Z^X$ at the point $x \in X$ are defined by

$$\begin{aligned} \mathrm{osc}(f, x) &= \inf_U \sup_{y,z \in U} d(f(y), f(z)) \\ \mathrm{osc}^*(f, x) &= \inf_U \sup_{y \in U} d(f(y), f(x)) \end{aligned} \tag{1.5}$$

where the infimum is taken over the neighborhoods U of x in X. Clearly we have the inequalities

$$\mathrm{osc}^*(f, x) \leq \mathrm{osc}(f, x) \leq 2\,\mathrm{osc}^*(f, x) \tag{1.6}$$

We write $\mathrm{osc}(f) = \sup_{x \in X} \mathrm{osc}(f, x)$ and $\mathrm{osc}^*(f) = \sup_{x \in X} \mathrm{osc}^*(f, x)$.

2. Iterated limits vs. distance to spaces of continuous functions

This first section is devoted to establish a relationship between distances between iterated limits, oscillations of functions and distances to spaces of continuous functions. The next lemma will allow us to replace nets by sequences in some computations involving the ε-interchanging limit property.

Lemma 2.1. *Let (Z,d) be a metric space, X a set and (x_α) and (f_β) nets in X and Z^X, respectively. If the iterated limits*

$$\lim_\alpha \lim_\beta f_\beta(x_\alpha) \quad \textit{and} \quad \lim_\beta \lim_\alpha f_\beta(x_\alpha)$$

exist, then there are increasing sequences (α_n) and (β_m) of indices such that

$$\lim_n \lim_m f_{\beta_m}(x_{\alpha_n}) = \lim_\alpha \lim_\beta f_\beta(x_\alpha),$$

$$\lim_m \lim_n f_{\beta_m}(x_{\alpha_n}) = \lim_\beta \lim_\alpha f_\beta(x_\alpha)$$

Proof. Take $p_\beta = \lim_\alpha f_\beta(x_\alpha)$, $q_\alpha = \lim_\beta f_\beta(x_\alpha)$, $p = \lim_\beta p_\beta$ and $q = \lim_\alpha q_\alpha$. We shall define inductively the sequences of indices (α_n) and (β_n) in such a way that

$$\lim_n f_{\beta_m}(x_{\alpha_n}) = p_{\beta_m}, \text{ for every } m \in \mathbb{N},$$
$$\lim_m f_{\beta_m}(x_{\alpha_n}) = q_{\alpha_n}, \text{ for every } n \in \mathbb{N}$$

and

$$\lim_m p_{\beta_m} = p \qquad \lim_n q_{\alpha_n} = q.$$

Take α_1 such that $d(q_{\alpha_1}, q) < 1$. Take β_1 such that $d(p_{\beta_1}, p) < 1$ and satisfying also $d(f_{\beta_1}(x_{\alpha_1}), q_{\alpha_1}) < 1$. Assume that α_k and β_k have been already chosen for $k < n$. Take $\alpha_n > \alpha_{n-1}$ such that $d(q_{\alpha_n}, q) < n^{-1}$ and $d(f_{\beta_k}(x_{\alpha_n}), p_{\beta_k}) < n^{-1}$ for every $k < n$. Take $\beta_n > \beta_{n-1}$ satisfying simultaneously $d(p_{\beta_n}, p) < n^{-1}$ and $d(f_{\beta_n}(x_{\alpha_k}), q_{\alpha_k}) < n^{-1}$ for every $k \leq n$. Clearly (α_n) and (β_n) satisfy the requirements above. □

Given a bounded map f from the topological space X into (Z,d) and a point $x \in X$, observe that we can always find a net (x_α) converging to x in X in such a way that $\operatorname{osc}^*(f,x) = \lim_\alpha d(f(x_\alpha), f(x))$. Indeed, write $\mathcal{U}_x$ for the family of neighborhoods of x and given $U \in \mathcal{U}_x$ and $\delta > 0$ take $x_{U,\delta} \in U$ satisfying

$$\sup_{y \in U} d(f(y), f(x)) - \delta \leq d(f(x_{U,\delta}), f(x)).$$

If the set $\mathcal{U}_x \times (0, +\infty)$ is directed by the binary relation $(U,\delta) \geq (U',\delta')$ if, and only if, $U \subset U'$ and $\delta \leq \delta'$, then $\operatorname{osc}^*(f,x) = \lim_{U,\delta} d(f(x_{U,\delta}), f(x))$.

The following easy fact will be used below: if (x_n) has a cluster point x in X and there exists $\lim_n f(x_n) = z$ then $d(f(x), z) \leq \operatorname{osc}^*(f,x)$.

Proposition 2.2. *Let (Z,d) be a compact metric space, X a topological space and H a subset of $C(X,Z)$. The following properties hold:*

(i) *if H ε-interchanges limits with X (in Z), then $\operatorname{osc}^*(f) \leq \varepsilon$ for every $f \in \operatorname{cl}_{Z^X}(H)$; in particular, $\operatorname{osc}(f) \leq 2\varepsilon$ for every $f \in \operatorname{cl}_{Z^X}(H)$;*
(ii) *conversely, if X is countably compact and $\operatorname{osc}^*(f) \leq \varepsilon$ for every f in $\operatorname{cl}_{Z^X}(H)$, then H ε-interchanges limits with X (in Z).*

Proof. Let us prove (i). Take $f \in \operatorname{cl}_{Z^K}(H)$ and fix $x \in X$. Take a net (x_α) in X converging to x such that

$$\lim_\alpha d(f(x_\alpha), f(x)) = \operatorname{osc}^*(f,x)$$

and take a net (f_β) in H converging to f in Z^X. Since Z is compact, we may assume that $f(x_\alpha)$ converges to some z in Z (see [3, Prop. 1.6.1, Thm. 3.1.23]). Thus we have

$$\lim_\alpha \lim_\beta f_\beta(x_\alpha) = \lim_\alpha f(x_\alpha) = z$$
$$\lim_\beta \lim_\alpha f_\beta(x_\alpha) = \lim_\beta f_\beta(x) = f(x)$$

Hence $d(z, f(x)) = \operatorname{osc}^*(f, x) \leq \varepsilon$ by applying Lemma 2.1.

The proof of (ii) is as follows. Take sequences (x_n) in X and (f_m) in H for which the limits below exist

$$d(\lim_n \lim_m f_m(x_n), \lim_m \lim_n f_m(x_n)) = D$$

Since X is countably compact we can take a cluster point x of (x_n) in X. Let $f \in \operatorname{cl}_{Z^X}(H)$ be a cluster point of (f_m) in Z^X. We have

$$\lim_n \lim_m f_m(x_n) = \lim_n f(x_n) = z$$
$$\lim_m \lim_n f_m(x_n) = \lim_m f_m(x) = f(x)$$

and therefore $D = d(f(x), z) \leq \operatorname{osc}^*(f, x)$. □

The estimates for oscillations in Proposition 2.2 can be obtained even when we only have the ε-interchanging limit property with dense subspaces.

Lemma 2.3. *Let f be a map of a topological space X into a metric space (Z, d), D be a dense subset of X, and $\varepsilon \geq 0$. If every point $x \in X$ has a neighborhood U such that $\sup_{d \in U \cap D} d(f(x), f(d)) \leq \varepsilon$ then $\operatorname{osc}^*(f) \leq 2\varepsilon$.*

Proof. Fix $x \in X$ and take a neighborhood U of x such that

$$\sup_{d \in U \cap D} d(f(x), f(d)) \leq \varepsilon.$$

For each $y \in U$ we can find a neighborhood V of y, contained in U, and such that $\sup_{d \in V \cap D} d(f(y), f(d)) \leq \varepsilon$. Now, we can pick any point $d \in V \cap D$ to estimate the distance $d(f(x), f(y)) \leq d(f(x), f(d)) + d(f(d), f(y)) \leq 2\varepsilon$. □

Proposition 2.4. *Let (Z, d) be a compact metric space, X be a topological space, and H be a subset of $C(X, Z)$. If H ε-interchanges limits with a dense subset D of X, then for every $f \in \operatorname{cl}_{Z^X}(H)$, $\operatorname{osc}^*(f) \leq 2\varepsilon$, hence $\operatorname{osc}(f) \leq 4\varepsilon$.*

Proof. We will prove that for each $f \in \operatorname{cl}_{Z^X}(H)$, each $\delta > \varepsilon$, and each point $x \in X$ there exist a neighborhood U of x such that $\sup_{d \in U \cap D} d(f(x), f(d)) \leq \delta$; then Lemma 2.3 will allow us to finish the proof of the proposition. Our reasoning is by contradiction: if for some $f \in \operatorname{cl}_{Z^X}(H)$ and some point $x \in X$ we have $\sup_{d \in U \cap D} d(f(x), f(d)) > \delta$ for some $\delta > \varepsilon$ and for each neighborhood U of x, then we can produce a net (d_α) in D converging to x in X such that $d(f(d_\alpha), f(x)) > \delta$ for every α. Since Z is compact we can assume that there exists $\lim_\alpha f(d_\alpha) = z$ in Z. On the other hand, let us take (f_β) in H such that (f_β) converges to f in Z^X. Since one computes

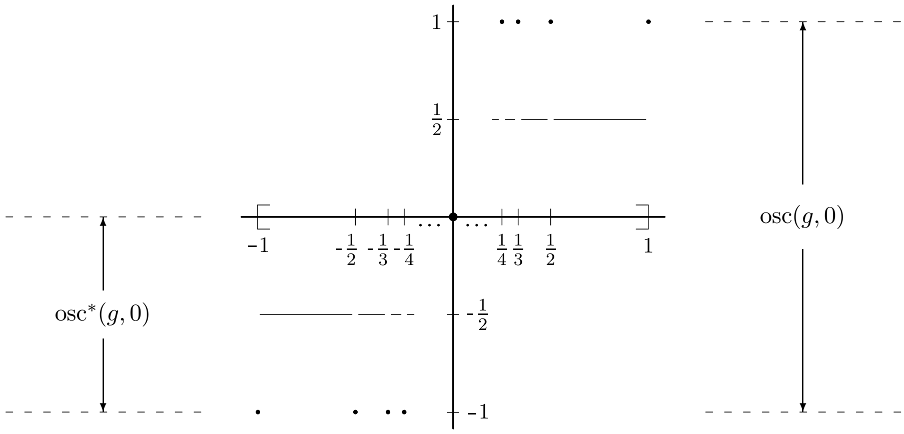
$$\lim_\alpha \lim_\beta f_\beta(d_\alpha) = \lim_\alpha f(d_\alpha) = z$$
$$\lim_\beta \lim_\alpha f_\beta(d_\alpha) = \lim_\beta f_\beta(x) = f(x)$$

we obtain $d(\lim_\alpha \lim_\beta f_\beta(d_\alpha), \lim_\beta \lim_\alpha f_\beta(d_\alpha)) \geq \delta$ that contradicts Lemma 2.1, if we bear in mind that H ε-interchanges limits with D and $\delta > \varepsilon$. □

Remark 2. The following simple example shows that the constant 4 in Proposition 2.4 cannot be improved. Take $X = Z = [-1, 1]$ (with the usual metric). Let $g : [-1, 1] \to [-1, 1]$ be defined as follows

$$g(x) = \begin{cases} -1 & \text{if } x \in \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}, \\ -\frac{1}{2} & \text{if } x \in [-1, 0) \setminus \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}, \\ 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x \in (0, 1] \setminus \{1, \frac{1}{2}, \frac{1}{3}, \dots\}, \\ 1 & \text{if } x \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \end{cases}$$

for $x \in [-1, 1]$.



The function g

One can easily verify that the function g is of the first Baire class (e.g. one may use the fact that, for each open $U \subset [-1, 1]$, the inverse image $f^{-1}(U)$ is an F_σ-set in $[-1, 1]$). Hence, we can take a sequence (f_n) of continuous functions from $[-1, 1]$ into $[-1, 1]$ converging pointwise to g. Define $H = \{f_n : n \in \mathbb{N}\}$ and $D = [-1, 1] \setminus \{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \dots\}$. A routine verification shows that H (1/2)-interchanges limits with D and 1-interchanges limits with X. For the function g, which belongs to the closure of H, we have $\operatorname{osc}(g) = 2$ and $\operatorname{osc}^*(g) = 1$. This also shows that the constants in Proposition 2.2 cannot be improved either. □

If X is countably compact space and H is a τ_p-relatively countably compact of $C(X, Z)$ it is readily seen that H interchanges limits with X. Therefore our Propositions 2.2 and 2.4 are the *quantitative* versions of Eberlein-Grothendieck's result below, see [8, Theorem 8.18] and [5, p. 12], that we simply obtain taking $\varepsilon = 0$ in our previous results.

Corollary 2.5. *Let X be a countably compact space, Z a compact metric space and H a subset of C(X,Z). The following statements are equivalent:*

(i) *H is τ_p-relatively countably compact in $C(X, Z)$;*
(ii) *H interchanges limits with X;*
(iii) *H interchanges limits with some dense subset D of X;*
(iv) $\mathrm{cl}_{Z^X}(H) \subset C(X, Z)$*;*
(v) *H is τ_p-relatively compact in $C(X, Z)$.*

Equations (1.6), (1.2), and (1.3) together with Proposition 2.2 allow us to obtain the following consequences for distances to spaces of continuous functions.

Corollary 2.6. *Let X be a topological space and let H be a uniformly bounded subset of $C^*(X)$. The following properties hold:*

(i) *if X is normal and the set H ε-interchanges limits with X, then*
$$\hat{d}(\mathrm{cl}_{\mathbb{R}^X}(H), C^*(X)) \leq \varepsilon\,.$$

(ii) *if X is countably compact and $\hat{d}(\mathrm{cl}_{\mathbb{R}^X}(H), C^*(X)) \leq \varepsilon$, then H 2ε-interchanges limits with X.*

Remark 3. Let us point out that the constant 2ε in the item (ii) of the above corollary cannot be replaced by a smaller one, even if X is a countable compact (hence metrizable) space. This is demonstrated by the following simple example. Let $X = \{0\} \cup \{1/k : k \in \mathbb{N}\}$ (with the standard Euclidean topology) and $Z = [0, 1]$. For each $n \in \mathbb{N}$, let $f_n : X \to Z$ be defined by $f_n(t) = (1-t)^n$ for $t \in X$. Take $H = \{f_n : n \in \mathbb{N}\}$. The sequence (f_n) converges pointwise to the characteristic function $\chi_{\{0\}}$ of the singleton $\{0\}$, hence $\mathrm{cl}_{Z^K}(H) = H \cup \{\chi_{\{0\}}\}$. One can easily compute (see Result 1.1) that $\hat{d}(\mathrm{cl}_{Z^K}(H), C(X, Z)) = 1/2$. On the other hand, for $x_k = 1/k \in X$, we have $\lim_n \lim_k f_n(x_k) = 1$ and $\lim_k \lim_n f_n(x_k) = 0$. $\square$

Now, we will extend Corollary 2.6 to the case of Z-valued functions with Z a compact convex set of a normed space. To that end we need the lemma below, that is a counterpart to Result 1.1 and seems to be well known. However, we were not able to find a reference for it in the literature, hence we include its proof. We will use a few facts about paracompact spaces that the reader can find in [3, Sec. 5.1]. Recall that if $\mathcal{A} = \{A_s\}_{s\in S}$ is a cover of a set X, the *star* of a point $x \in X$ with respect to $\mathcal{A}$ is the set $\mathrm{St}(x, \mathcal{A}) := \bigcup\{A_s : x \in A_s\}$; we say that a cover $\mathcal{V}$ is a *barycentric refinement* of $\mathcal{A}$ if for every $x \in X$ there is $s(x) \in S$ such that $\mathrm{St}(x, \mathcal{V}) \subset A_{s(x)}$. A topological space is paracompact if, and only if, every open cover of the space has an open barycentric refinement, see [3, Theorem 1.1.12].

Lemma 2.7. *Let X be a paracompact space and let Z be a convex subset of a normed space E. For every bounded map $f : X \to Z$ we have*
$$\frac{1}{2}\,\mathrm{osc}(f) \leq d(f, C^*(X, Z)) \leq \mathrm{osc}(f)\,.$$

Proof. The first inequality is given by (1.3). Let us prove the second inequality. Put $s = \mathrm{osc}(f)$ and fix $\varepsilon > 0$. We will construct $g \in C^*(X, Z)$ satisfying $d(f, g) \leq s+\varepsilon$. For each $x \in X$ find an open neighborhood U_x of x such that $\mathrm{diam}\, f(U_x) < s + \varepsilon$. Let $\mathcal{V}$ be an open barycentric refinement of the cover $\mathcal{A} = \{U_x : x \in X\}$, and let $\{p_a : a \in A\}$ be a locally finite partition of unity subordinated to $\mathcal{V}$. For every $a \in A$ take a point $x_a \in p_a^{-1}((0, 1])$ and put $z_a = f(x_a)$. We define $g(x) = \sum_{a\in A} p_a(x) z_a$. Obviously, the map g is continuous. Fix $x \in X$. Let $B = \{a \in A : p_a(x) > 0\}$. Our choice of $\mathcal{V}$ and $\{p_a : a \in A\}$ guaranties that there exists $y \in X$ such that
$$\bigcup\{p_a^{-1}((0, 1]) : a \in B\} = \mathrm{St}(x, \{p_a^{-1}((0, 1]) : a \in A\}) \subset \mathrm{St}(x, \mathcal{V}) \subset U_y.$$
Therefore $x \in U_y$ and $x_a \in U_y$, for $a \in B$. It follows that both vectors $g(x)$ and $f(x)$ belong to $\mathrm{conv}(f(U_y))$. Since $\mathrm{diam}\,\mathrm{conv}(f(U_y)) = \mathrm{diam}\, f(U_y) < s+\varepsilon$ we infer that $\|g(x) - f(x)\| < s + \varepsilon$ and $d(f, g) \leq s + \varepsilon$ (this shows that g is bounded). Because ε was arbitrary, this proves the required inequality. $\square$

Remark 4. We exhibit now an example showing that in the above lemma the inequality $d(f, C^*(X,Z)) \leq \operatorname{osc}(f)$ cannot be improved: let $X = [0,1]$ and $Z = \operatorname{conv}\{e_n : n \in \mathbb{N}\} \subset \ell_1$, where $(e_n)_{n\in\mathbb{N}}$ are the standard unit vectors in ℓ_1. We take a partition of $[0,1]$ into countably many dense sets $A_1, A_2, \dots, A_n, \dots$ and we define $f : [0,1] \to Z$ to be the map which takes value e_n on A_n, for $n \in \mathbb{N}$. Clearly $\operatorname{osc}(f) = 2$. On the other hand we can easily calculate that $d(f, C^*([0,1], Z)) = 2$. To show this we will prove that for every $g \in C^*([0,1], Z)$ and every $\delta > 0$ there is $y \in [0,1]$ such that $(d(f,g) \geq)\|g(y) - f(y)\|_1 \geq 2 - \delta$. Indeed, we know that $g(0) = \sum_{k=1}^m \lambda_k e_k$ with $\sum_{k=1}^m \lambda_k = 1$ and $\lambda_k \geq 0$, $k = 1, 2, \dots, m$. We fix now $n > m$ and take $y \in A_n$ such that $\|g(0) - g(y)\|_1 \leq \delta$. Then

$$\|f(y)-g(y)\|_1 \geq \|f(y)-g(0)\|_1 - \|g(0)-g(y)\|_1 = 2 - \|g(0)-g(y)\|_1 \geq 2-\delta,$$

as we wanted to prove. □

Inequalities (1.3) and (1.6) together with Propositions 2.2 and 2.7 allow us to obtain:

Corollary 2.8. *Let X be a topological space, Z is a compact convex subset of a normed space and H a subset $C(X,Z)$. The following properties hold:*

(i) *if X is paracompact and the set H ε-interchanges limits with X, then $\hat{d}(\operatorname{cl}_{Z^X}(H), C(X,Z)) \leq 2\varepsilon$;*
(ii) *if X is countably compact and $\hat{d}(\operatorname{cl}_{Z^K}(H), C(X,Z)) \leq \varepsilon$ then H 2ε-interchanges limits with X.*

Next example shows that convexity of the set Z is essential in Corollary 2.8.

Example 2.9. *For each $n \in \mathbb{N}$, there exist a compact metric space (Z,d), a compact space K, and a subset H of $C(K,Z)$ such that $\hat{d}(\operatorname{cl}_{Z^K}(H), C(K,Z)) = 1$ and H $(1/n)$-interchanges limits with K.*

The construction. Let

$$Z = [-1,1] \times \{0\} \cup \bigcup_{i=-n}^{n} \{\frac{i}{n}\} \times [0,1] \subset \mathbb{R}^2 .$$

We equip Z with the standard Euclidean metric d. Take $K = [-1,1]$ with the Euclidean topology. Let $g : K \to Z$ be defined by the formula:

$$g(t) = \begin{cases} (\frac{i}{n}, 1) & \text{for } t \in [\frac{i}{n}, \frac{i+1}{n}),\ i = -n, -n+1, \dots, n-1, \\ (1,1) & \text{for } t = 1. \end{cases}$$

Observe that, for each $f \in C(K,Z)$, either $f(t) = (s,0)$ for some $s, t \in [-1,1]$, or $f(K) \subset \{i/n\} \times [0,1]$ for some i. In both cases $d(g,f) \geq 1$. Since $d(g, f_0) = 1$ for the constant function f_0 taking value $(0,1) \in Z$, we have $d(g, C(K,Z)) = 1$.

For $k > n$, let $A_k = \bigcup_{i=-n}^{n} [\frac{i}{n}, \frac{i+1}{n} - \frac{1}{k}] \cup \{1\} \subset K$. Take a continuous piece-wise linear function f_k from K to Z such that $f_k|A_k \equiv g|A_k$. Clearly the sequence $(f_k)_{k>n}$ converges pointwise to g. Define $H = \{f_k : k > n\}$. Then $\operatorname{cl}_{Z^K}(H) \setminus H = \{g\}$, so $\hat{d}(\operatorname{cl}_{Z^K}(H), C(K,Z)) = 1$. Let us verify that H $(1/n)$-interchanges limits with K. Take a sequence (h_i) in H and a sequence (x_j) in K such that the following limits exist

$$\lim_i \lim_j h_i(x_j), \quad \lim_j \lim_i h_i(x_j) .$$

Without loss of generality we may assume that (h_i) converges pointwise to g and (x_j) converges to some x in K. Then $\lim_i \lim_j h_i(x_j) = \lim_i h_i(x) = g(x)$ and $\lim_j \lim_i h_i(x_j) = \lim_j g(x_j)$. It remains to observe that $d(g(x), \lim_j g(x_j)) \leq 1/n$ since $\lim_j x_j = x$. $\square$

3. Distances to the convex hulls

The goal of this section is to prove Theorem 3.3 saying that ε-interchanging limit property is preserved when taking convex hulls. For subsets of Banach spaces this has been done in [4, Theorem 13] using Ptak's combinatorial lemma, see [9, §24.6]. Our Theorem 3.3 is more general than [4, Theorem 13] and in its proof we replace Ptak's lemma by some ideas from the proof of the Krein-Smulyan theorem in Kelley-Namioka's book [8, Ch 5. Sec. 17]. We are grateful to Prof. Namioka who brought to our attention this line of argument to prove the Krein-Smulyan theorem.

For a given set X, $\mathcal{P}(X)$ denotes the power set of X.

Result 3.1 (Kelley-Namioka, [8, Lemma 17.9])**.** *Let μ be a finitely additive (finite) measure defined on an algebra of sets $\mathcal{A}$, and let (A_k) a sequence of sets from $\mathcal{A}$ such that $\mu(A_k) > \delta$ for some $\delta > 0$ and every $k \in \mathbb{N}$. Then there exists a subsequence (A_{k_i}) such that $\mu(\cap_{i=1}^n A_{k_i}) > 0$ for every $n \in \mathbb{N}$.*

Next lemma is a slight reformulation of Lemma 17.10 from [8].

Lemma 3.2. *Let (I_n) be a sequence of pairwise disjoint finite nonempty sets and let μ_n be a probability measure on $\mathcal{P}(I_n)$ for each n. Let (A_k) be a sequence of subsets of $I = \bigcup_{n\in\mathbb{N}} I_n$ such that, for some $\delta > 0$, $\liminf_n \mu_n(A_k \cap I_n) > \delta$ holds for every $k \in \mathbb{N}$. Then there is a subsequence (A_{k_i}) such that $\bigcap_{i\leq j} A_{k_i} \neq \emptyset$ for each $j \geq 1$.*

Proof. Let $\mathcal{A}$ be the subalgebra of $\mathcal{P}(I)$ generated by sets A_k, $k \in \mathbb{N}$. Since $\mathcal{A}$ is countable we can find an increasing sequence (i_j) of positive integers such that $\lim_j \mu_{i_j}(A \cap I_{i_j})$ exists for every $A \in \mathcal{A}$. Then the function $\mu : \mathcal{A} \to [0,1]$ defined by $\mu(A) = \lim_j \mu_{i_j}(A \cap I_{i_j})$, for $A \in \mathcal{A}$, is a finitely additive measure on $\mathcal{A}$. For all k, we have $\mu(A_k) > \delta$. The desired conclusion follows easily from 3.1. $\square$

Theorem 3.3. *Let Z be a compact convex subset of a normed space E, let K be a set, and let H be a subspace of the product Z^K. Then, for each $\varepsilon \geq 0$, H ε-interchanges limits with K if, and only if, $\operatorname{conv}(H)$ ε-interchanges limits with K.*

Proof. Obviously, if $\operatorname{conv}(H)$ ε-interchanges limits with K then H also does. Let us prove the reverse implication. Let (f_n) and (x_k) be sequences in $\operatorname{conv}(H)$ and K, respectively, such that the limits $\lim_n \lim_k f_n(x_k)$, $\lim_k \lim_n f_n(x_k)$ exist. Put

$$\gamma = \| \lim_n \lim_k f_n(x_k) - \lim_k \lim_n f_n(x_k) \| . \tag{3.1}$$

For each $n \in \mathbb{N}$ we have $f_n = \sum_{a\in I_n} t_a g_a$, where $g_a \in H$, $t_a \in [0,1]$, for all a in the finite set I_n, and $\sum_{a\in I_n} t_a = 1$. We assume that the index sets I_n are pairwise disjoint. Let $I = \bigcup_{n\in\mathbb{N}} I_n$. We may select a subsequence of (x_k) (denoted again by (x_k)) such that, for every $a \in I$, the sequence $(g_a(x_k))_k$ converges to some

$q_a \in Z$. Then, for each n,

$$p_n = \lim_k f_n(x_k) = \sum_{a \in I_n} t_a q_a \,. \tag{3.2}$$

By (3.1) we can find a functional $e^* \in B_{E^*}$ such that

$$\begin{aligned} \gamma &= e^*(\lim_n \lim_k f_n(x_k) - \lim_k \lim_n f_n(x_k)) \\ &= e^*(\lim_n p_n - \lim_k \lim_n f_n(x_k)) = \lim_k e^*(\lim_n p_n - \lim_n f_n(x_k)) \,. \end{aligned} \tag{3.3}$$

Fix $\delta > 0$. By removing finitely many k, we may assume that

$$e^*(\lim_n p_n - \lim_n f_n(x_k)) = \lim_n e^*(p_n - f_n(x_k)) > \gamma - \delta \tag{3.4}$$

holds for every k. Hence, for each k, we can find n_k such that if $n \geq n_k$, then

$$e^*(p_n - f_n(x_k)) > \gamma - \delta \,. \tag{3.5}$$

For every n, let μ_n be the probability measure on $\mathcal{P}(I_n)$ defined by

$$\mu_n(A) = \sum_{a \in A} t_a$$

for $A \subset I_n$. For every $k \geq 1$, put

$$A_k = \{a \in I : e^*(q_a - g_a(x_k)) > \gamma - 2\delta\}$$

Let M denote the diameter of the set Z (obviously, we may assume that $M > 0$). Using (3.2) and (3.5) we obtain for every $k \in \mathbb{N}$ and every $n \geq n_k$ the following inequality:

$$\begin{aligned} \gamma - \delta &< e^*(p_n - f_n(x_k)) = e^*(\sum_{a \in I_n} t_a q_a - \sum_{a \in I_n} t_a g_a(x_k)) \\ &= \sum_{a \in I_n} t_a e^*(q_a - g_a(x_k)) = \sum_{a \in I_n \cap A_k} t_a e^*(q_a - g_a(x_k)) \\ &\quad + \sum_{a \in I_n \setminus A_k} t_a e^*(q_a - g_a(x_k)) \leq \sum_{a \in I_n \cap A_k} t_a M + \gamma - 2\delta \\ &= \mu_n(I_n \cap A_k) M + \gamma - 2\delta \,. \end{aligned} \tag{3.6}$$

It follows that $\mu_n(I_n \cap A_k) > \delta/M$, and therefore $\liminf_n \mu_n(I_n \cap A_k) \geq \delta/M$ for every k. Lemma 3.2 implies that there exists a subsequence (A_{k_i}) such that $\bigcap_{i \leq j} A_{k_i} \neq \emptyset$ for each $j \geq 1$. This means that, for every $j \geq 1$, we can find $a_j \in I$ such that $e^*(q_{a_j} - g_{a_j}(x_{k_i})) > \gamma - 2\delta$ for all $i \leq j$. Put $h_j = g_{a_j}$, $r_j = q_{a_j}$, and $y_i = x_{k_i}$ for $i, j \geq 1$. Then we have $\lim_i h_j(y_i) = r_j$ for each $j \geq 1$, and

$$e^*(r_j - h_j(y_i)) > \gamma - 2\delta \tag{3.7}$$

for all $i \leq j$. We can find a subsequence of (h_j) (denoted again by (h_j)) such that, for every i, the sequence $(h_j(y_i))_j$ converges to some s_i in Z, and the corresponding sequence (r_j) converges to $r \in Z$. Next, we may select a subsequence of (y_i) (denoted again by (y_i)) such that the corresponding sequence (s_i) converges to some $s \in Z$. Then $\lim_j \lim_i h_j(y_i) = \lim_j r_j = r$ and $\lim_i \lim_j h_j(y_i)) = \lim_i s_i = s$. For each $i \geq 1$, inequality (3.7) implies that $\lim_j e^*(r_j - h_j(y_i)) =$

$e^*(r - s_i) \geq \gamma - 2\delta$. Therefore $\lim_i e^*(r - s_i) = e^*(r - s) \geq \gamma - 2\delta$. Since $e^* \in B_{E^*}$ we infer that

$$\|r - s\| = \|\lim_j \lim_i h_j(y_i) - \lim_i \lim_j h_j(y_i))\| \geq \gamma - 2\delta\,.$$

From our assumption that H ε-interchanges limits with K we obtain that $\varepsilon \geq \gamma - 2\delta$. Because δ was arbitrary, this gives us the required inequality $\varepsilon \geq \gamma$. □

From Corollary 2.6 and Theorem 3.3 we immediately deduce the following:

Corollary 3.4. *For a normal countably compact space K and a uniformly bounded subset $H \subset C(K)$ we have*

$$\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H)), C(K)) \leq 2\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(H), C(K))\,.$$

If H is not necessarily contained in $C(K)$ then we obtain the following estimate.

Theorem 3.5. *For a normal countably compact space K and a uniformly bounded subset H of $\mathbb{R}^K$ we have*

$$\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H)), C(K)) \leq 5\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(H), C(K))\,.$$

Proof. Since $\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(\mathrm{cl}_{\mathbb{R}^K} H)) = \mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H))$ we may assume without the loss of generality that $\mathrm{cl}_{\mathbb{R}^K}(H) = H$. So assume H is a uniformly bounded and compact subset of $\mathbb{R}^K$ and let us prove the corollary by establishing the inequality

$$\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H)), C(K)) \leq 5\hat{d}(H, C(K)). \tag{3.8}$$

Fix $\varepsilon > \hat{d}(H, C(K))$. For each $f \in H$ we have $d(f, C(K)) < \varepsilon$ and thus we can pick a function $g(f) \in C(K)$ such that

$$\|f - g(f)\|_\infty < \varepsilon. \tag{3.9}$$

We claim that the uniformly bounded set $H_0 := \{g(f) : f \in H\}$ 4ε-interchanges limits with K. Indeed, we have $H_0 \subset H + B[0, \varepsilon]$, where the (closed!) balls are taken in $\ell_\infty(K)$. Since the last set is compact in $\mathbb{R}^K$, we get that $\mathrm{cl}_{\mathbb{R}^K}(H_0) \subset H + B[0, \varepsilon]$. On the other hand, we have $H \subset H_0 + B[0, \varepsilon]$, and therefore we obtain $\mathrm{cl}_{\mathbb{R}^K}(H_0) \subset H_0 + B[0, 2\varepsilon]$. The last inclusion is read as

$$\hat{d}(\mathrm{cl}_{\mathbb{R}^K}(H_0), C(K)) \leq 2\varepsilon$$

and by Corollary 2.6 we obtain that H_0 4ε-interchanges limits with K. We have now that $\mathrm{conv}(H_0)$ 4ε-interchanges limits with K by Theorem 3.3. We can use Corollary 2.6 to conclude that

$$\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H_0)) \subset C(K) + B[0, 4\varepsilon].$$

Finally the inequality (3.9) allows us to deduce that

$$\mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H)) \subset \mathrm{cl}_{\mathbb{R}^K}(\mathrm{conv}(H_0)) + B[0, \varepsilon] \subset C(K) + B[0, 5\varepsilon],$$

that clearly implies (3.8) and the proof is over. □

If we use Corollary 2.8 instead of Corollary 2.6 along with Theorem 3.3 we obtain the result below.

Theorem 3.6. *Let K be a compact space, Z be a compact convex subset of a normed space E, and H be a subspace of $C(K, Z)$. Then*

$$\hat{d}(\mathrm{cl}_{Z^K}(\mathrm{conv}(H)), C(K, Z)) \leq 4\hat{d}(\mathrm{cl}_{Z^K}(H), C(K, Z))\,.$$

4. DISTANCES TO SPACES OF AFFINE CONTINUOUS FUNCTIONS

Given a compact convex subset K of a locally convex space, we denote by $\mathcal{A}(K)$ the space of affine real-valued functions defined on K, and by

$$\mathcal{A}^C(K) = C(K) \cap \mathcal{A}(K)$$

the space of continuous affine functions on K.

The distance of an affine bounded function to the space of continuous functions is the same as the distance to the space of affine continuous functions.

Proposition 4.1. *Let K be a compact convex subset of a locally convex space. Then for any bounded function f in $\mathcal{A}(K)$ we have*

$$d(f, C(K)) = d(f, \mathcal{A}^C(K)) .$$

Proof. Our proof uses slightly modified arguments from the proof of [1, Proposition 1.18] (Result 1.1). Since we have $d(f, C(K)) = \frac{1}{2}\operatorname{osc}(f)$ it is enough to prove that $d(f, \mathcal{A}^C(K)) \leq \frac{1}{2}\operatorname{osc}(f)$. Fix $\delta > \frac{1}{2}\operatorname{osc}(f)$ and define

$$f_1(x) = \inf_U \sup\{f(z) : z \in U\} - \delta$$

$$f_2(x) = \sup_U \inf\{f(z) : z \in U\} + \delta$$

where the infimum and supremum are taken over the neighborhoods U of x. We claim that f_1 is concave upper semicontinuous and f_2 is convex lower semicontinuous. Indeed, we shall show the concavity of f_1 (the other proof is similar). Take $\eta > 0$, points $x, y \in K$ and $\lambda \in (0, 1)$. Take U a neighborhood of $\lambda x + (1-\lambda)y$ such that

$$\sup\{f(z) : z \in U\} - \delta \leq f_1(\lambda x + (1-\lambda)y) + \eta.$$

Take V and W neighborhoods of x and y, respectively, such that

$$\lambda V + (1-\lambda)W \subset U.$$

Then we have

$$\begin{aligned}
&\lambda f_1(x) + (1-\lambda) f_1(y) \\
&\leq \lambda \sup\{f(z) : z \in V\} + (1-\lambda)\sup\{f(z) : z \in W\} - \delta \\
&\leq \sup\{f(z) : z \in U\} - \delta \\
&\leq f_1(\lambda x + (1-\lambda)y) + \eta.
\end{aligned}$$

Since $\eta > 0$ is arbitrary, we get that f_1 is concave. The definition of oscillation gives us that $f_1 < f_2$ by and Theorem 21.20 in [2] can be applied to deduce the existence of a continuous affine function h defined on K such that

$$f_1(x) < h(x) < f_2(x)$$

for every $x \in K$. We conclude now that $h(x) - \delta < f(x) < h(x) + \delta$, for every $x \in K$, hence $\|f - h\|_\infty \leq \delta$. It follows that $d(f, \mathcal{A}^C(K)) \leq \delta$ and the proof is over. □

Corollary 4.2. *Let E be a Banach space and let B_{E^*} be the closed unit ball in the dual E^* endowed with the w^*-topology. Let $i : E \to E^{**}$ and $j : E^{**} \to \ell_\infty(B_{E^*})$ be the canonical embedding. Then, for every $x^{**} \in E^{**}$ we have:*

$$d(x^{**}, i(E)) = d(j(x^{**}), C(B_{E^*})) .$$

Proof. Consider x^{**} as an affine function on B_{E^*}. By the former result, for every $\delta > d(f, C(B_{E^*}))$ there is a w*-continuous affine function h_1 defined on B_{E^*} such that $\|x^{**} - h_1\| \leq \delta$. Define $h_2(x^*) = -h_1(-x^*)$, for every $x^* \in B_{E^*}$. Since B_{E^*} is symmetric we deduce that $\|x^{**} - h_2\| \leq \delta$. Now the function $g : B_{E^*} \to \mathbb{R}$ defined by $g = \frac{1}{2}(h_1 + h_2)$ is affine, w*-continuous and satisfies $g(0) = 0$. Hence, g is the restriction to B_{E^*} of a linear form y^{**} defined on the whole E^*. Since $y^{**}|_{B_{E^*}} = g$ is w*-continuous we can use Grothendieck's completeness theorem, [9, §21.9.(4)] to obtain that $y^{**} = i(x)$ for some $x \in E$. Clearly we have $\|x^{**} - x\| \leq \delta$ and the proof is over. ☐

All the previous results for $C(K) \subset \mathbb{R}^K$ can be applied for Banach spaces: take a Banach space E and a bounded set $H \subset E^{**}$ (we identify E with a subspace of E^{**} and E^{**} with a subset of $\mathbb{R}^{B_{E^*}}$). The pointwise closure of H in $\mathbb{R}^{B_{E^*}}$ is simply its w*-closure w*-cl(H) in E^{**}. It is clear that after the above identifications we have

$$\hat{d}(H, E) = \sup_{y \in H} \inf_{x \in E} \|y - x\|,$$

where $\|\cdot\|$ is the canonical norm in the bidual space E^{**}.

As a straightforward consequence of Proposition 2.2, Corollary 4.2 and Corollary 3.4 we obtain some of the main results proved in the interesting recent paper by M. Fabian, P. Hajék, V. Montesinos and V. Zizler, [4].

Corollary 4.3 ([4, Proposition 8]). *Let E be a Banach space and let H be a bounded subset of E. Then following properties hold:*

(i) *if H ε interchanges limits with B_{E^*}, then $\hat{d}(\mathrm{w}^*\text{-}\,\mathrm{cl}(H), E) \leq \varepsilon$;*
(ii) *if $\hat{d}(\mathrm{w}^*\text{-}\,\mathrm{cl}(H), E) \leq \varepsilon$, then H 2ε interchanges limits with B_{E^*}.*

Corollary 4.4 ([4, Theorem 2]). *Let E be a Banach space and let H be a bounded subset of E. If $\hat{d}(\mathrm{w}^*\text{-}\,\mathrm{cl}(H), E) \leq \varepsilon$, then $\hat{d}(\mathrm{w}^*\text{-}\,\mathrm{cl}(\mathrm{conv}(H)), E) \leq 2\varepsilon$.*

Theorem 3.5 and Corollary 4.2 combined together give the following inequality for Banach spaces. This inequality has been recently proved by A. S. Granero in [6] using completely different techniques.

Corollary 4.5 ([6, Theorem 5]). *Let E be a Banach space and let H be a w*-compact subset of E^{**}. Then $\hat{d}(\mathrm{w}^*\text{-}\,\mathrm{cl}(\mathrm{conv}(H)), E) \leq 5\hat{d}(H, E)$.*

5. Approximation by sequences

In this section we show that ε-interchanging limits property implies ε-approximation by sequences as presented in Theorem 5.2.

Lemma 5.1. *Let (Z, d) be a compact metric space and K be a set. For given functions $f_1, \dots, f_n \in Z^K$ and $\delta > 0$, there is a finite subset $L \subset K$ such that for every $x \in K$ there is $y \in L$ verifying*

$$d(f_k(y), f_k(x)) < \delta$$

for every $1 \leq k \leq n$.

Proof. The metric

$$d_\infty\big((x_k), (y_k)\big) := \sup_{1 \leq k \leq n} d(x_k, y_k),$$

$(x_k), (y_k) \in Z^n$, defines the product topology of the compact space Z^n. Let $B := \{(f_1(x), \dots, f_n(x)) : x \in K\}$. The space (B, d_∞), being a subspace of the compact space (Z^n, d_∞) is totally bounded (see [3, Theorems 4.3.2, 4.3.27]). Hence there is a finite set $L \subset K$ such that the set $\{(f_1(x), \dots, f_n(x)) : x \in L\}$ is δ-dense in (B, d_∞). □

Proposition 5.2. *Let (Z, d) be a compact metric space, K a set, and $H \subset Z^K$ a set which ε-interchanges limits with K. Then for any $f \in \operatorname{cl}_{Z^K} H$, there is a sequence $(f_n) \subset H$ such that*

$$\sup_{x \in K} d(g(x), f(x)) \le \varepsilon \tag{5.1}$$

for any cluster point g of (f_n) in Z^K.

Proof. Define $f_1 := f$. If we apply Lemma 5.1 to f_1 and ε we can find a finite set $L_1 \subset K$ such that

$$\min_{y \in L_1} d\big(f_1(x), f_1(y)\big) < 1 \text{ for each } x \in K.$$

Since $f \in \operatorname{cl}_{Z^K}(H)$, there is $f_2 \in H$ such that

$$d\big(f_2(y), f_1(y)\big) < \frac{1}{2} \text{ for each } y \in L_1.$$

An inductive argument provide us with functions $f_1, f_2, \dots, f_n, \dots$, $f_n \in H$ for $n \ge 2$, and finite subsets $L_1, L_2, \dots, L_n, \dots$ of K such that

$$\min_{y \in L_n} \max_{k \le n} \{d\big(f_k(x), f_k(y)\big)\} < \frac{1}{n} \text{ for every } x \in K$$

and

$$d\big(f_{n+1}(y), f_1(y)\big) < \frac{1}{n+1} \text{ for every } y \in \bigcup_{k=1}^{n} L_k.$$

Let us define $D := \bigcup_{k=1}^{\infty} L_n$. The following statements hold:

(a) $\lim_{k \to \infty} f_k(y) = f_1(y)$, for every $y \in D$;
(b) for each $x \in X$ and every $n \in \mathbb{N}$ there is $y_n \in D$ such that

$$\max_{k \le n} \{d\big(f_k(x), f_k(y_n)\big)\} < \frac{1}{n}.$$

For a fixed $x \in K$, observe that the sequence (y_n) constructed in (b) satisfies

$$\lim_{n \to \infty} f_k(y_n) = f_k(x) \text{ for every } k = 1, 2, \dots$$

Fix now a cluster point g of (f_k) in Z^K. Choose a subsequence (f_{k_j}) such that at the fixed point x we have $\lim_j f_{k_j}(x) = g(x)$. On the one hand we can compute

$$\lim_j \lim_n f_{k_j}(y_n) = \lim_j f_{k_j}(x) = g(x)$$

and on the other hand we have

$$\lim_n \lim_j f_{k_j}(y_n) = \lim_n f_1(y_n) = f_1(x) = f(x).$$

Consequently $d(g(x), f(x)) \le \varepsilon$ and the proof is over. □

Corollary 5.3. *Let (Z, d) be a compact metric space, K a set, and H a subset of Z^K that interchanges limits with K ($\varepsilon = 0$). Then for any $f \in \mathrm{cl}_{Z^K} H$, there is a sequence $(f_n) \subset H$ such that*

$$\lim_n f_n(x) = f(x)$$

for every $x \in K$.

Proof. We use Proposition 5.2 for $\varepsilon = 0$ and produce a sequence (f_n) such that for every cluster point g of (f_n) in Z^K we have $g = f$. Since Z^K is compact we conclude that (f_n) converges to its unique cluster point f in Z^K. □

The above corollary appears as Theorem 8.20 in [8] in the case of continuous functions defined in a topological compact space. It also appears in [5, p. 31] in a more general situation attributed to M. de Wilde.

We finally remark that Corollaries 2.5 and 5.3 imply that for any topological compact space K the space $(C(K), \tau_p)$ is angelic, as the interested reader can verify in [5, p. 36].

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