Dentability indices with respect to measures of non compactness

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Dedicated to the memory of my father

Abstract

We study the relationships between the ordinal indices of set derivations associated to several measures of non compactness. We obtain applications to the Szlenk index, improving a result of Lancien, and LUR renorming, providing a non probabilistic proof of a result of Troyanski.

1 Introduction

There are several quantities in analysis under the name of measures of non compactness which quantify how far a set is from a given class of compacta. For instance, we may consider the diameter $\operatorname{diam}(A)$ as the simplest one. It measures how far is a set of being a singleton. The classical Kuratowski measure of non compactness $\alpha(A)$ is the infimum of the numbers $r > 0$ such that A can be covered by finitely many sets of diameter less than r, and it measures how far is A of being relatively metric compact. For a set $A \subset X$ in a Banach space the number

$$w(A) = \inf\{r > 0 : \overline{A}^{w^*} \subset X + rB_{X^{**}}\}$$

measures how far is A of being relatively weakly compact (the weak*-closure is taken in the bidual X^{**}). There are more sophisticated measures of non compactness: for a convex bounded set A, we define $\beta_1(A)$ as the infimum of the numbers $\varepsilon > 0$ such that there is $N_1(A, \varepsilon) \in \mathbb{N}$ verifying that any martingale $(M_n)_{0 \le n \le N} \subset L_1([0,1], X)$ with values in A and $\|M_n - M_{n-1}\|_1 \ge \varepsilon$ must have length N less than $N_1(A, \varepsilon)$. The measure $\beta_{æ}(A)$ and the numbers $N_{æ}(A, \varepsilon)$ are defined similarly asking the martingales to satisfy $\|M_n - M_{n-1}\| \ge \varepsilon$ almost everywhere. It is not difficult to see that $\beta_{æ}(A) \le \beta_1(A)$. These measures are related to the notion of superreflexive Banach space, see [11]. Troyanski [13] called β_1 the index of non-superreflexivity.

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For a measure of non compactness η we define the "slice derivation" as the following set operation

$$[A]'_\varepsilon = \{x \in A : \forall\, H\; w^*\text{-open halfspace containing } x,\ \eta(A \cap H) \geq \varepsilon\}$$

for $\varepsilon > 0$, that can be understand as removing the slices of A which are ε-small with respect to η. For any ordinal γ, the sets $[A]^\gamma_\varepsilon$ are defined in the natural way, taking intersection in the case of limit ordinals. Define for any subset $E \subset A$ the ordinal index

$$\mathfrak{D}_\eta(E, A)_\varepsilon = \inf\{\gamma : [A]^\gamma_\varepsilon \cap E = \emptyset\}$$

and take $\mathfrak{D}_\eta(A)_\varepsilon = \mathfrak{D}_\eta(A, A)_\varepsilon$. The existence of those indices is no always guarantied, for instance $\mathfrak{D}_{\text{diam}}(B_X)_\varepsilon$ exists if, and only if, X has the Radon-Nikodym property (bounded subsets are dentable, see [2, Theorem 2.3.6]).

The norm $\|.\|$ of a Banach space X is said to be locally uniformly rotund (LUR) if for every $x, x_k \in X$, such that $\lim_k \|x_k\| = \|x\|$ and $\lim_k \|x + x_k\| = 2\|x\|$, then $\lim_k \|x - x_k\| = 0$. Troyanski proved this result about LUR renormings.

Theorem 1.1 (Troyanski [13]) *Let X be a Banach space such that for every $x \in S_X$ and every $\varepsilon > 0$, there is a halfspace H containing x and such that $\beta_1(B_X \cap H) < \varepsilon$. Then X has an equivalent LUR norm.*

As a Corollary, Troyanski showed that β_1 can be replaced by the Kuratowski measure α since $\beta_1(A) \leq 2\alpha(A)$ for every bounded convex set A [13, Corollary 2.4]. The original proof of Theorem 1.1 employs probabilistic methods and is rather involved. We shall prove by means of geometrical arguments the following result that implies Theorem 1.1.

Theorem 1.2 *Let X be a Banach space such that for every $x \in S_X$ and every $\varepsilon > 0$, there is a halfspace H containing x and such that $\beta_{æ}(B_X \cap H) < \varepsilon$. Then $\mathfrak{D}_{diam}(S_X, B_X)_\varepsilon \leq \omega^\omega$ for any $\varepsilon > 0$.*

The idea of obtaining a LUR equivalent norm by slice derivation of the unit ball was first considered by Lancien [6]. The fact that it is enough to "eat" the points of the unit sphere to get a LUR norm allows the use of this technique in more situations, see [12, 10, 3]. In all these cases, including as well Theorem 1.1, the LUR norm can be obtained as a convergent series of weighted square powers of the Minkowski functionals of a suitable countable family of symmetric convex sets obtained by slicing B_X, see [12] and the last section. In [3] the authors proved by slice derivation Theorem 1.1 for the Kuratowski measure α instead of β_1, also giving (implicitly) ω^ω as an estimate.

Let X be Banach space and consider its dual space X^*. For any $A \subset X^*$ bounded define the "fragment derivation"

$$\langle A\rangle'_\varepsilon = \{x^* \in A : \forall\, U\; w^*\text{-neighbourhood of } x^*, \eta(A \cap U) \geq \varepsilon\}$$

Define inductively for ordinals $\langle A\rangle_\varepsilon^\gamma$ in the obvious way. Take

$$Sz(X)_\varepsilon = \inf\{\gamma : \langle B_{X^*}\rangle_\varepsilon^\gamma = \emptyset\}$$

and $Sz(X) = \sup_{\varepsilon>0} Sz(X)_\varepsilon$ and $Dz(X) = \sup_{\varepsilon>0} \mathfrak{D}_{\text{diam}}(B_{X^*})_\varepsilon$ where the index $\mathfrak{D}_{\text{diam}}$ is defined as above but using weak*-open halfspaces. These indices are known as the Szlenk index and the weak* dentability index respectively, see the survey paper [8]. Both indices are defined if, and only if, X is an Asplund Banach space, see [2] for definition and characterizations. In that case, the sequences of derived sets are strictly decreasing (bounded subsets of the dual of an Asplund space are weak*-dentable, see [2, Theorem 4.2.13]). Obviously we have $Sz(X) \leq Dz(X)$.

Lancien proved in [7] that $Sz(X) < \omega_1$ implies $Dz(X) < \omega_1$. His proof uses a reduction to the separable case and deep results of descriptive set theory to show the existence of an universal function $\psi : [0, \omega_1) \to [0, \omega_1)$ such that

$$Dz(X) \leq \psi(Sz(X))$$

for every Asplund Banach space X with $Sz(X) < \omega_1$. We shall give a constructive proof of Lancien's result by means of geometrical arguments showing that the universal function is of exponential form without restriction on the cardinality.

Theorem 1.3 *Let X be an Asplund Banach space, then $Dz(X) \leq \omega^{Sz(X)}$.*

This bound can be sharpened in certain classes of spaces, see [8, 4].

Although the results showed in this introduction involves the essentially the measure diam, the techniques are valid for many others measures of non compactness, as the mentioned in the introduction, or measures derived from diam by iteration, see Definition 2.7. For that reason, the second section of this paper is developed with the full generality for an abstract measure of non compactness. The last section contains applications to several ordinal indices in Banach space and LUR renormings.

2 Slicing and eating

To get a suitable level of generality, along this section X will denote a locally convex space with topology $\mathcal{V}$. For $x \in X$ a given point $\mathcal{V}_x$ denotes the neighbourhoods of x and $\mathcal{H}_x$ the family of open halfspaces containing x.

Definition 2.1 *A measure of non compactness is a non negative function η defined on some class of the bounded subsets of X satisfying the following properties whenever all the sets considered lies in that class:*

1. *If $A \subset B$, then $\eta(A) \leq \eta(B)$.*
2. *If $A \subset X$, then $\eta(\overline{A}) = \eta(A)$.*

3. *There there exists $\kappa \geq 1$ such that for every bounded symmetric convex set B there is $b > 0$ verifying $\eta(conv(A) + \lambda B) \leq \kappa\eta(A) + \lambda b$ for every $A \subset X$ and every $\lambda \geq 0$.*

The definition lists the useful properties that are common to all the considered examples. We do not give details about the class where η is defined, obviously it should be stable by a certain number of set operations used in the proofs, but in applications we shall consider just two: the bounded sets, and the sets such that its closed convex hull is compact. Property *3* combines two facts that are more easy to understand separately, as the existence of an universal bound for convex hulls, and a "Lipschitz" property with respect to perturbations by sums of balls. We put both properties together in one formula to simplify some arguments below. For the diameter and the Kuratowski measure $\kappa = 1$. For the measure of non weak compactness w mentioned in the introduction $\kappa = 2$, see [5]. Another example of measure of non compactness in the sense of Definition 2.1 is $\mathrm{osc}_f(A) = \mathrm{diam}(f(A))$ defined for the bounded subsets of a normed space if f is a Lipschitz map with values into a metric space.

We recall the derivations defined in the introduction in this more general setting. For any $A \subset X$ bounded consider the following sets

$$[A]'_\varepsilon = \{x \in A : \forall\, H \in \mathcal{H}_x,\ \eta(A \cap H) \geq \varepsilon\}$$

$$\langle A\rangle'_\varepsilon = \{x \in A : \forall\, U \in \mathcal{V}_x,\ \eta(A \cap U) \geq \varepsilon\}$$

For any ordinal γ, the sets $\langle A\rangle^\gamma_\varepsilon$ and $[A]^\gamma_\varepsilon$ are defined in the obvious way. Define for any subset $E \subset A$ the ordinal indices

$$\mathfrak{D}_\eta(E, A)_\varepsilon = \inf\{\gamma : [A]^\gamma_\varepsilon \cap E = \emptyset\}$$

$$\mathfrak{F}_\eta(E, A)_\varepsilon = \inf\{\gamma : \langle A\rangle^\gamma_\varepsilon \cap E = \emptyset\}$$

if such ordinals exists, if not the index is ∞. Finally take $\mathfrak{D}_\eta(A)_\varepsilon = \mathfrak{D}_\eta(A, A)_\varepsilon$, and $\mathfrak{F}_\eta(A)_\varepsilon = \mathfrak{F}_\eta(A, A)_\varepsilon$. Along this section the brackets [ ] and $\langle\ \rangle$ are reserved for the derivations with respect to η, but the indices may be referred to other measures. The constant κ is fixed for η from now on.

The following result is based on the so called Bourgain-Namioka Lemma, see [2, Theorem 3.4.1]. The iteration argument was provided kindly by J. Orihuela for our paper [12] and it was also used in [3] dealing with the Kuratowski index of non compactness.

Lemma 2.2 *Let $A \subset X$ be a bounded set, let H be an open half space and take $\varepsilon > \kappa\,\eta(H \cap A)$. Then the sequence (A_n) defined recursively by $A_1 = \overline{conv}(A)$ and $A_{n+1} = \overline{conv}((A \setminus H) \cup (A \cap [A_n]'_\varepsilon))$ verifies $(H \cap A) \subset \bigcup_{n=1}^\infty (A_n \setminus [A_n]'_\varepsilon)$ and*

$$H \cap \bigcap_{n=1}^{\infty} A_n = \emptyset$$

Proof. We may assume that $H = \{x \in X : f(x) > a\}$ with $f \in X^*$. Take the symmetric convex set $B = \text{conv}(A) - \text{conv}(A)$. Fix $r \in (0,1)$ such that $\eta(\text{conv}(H \cap A) + rB) < \varepsilon$. For any subset $E \subset A$ we claim that

$$\sup\{f(x) : x \in [\overline{\text{conv}}(E)]'_\varepsilon\} \leq ra + (1-r)\sup\{f(x) : x \in E\}$$

if $[\overline{\text{conv}}(E)]'_\varepsilon \neq \emptyset$. Indeed, discard the "extreme" cases: if $E \cap H = \emptyset$ then the inequality is obvious, and $E \setminus H = \emptyset$ implies $[\overline{\text{conv}}(E)]'_\varepsilon = \emptyset$. Otherwise, take $E_1 = \text{conv}(E \cap H)$, $E_2 = \text{conv}(E \setminus H)$. Define

$$D = \{(1-\lambda)x_1 + \lambda x_2 : x_1 \in E_1, x_2 \in E_2, \lambda \in [r,1]\}$$

If $x \in \text{conv}(E) \setminus D$ then $x = (1-\lambda)x_1 + \lambda x_2$ with $x_1 \in E_1$, $x_2 \in E_2$ and $\lambda \in [0,r]$. Since $x - x_1 = \lambda(x_2 - x_1)$, we have

$$\text{conv}(E) \setminus D \subset E_1 + \lambda B \subset \text{conv}(H \cap A) + rB$$

and thus $\eta(\text{conv}(E) \setminus D) < \varepsilon$. If G is an open halfspace such that $\overline{G} \cap \overline{D} = \emptyset$ then

$$\overline{\text{conv}}(E) \cap G \subset \overline{\text{conv}(E) \cap G} \subset \overline{\text{conv}(E) \setminus D}$$

We deduce $\eta(\overline{\text{conv}}(E) \cap G) < \varepsilon$, therefore $[\overline{\text{conv}}(E)]'_\varepsilon \subset \overline{D}$ and

$$\sup\{f(x) : x \in [\overline{\text{conv}}(E)]'_\varepsilon\} \leq \sup\{f(x) : x \in D\}$$

and an easy estimation of $\sup\{f(x) : x \in D\}$ finishes the proof of the claim.
Now, we shall consider the sequence (A_n). If $A \setminus H = \emptyset$, then $[A_1]'_\varepsilon = \emptyset$. Thus $A_n = \emptyset$ for $n \geq 2$ and we are done. If $A \setminus H \neq \emptyset$, then for every $n \in \mathbb{N}$ is defined

$$s_n = \sup\{f(x) : x \in A_n\}$$

We have $s_n \geq a$ and the sequence (s_n) is decreasing. If $s_n = a$ for some n, we are done because $A_n \cap H = \emptyset$. If $s_n > a$ for every $n \in \mathbb{N}$, then the hypothesis to apply the former claim holds. Taking $E = (A \setminus H) \cup (A \cap [A_{n-1}]'_\varepsilon)$ we get

$$a \leq s_{n+1} \leq ra + (1-r)s_n$$

and thus $0 \leq s_{n+1} - a \leq (1-r)(s_n - a)$, implying the convergence of (s_n) to a. Therefore

$$\sup\{f(x) : x \in \bigcap_{n=1}^{\infty} A_n\} \leq a$$

implying that $H \cap \bigcap_{n=1}^\infty A_n = \emptyset$. Now, if $x \in H \cap A$ there is $n \in \mathbb{N}$ such that $x \in A_n$ and $x \notin A_{n+1}$. Thus $x \notin [A_n]'_\varepsilon$, and therefore $x \in A_n \setminus [A_n]'_\varepsilon$. ■

Proposition 2.3 *Let $A \subset X$ be a bounded set and $\delta > 0$. Take $\varepsilon > \kappa\delta$. Then the sequence (A_n) defined recursively by $A_1 = \overline{conv}(A)$ and $A_{n+1} = \overline{conv}(A \cap [A_n]'_\varepsilon)$ verifies $(A \setminus [A]'_\delta) \subset \bigcup_{n=1}^\infty (A_n \setminus [A_n]'_\varepsilon)$.*

Proof. If $x \in A \setminus [A]'_\delta$, there is a half space $H \in \mathcal{H}_x$ such that $\eta(H \cap A) < \delta$. Let (E_n) the sequence given by Lemma 2.2. It is easy to see that $A_n \subset E_n$. It follows $H \cap \bigcap_{n=1}^\infty A_n = \emptyset$. Consequently, there is $n \in \mathbb{N}$ such that $x \in A_n$ and $x \notin A_{n+1}$. Thus $x \notin [A_n]'_\varepsilon$, and therefore $x \in A_n \setminus [A_n]'_\varepsilon$. ■

The following notion is one of the two technical hypothesis that we shall need to prove the main results of this section about relationships between different ordinal indices.

Definition 2.4 *We say that a measure of non compactness η is regular if there exists $\varsigma \geq 1$ such that $\eta(A) < \varepsilon\varsigma$ for every closed convex set A such that $[A]'_\varepsilon = \emptyset$.*

It is easy to see that diam and osc_f are regular with $\varsigma = 2$. The Kuratowski measure α is not regular in general (for instance, ℓ_1 is a counterexample), but in a dual Banach space endowed with the weak* topology it is regular with $\varsigma = 1$. There are other examples of measures which become regular when their use is restricted to compact convex sets. The constant ς is fixed when is η assumed to be regular.

The proof of the next result uses a recursive argument similar to [3, Lemma 3.3].

Lemma 2.5 *Assume η is regular, $\delta > 0$ and $\varepsilon > \kappa\varsigma\delta$. Let $C \subset X$ be bounded closed convex, H open halfspace such that $[\overline{H \cap C}]^n_\delta = \emptyset$. Then $H \cap [C]^{\omega^{n-1}}_\varepsilon = \emptyset$.*

Proof. The proof will be by induction on n for every bounded closed convex and every open halfspace satisfying the hypothesis with given δ and ε. If $n = 1$, then it is obvious. Suppose it is true for n (main induction hypothesis) and we want to prove it for $n+1$. So we may assume $[\overline{H \cap C}]^{n+1}_\delta = \emptyset$ and $[\overline{H \cap C}]^n_\delta \neq \emptyset$. By the regularity we have $\eta([\overline{H \cap C}]^n_\delta) < \delta\varsigma$. Let (A_k) the sequence given by Lemma 2.2 for $A = [\overline{H \cap C}]^n_\delta \cup (C \setminus H)$. Notice that $[A_k]'_\varepsilon \cap [\overline{H \cap C}]^n_\delta \subset A_{k+1}$ by the definition of the sequence (A_k). We claim that $[C]^{\omega^{n-1}\cdot k}_\varepsilon \subset A_k$ for every $k \in \mathbb{N}$. Proof by induction on k. If $k = 1$, then any open halfspace G with $\overline{G} \cap A_1 = \emptyset$ verifies $\overline{G} \cap [\overline{H \cap C}]^n_\delta = \emptyset$. As $G \cap C \subset H \cap C$, we have $[\overline{G \cap C}]^n_\delta = \emptyset$. So by the main induction hypothesis $G \cap [C]^{\omega^{n-1}}_\varepsilon = \emptyset$. As G was arbitrary, we deduce that $[C]^{\omega^{n-1}}_\varepsilon \subset A_1$. Assume that the claim is proven for k, then $[C]^{\omega^{n-1}\cdot k+1}_\varepsilon \subset [A_k]'_\varepsilon$. If G is an open halfspace with $\overline{G} \cap A_{k+1} = \emptyset$, then $\overline{G} \cap [A_k]'_\varepsilon \cap [\overline{H \cap C}]^n_\delta = \emptyset$. As $G \cap [A_k]'_\varepsilon \subset H \cap C$, we have $[\overline{G \cap [A_k]'_\varepsilon}]^n_\delta = \emptyset$. By the main induction hypothesis $G \cap [[A_k]'_\varepsilon]^{\omega^{n-1}}_\varepsilon = \emptyset$. As G was arbitrary, $[[A_k]'_\varepsilon]^{\omega^{n-1}}_\varepsilon \subset A_{k+1}$. Then we have

$$[C]^{\omega^{n-1}\cdot(k+1)}_\varepsilon = [[C]^{\omega^{n-1}\cdot k+1}_\varepsilon]^{\omega^{n-1}}_\varepsilon \subset A_{k+1}$$

which finishes the proof of the claim. Using other property of the sequence (A_k), we have

$$H \cap \bigcap_{k=1}^{\infty} [C]^{\omega^{n-1}\cdot k}_\varepsilon \subset H \cap \bigcap_{k=1}^{\infty} A_k = \emptyset$$

obtaining that $H \cap [C]^{\omega^n}_\varepsilon = \emptyset$ as we wanted. ■

Lemma 2.6 *Assume that η is regular. Let A be bounded closed convex, let B be bounded convex symmetric and b its associated constant. Then*

$$\eta(\overline{(A+\lambda B)}\cap H) < \varepsilon\kappa\varsigma + \lambda b$$

for every $\lambda > 0$ and every open halfspace $H = \{x \in X : f(x) > a\}$ such that $a > \sup\{f(x) : x \in [A]'_\varepsilon + \lambda B\}$.

Proof. First notice that $\eta(A \cap G) < \varepsilon\varsigma$ for any open halfspace G such that $\overline{G} \cap [A]'_\varepsilon = \emptyset$ by the regularity of η. We may assume $\lambda = 1$ without loss of generality. Take $s = \sup\{f(z) : z \in B\}$ and consider $G = \{y \in A : f(y) > a - s\}$. Clearly, we have $(A + B) \cap H \subset A \cap G + B$. Take a_1 such that

$$a > a_1 > \sup\{f(x) : x \in [A]'_\varepsilon + B\}$$

and define $G_1 = \{y \in A : f(y) > a_1 - s\}$. We claim that $[A]'_\varepsilon \cap G_1 = \emptyset$. Indeed, if not take $y \in [A]'_\varepsilon \cap G_1$ and $v \in B$ such that $f(v) > s - (f(y) - (a_1 - s))$, so $f(y+v) > a_1$ and $y+v \in [A]'_\varepsilon + B$ which is a contradiction. Clearly $\overline{G} \cap A \subset G_1 \cap A$ and so $\eta(\overline{G} \cap A) < \varsigma\varepsilon$ since $\overline{G} \cap [A]'_\varepsilon = \emptyset$. Therefore

$$\eta(\overline{(A+B)}\cap H) \leq \eta(\overline{(A+B)\cap H}) \leq \eta(A \cap G + B) \leq \varepsilon\kappa\varsigma + b$$

finishing the proof. ■

Definition 2.7 *For a measure of non compactness and an ordinal γ we define*

$$\eta^\gamma(A) = \inf\{\varepsilon > 0 : \mathfrak{D}_\eta(\overline{conv}(A))_\varepsilon < \gamma\}$$

Proposition 2.8 *If η is a regular measure of non compactness, then $\eta^\omega(A)$ is a measure of non compactness. Moreover, $\eta^{\omega_1}(A)$ is a measure of non compactness provided that its use is restricted to sets whose closed convex hulls are compact.*

Proof. Clearly, it is enough to verify property *3* of Definition 2.1. Since $\eta^\omega(A) = \eta^\omega(\mathrm{conv}(A))$ we may assume that A is bounded convex. Given B symmetric bounded convex and using Lemma 2.6, for any ordinal γ we have

$$\overline{[[A]^\gamma_\varepsilon + \lambda B]'_{\varepsilon\kappa\varsigma+\lambda b}} \subset \overline{[A]^{\gamma+1}_\varepsilon + \lambda B}$$

implying by finite induction $\eta^\omega(A + \lambda B) \leq \kappa\varsigma\,\eta^\omega(A) + \lambda b$ which proves the first part of the proposition. The second part follows by transfinite induction using that

$$\bigcap_{\xi<\gamma}([A]^\xi_\varepsilon + \lambda B) = [A]^\gamma_\varepsilon + \lambda B$$

if γ is a limit ordinal, and besides A is compact and B is closed. ■

We arrive to one of the main results of the section.

Theorem 2.9 *Assume that η is regular, $\delta > 0$ and $\varepsilon > \kappa\varsigma\delta$. Then*

$$\mathfrak{D}_\eta(E, C)_\varepsilon \leq \omega^\omega \cdot \mathfrak{D}_{\eta^\omega}(E, C)_\delta$$

for every bounded closed convex set C and every subset $E \subset C$.

Proof. By Lemma 2.5 we have

$$[C]_\varepsilon^{\omega^\omega} \subset \{x \in C : \forall\, H \in \mathcal{H}_x,\ \eta^\omega(C \cap H) \geq \delta\}$$

being the last set the first step of the slice derivation with respect to η^ω. The result follows by transfinite iteration. ■

Until the end of the section we shall take advantage of the compactness to relate the indices $\mathfrak{D}_\eta$ and $\mathfrak{F}_\eta$. A main tool will be the next result inspired in ideas from [10].

Lemma 2.10 *If $C \subset X$ is convex and compact then ext$([C]_\varepsilon^\omega) \subset \langle C\rangle'_\varepsilon$.*

Proof. Suppose it is not the case. If $x \in \text{ext}([C]_\varepsilon^\omega) \setminus \langle C\rangle'_\varepsilon$, there is $U \in \mathcal{V}_x$ such that $\eta(C \cap U) < \varepsilon$. As x is extreme, by Choquet's Lemma, there is $H \in \mathcal{H}_x$ such that $\overline{H} \cap [C]_\varepsilon^\omega \subset U$. Since $[C]_\varepsilon^\omega = \bigcap_{n=1}^\infty [C]_\varepsilon^n$, by compactness there is $n \in \mathbb{N}$ such that $\overline{H} \cap [C]_\varepsilon^n \subset U$. We have $\eta(H \cap [C]_\varepsilon^n) < \varepsilon$, thus $x \notin [C]_\varepsilon^{n+1}$ which is a contradiction. ■

The second technical hypothesis of the section is the following.

Definition 2.11 *We say that a measure of non compactness η is normal if the set function η_{N} is a measure of non compactness, where*

$$\eta_{N}(A) = \inf\{\varepsilon > 0 : \exists U_i \in \mathcal{V},\ A \subset \bigcup_{i=1}^n U_i,\ \eta(U_i \cap A) < \varepsilon\}$$

The associated constant to η_{N} will be denoted κ_{N}.

We see that diam_{N} is close to the Kuratowski measure α, implying that diam is normal, see Lemma 3.2 (stated for the weak* topology) for more details. The measures α and w are normal since $\alpha_{N} = \alpha$ and $w_{N} = w$.

Lemma 2.12 *Assume that η is normal, $\delta > 0$ and $\varepsilon > \kappa_{N}\delta$. Let $C \subset X$ be convex compact, H open halfspace such that $\langle\overline{H \cap C}\rangle_\delta^\gamma = \emptyset$. Then $H \cap [C]_\varepsilon^{\omega^\gamma} = \emptyset$.*

Proof. We shall use induction on γ for every compact convex and every open halfspace satisfying the hypothesis with given δ and ε. For $\gamma = 1$ we have $\langle\overline{H \cap C}\rangle'_\delta = \emptyset$, and so $H \cap \langle C\rangle'_\delta = \emptyset$. Applying Lemma 2.10 we have $H \cap [C]_\varepsilon^\omega = \emptyset$. Now suppose $\langle\overline{H \cap C}\rangle_\delta^\gamma = \emptyset$. By compactness γ cannot be a limit ordinal, so assume $\gamma = \xi + 1$ with $\langle\overline{H \cap C}\rangle_\delta^\xi \neq \emptyset$ and that the statement is proven for every $\vartheta \leq \xi$. By compactness $\eta_{N}(\langle\overline{H \cap C}\rangle_\delta^\xi) < \delta$. Since η_{N} is measure we may apply Lemma 2.2 for it with δ, ε and $A = \langle\overline{H \cap C}\rangle_\delta^\xi \cup (C \setminus H)$. The given sequence (A_n) verifies that

$$\langle\overline{H \cap C}\rangle_\delta^\xi \cap [A_n]_\varepsilon^\omega \subset A_{n+1}$$

Indeed, suppose not. If $x \in \langle\overline{H \cap C}\rangle_\delta^\xi \cap [A_n]_\varepsilon^\omega \setminus A_{n+1}$, then there is $G \in \mathcal{H}_x$ such that $G \cap A_{n+1} = \emptyset$ and $\eta_{N}(G \cap A_n) < \varepsilon$. That implies $G \cap \langle A_n\rangle'_\varepsilon = \emptyset$, and by Lemma 2.10, $G \cap [A_n]_\varepsilon^\omega = \emptyset$ which is a contradiction. We claim that

$$[C]_\varepsilon^{\omega^\xi \cdot n} \subset A_n$$

for every $n \in \mathbb{N}$. Proof by induction on n. If $n = 1$, then any open halfspace G with $\overline{G} \cap A_1 = \emptyset$ verifies $\overline{G} \cap \langle \overline{H \cap C} \rangle_\delta^\xi = \emptyset$. As $G \cap C \subset H \cap C$, we have $\langle \overline{G \cap C} \rangle_\delta^\xi = \emptyset$. So by the transfinite induction hypothesis then $G \cap [C]_\varepsilon^{\omega^\xi} = \emptyset$ and so $[C]_\varepsilon^{\omega^\xi} \subset A_1$ since G was arbitrary. Assume it is proven for n, then $[C]_\varepsilon^{\omega^\xi \cdot n + \omega} \subset [A_n]_\varepsilon^\omega$. If G is an open halfspace with $\overline{G} \cap A_{n+1} = \emptyset$, then $\overline{G} \cap [A_n]_\varepsilon^\omega \cap \langle \overline{H \cap C} \rangle_\delta^\xi = \emptyset$. Since $G \cap [A_n]_\varepsilon^\omega \subset H \cap C$, we have $\langle \overline{G \cap [A_n]_\varepsilon^\omega} \rangle_\delta^\xi = \emptyset$. By the transfinite induction hypothesis

$$G \cap [[A_n]_\varepsilon^\omega]_\varepsilon^{\omega^\xi} = \emptyset$$

Since G was arbitrary $[[A_n]_\varepsilon^\omega]_\varepsilon^{\omega^\xi} \subset A_{n+1}$. Therefore we have

$$[C]_\varepsilon^{\omega^\xi \cdot (n+1)} = [[C]_\varepsilon^{\omega^\xi \cdot n + \omega}]_\varepsilon^{\omega^\xi} \subset A_{n+1}$$

This finish the proof of the claim. Using other property of the sequence (A_n), we have

$$H \cap \bigcap_{n=1}^{\infty} [C]_\varepsilon^{\omega^\xi \cdot n} \subset H \cap \bigcap_{n=1}^{\infty} A_n = \emptyset$$

obtaining that $H \cap [C]_\varepsilon^{\omega^\gamma} = \emptyset$ as we wanted. ■

The second main result of the section is the following.

Theorem 2.13 *Assume that η is normal. Define $\psi(\gamma) = \omega^\gamma$. Let C be convex compact and $\delta > 0$. If $\varepsilon > \kappa_N \delta$, then*

$$\mathfrak{D}_\eta(C)_\varepsilon \leq \psi(\mathfrak{F}_\eta(C)_\delta)$$

Proof. Follows easily from the previous Lemma. ■

Theorem 2.14 *Assume that η is regular. Let $C \subset X$ be a convex compact set and take $\varepsilon > \kappa\varsigma\delta$. Then for any subset $E \subset C$, if $\mathfrak{D}_{\eta^{\omega_1}}(E, C)_\delta < \omega_1$, then $\mathfrak{D}_\eta(E, C)_\varepsilon < \omega_1$.*

Proof (Sketch). Just mimic the proof of Lemma 2.5 using transfinite induction like in the proof of Lemma 2.12. In this case, the bound for the countable ordinal is not very nice, so we omitted it in the statement. ■

3 Banach spaces and renorming

This section is devoted to apply the general results of the former section to the measure diam, other related measures of non compactness and several ordinal indices appearing in Banach spaces. These results are applied to LUR renorming.

Lemma 3.1 *For every bounded closed convex subset A of a Banach space*

$$\beta_{æ}(A) \leq diam^\omega(A) \leq 2\beta_{æ}(A)$$

Proof. Let A be convex bounded. Take $\varepsilon > \mathrm{diam}^\omega(A)$. Then no ε-separated (almost everywhere) martingale has length greater than $\mathfrak{D}_{\mathrm{diam}}(A)_\varepsilon$, because it is easy to see inductively that the values of M_{n-k} lies essentially in $[A]_\varepsilon^k$. Therefore $\beta_{æ}(A) < \varepsilon$. On the other hand, if $\beta_{æ}(A) < \varepsilon$, we shall show that $\mathfrak{D}_{\mathrm{diam}}(A)_\delta \leq N_{æ}(A,\varepsilon)+1$ for any $\delta > 2\varepsilon$. This will imply $\mathrm{diam}^\omega(A) \leq 2\varepsilon$. Indeed, assume that $\mathfrak{D}_{\mathrm{diam}}(A)_\delta > N(A,\varepsilon)+1$ and take $N = \mathfrak{D}_{\mathrm{diam}}(A)_\delta - 1$. We shall build functions g_n for $0 \leq n \leq N$ defined on $[0,1]$ and valued in X, which are constant on a finite partition of $[0,1]$ into intervals, g_n take values in $[A]_\delta^{N-n}$, g_{n+1} is measurable with respect to the algebra $\mathcal{A}_n$ generated by g_n,

$$\|g_{n+1} - g_n\| > \delta/2 \quad \text{a.e. ,} \qquad \text{and}$$

$$\|\mathbb{E}(g_{n+1}|\mathcal{A}_n) - g_n\|_\infty < 2^{-n-3}(\delta - 2\varepsilon)$$

Take any $x_0 \in [A]_\delta^N$ and define $g_0(t) = x_0$. Assume g_n with $n < N$ is built, and take x any of the value of g_n, and let $I \in \mathcal{A}_n$ an interval such that $g_n|_I = x$. Since

$$x \in \overline{\mathrm{conv}}([A]_\delta^{N-n-1} \setminus B(x,\delta/2))$$

there are points $(x_i)_{i=1}^k \subset [A]_\delta^{N-n-1} \setminus B(x,\delta/2)$ and $\lambda_i > 0$ with $\sum_{i=1}^k \lambda_i = 1$ and

$$\|\sum_{i=1}^k \lambda_i x_i - x\| < 2^{-n-3}(\delta - 2\varepsilon)$$

Take a partition $I = \bigcup_{i=1}^k I_i$ into intervals such that $|I_i| = \lambda_i |I|$ and define $g_{n+1} = x_i$ on I_i. That finish the construction of (g_n). By [1, Lemma 5.10] there exists a martingale $(M_n)_{0\leq n\leq N}$ with $\|M_n - g_n\|_\infty < 2^{-2}(\delta - 2\varepsilon)$. We get that $\|M_{n+1} - M_n\| > \varepsilon$ a.e. which is a contradiction since $N > N_{æ}(A,\varepsilon)$. ■

Proof of Theorem 1.2. It is a consequence of Theorem 2.9 and the previous Lemma. ■

For the following results concerning dual Banach spaces, in order to apply the results of the former section, the working topology on X^* is the weak*.

Lemma 3.2 *For every bounded set A in a dual Banach space X^**

$$\alpha(A) \leq \mathit{diam}_{\scriptscriptstyle N}(A) \leq 2\alpha(A)$$

As a consequence, $\mathit{diam}_{\scriptscriptstyle N}$ is a measure of non compactness on dual Banach spaces when restricted to bounded sets with associate constant $\kappa_{\scriptscriptstyle N} = 2$.

Proof. Take $\varepsilon > \mathrm{diam}_{\scriptscriptstyle N}(A)$. Then A is covered by finitely many open sets U_i with $\mathrm{diam}(A \cap U_i) < \varepsilon$. We deduce that $\alpha(A) \leq \varepsilon$. Suppose now that $\alpha(A) \leq \varepsilon$. Then $A = \bigcup_{i=1}^n A_i$ with $\mathrm{diam}(A_i) < \varepsilon$. Define for every $I \subset \{1, 2, \ldots, n\}$ the weak* open $U_I = X^* \setminus \bigcup_{i\in I} \overline{A_i}^{w^*}$. We claim that for every $x^* \in A$, there is $I(x^*)$ such that $x^* \in U_{I(x^*)}$ and $\mathrm{diam}(U_{I(x^*)} \cap A) < 2\varepsilon$. Indeed, it is enough to take $I(x^*) = \{i : x^* \notin \overline{A_i}^{w^*}\}$. We have $x^* \in A \cap U \subset \bigcup_{i\notin I(x^*)} \overline{A_i}^{w^*} \subset B(x^*,\varepsilon)$. The weak* open sets of the form U_I, where $I = I(x^*)$ for some $x^* \in A$ provide a finite cover of A verifying that $\operatorname{diam}(U_I \cap A) < 2\varepsilon$, therefore $\operatorname{diam}_N(A) < 2\varepsilon$. Finally, given $A \subset X^*$ weak* compact

$$\operatorname{diam}_N(\overline{\operatorname{conv}}^{w^*}(A) + \lambda B_{X^*})$$

$$\leq 2\alpha(\overline{\operatorname{conv}}^{w^*}(A) + \lambda B_{X^*}) \leq 2\alpha(A) + 4\lambda \leq 2\operatorname{diam}_N(A) + 4\lambda$$

which proves that diam_N is a measure of non compactness. ■

Theorem 3.3 *Define $\psi(\gamma) = \omega^\gamma$. Let C be convex weak* compact If $\varepsilon > 0$, then*

$$\mathfrak{D}_{diam}(A)_{3\varepsilon} \leq \psi(\mathfrak{F}_{diam}(A)_\varepsilon)$$

Proof. It follows from Theorem 2.13 having in mind that $\kappa_N = 2$ in this case. ■

Proof of Theorem 1.3. We have $\mathfrak{D}_{\text{diam}}(B_{X^*})_\varepsilon \leq \psi(Sz(X))$ for every $\varepsilon > 0$ by the previous result. ■

Consider the slice derivation with respect to the Kuratowski measure in the dual of X and the associate index $Cz(X) = \sup_{\varepsilon>0} \mathfrak{D}_\alpha(B_{X^*})_\varepsilon$. It is clear that $Sz(X) \leq Cz(X) \leq Dz(X)$. We shall denote

$$(A)'_\varepsilon = \{x^* \in A : \forall\, H \in \mathcal{H}_{x^*}, \operatorname{diam}_N(A \cap H) \geq \varepsilon\}$$

Lemma 3.4 *Let $C \subset X^*$ be weak* compact convex. Then $[C]^\omega_\varepsilon \subset (C)'_\varepsilon$.*

Proof. Notice that $(C)'_\varepsilon = \overline{\operatorname{conv}}^{w^*}(\langle C\rangle'_\varepsilon)$ and apply Lemma 2.10. ■

The next result shows that, a priori, the indices $Dz(X)$ are $Cz(X)$ more closer between them than to the index $Sz(X)$.

Theorem 3.5 *Let X be an Asplund Banach space. Then $Dz(X) = Cz(X) \cdot \omega$ whenever $Cz(X) < \omega^\omega$. In other case we have $Dz(X) = Cz(X)$.*

Proof. We have by the preceding Lemma we have $\mathfrak{D}_{\text{diam}}(B_{X^*})_\varepsilon \leq \omega \cdot \mathfrak{D}_\alpha(B_{X^*})_\varepsilon$. Therefore $Dz(X) \leq \omega \cdot Cz(X)$ and, obviously, $Cz(X) \leq Dz(X)$. The result follows from the fact that $Cz(X)$ and $Dz(X)$ are of the form ω^ϱ [8, Proposition 2].■

In the remaining part of the section we shall apply the results to renormings of Banach spaces. We shall use the following criterion of [9] for LUR renormability, although for the formulation here we follow [12].

Theorem 3.6 (Moltó, Orihuela, Troyanski) *X has an equivalent LUR norm if, and only if, there is a sequence (A_n) of subsets of X such that for any point $x \in X$ (equivalently $x \in S_X$) and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathcal{H}_x$ such that diam$(A_n \cap H) < \varepsilon$.*

Let us remark that the construction of the norm is particularly simple when the sets (A_n) are convex, symmetric and contains 0 as interior point. In that case, the norm is any series of weighted square powers of the Minkowski functionals converging uniformly on bounded sets, see [12].

We obtain the following improvement of Theorem 3.6.

Corollary 3.7 *X has an equivalent LUR norm if, and only if, there is a sequence (A_n) of subsets of X such that for any point $x \in X$ (equivalently $x \in S_X$) and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathcal{H}_x$ such that $\mathrm{diam}^\omega(A_n \cap H) < \varepsilon$.*

Proof. Without loss of generality we may assume that the sets (A_n) are bounded. We also may assume that the sets (A_n) are also closed and convex, replacing each set by the sequence of sets given by Proposition 2.3 for $\varepsilon = m^{-1}$ for every $m \in \mathbb{N}$ using derivation with respect to diam^ω. Finally, using Theorem 2.9, the countable family $[\overline{\mathrm{conv}}(A_n)]^\gamma_{1/m}$, where the derivation is with respect to diam and $\gamma < \omega^\omega$ verifies the hypothesis of Theorem 3.6. ∎

To get a LUR norm lower semicontinuous with respect to the topology $\sigma(X, F)$ where $F \subset X^*$ is norming it is enough to ask the halfspaces in Theorem 3.6, as well as Corollary 3.7, to be $\sigma(X, F)$-open [12]. With this remark Corollary 3.7 extends to the measure diam^ω Theorem 1.3 of [3] proved for the Kuratowski measure. It is easy to see that $\mathrm{diam}^\omega(A) \leq \alpha(A)$ for every bounded convex set (A).

We shall finish with a result showing the spirit of covering characterization of renormings which is to concentrate on the unit sphere "ε-properties" spread on the space. We shall need the technical requirement of η to be homogeneous, that is, $\eta(\lambda A) = \lambda \eta(A)$ for $\lambda > 0$.

Theorem 3.8 *Let X be a Banach space and let η be an homogeneous regular measure of non compactness. The following properties are equivalent:*

i) There is an equivalent norm such that after endowing X with it, then for every $x \in S_X$ and every $\varepsilon > 0$, there is $\delta > 0$ such that $\eta(B_X \cap H) < \varepsilon$ for any $H \in \mathcal{H}_x$ disjoint with $(1 - \delta)B_X$.

ii) There is a sequence of subsets (A_n) of X such that for any point $x \in X$ and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathcal{H}_x$ such that $\eta(A_n \cap H) < \varepsilon$.

iii) There is a sequence of subsets (A_n) of X such that for any point $x \in X$ and $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathcal{H}_x$ such that $\eta^\omega(A_n \cap H) < \varepsilon$.

Proof (Sketch). We shall follow the main steps for the LUR renorming result of [12]. The technical details missing here can be found there.
i) $\Rightarrow$ *ii)* Assume X endowed with such a norm and take $A_n = a_n B_X$ where (a_n) is a enumeration of the positive rational numbers and use homogeneity.
ii) $\Rightarrow$ *iii)* It is obvious.
iii) $\Rightarrow$ *ii)* Proceed like in the proof of Corollary 3.7.

ii) $\Rightarrow$ *i*) Without loss of generality we may assume that the sets (A_n) are bounded, closed and convex as in the proof of Corollary 3.7. Let B denote the "old" unit ball of X. By considering the sets $A_n + m^{-1}B$ and Lemma 2.6 we may assume without loss of generality that the sets (A_n) verifies the property of the hypothesis with x interior to A_n. Take $a_n \in A_n$ an interior point and let f_n be the Minkowski functional of $A_n - a_n$. Let $f_{n,m,p}$ be the Minkowski functional of $A_{n,m,p} - a_{n,m,p}$ where $A_{n,m,p} = [A_n]'_{1/m} + p^{-1}B$ with $a_{n,m,p} \in A_{n,m.p}$ an interior point. Define a convex continuous function F on X by

$$F(x)^2 = \sum_n \lambda_n f_n(x)^2 + \sum_{n,m,p} \lambda_{n,m,p} f_{n,m,p}(x)^2$$

where the positive coefficients are taken to guarantee the uniform convergence on bounded sets. Define the a new equivalent norm $|||.|||$ as the Minkowski functional of the set

$$B_X = \{x \in X : F(x) + F(-x) \leq 3F(0)\}$$

Given $x \in X$ with $|||x||| = 1$ and $\varepsilon > 0$, fix $n, m, p \in \mathbb{N}$ such that $x \in A_n$ is interior, $m^{-1}\varsigma < \varepsilon$ and $x \notin A_{n,m,p}$. Therefore $f_n(x) < 1$ and $f_{n,m,p}(x) > 1$. By usual convexity arguments, we fix $\delta > 0$ such that $y \in B_X$ with $|||x + y||| > 2(1 - \delta)$ forces $f_n(y) < 1$ and $f_{n,m,p}(y) > 1$, obtaining that $y \in A_n \setminus A_{n,m,p}$. Take $H \in \mathcal{H}_x$ disjoint with $(1 - \delta)B_X$. We have $B_X \cap H \subset A_n \setminus A_{n,m,p}$. Since $B_X \cap H$ is convex and disjoint with the interior of $A_{n,m,p}$, using the regularity of η, we deduce that $\eta(B_X \cap H) < \varepsilon$ as we wanted. ■

It is possible to give several variations on the last theorem, for instance, to get the norm lower semicontinuous with respect to $\sigma(X, F)$ or to restrict the properties for some subset of X, instead of the whole space. Moreover, if we are dealing with a homogeneous measure η, which is regular with the help of the compactness, we may place X into its bidual X^{**} and prove *i*) $\Leftrightarrow$ *ii*). This is the case, for instance, of the measure of non weak compactness w.

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