Finitely dentable functions, operators and sets

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Abstract

We introduce the notion of finitely dentable map, which extends some properties of the uniform convexifying operators of Beauzamy to a non linear frame and provides a characterization of the functions that can be approximated by differences of convex Lipschitz functions.

1 Introduction

Maps with the *point of continuity property* appear related to the classical theorem of Baire. Recall that a map $f : X \to Y$ between topological spaces has the point of continuity property if the restriction of f to any nonempty closed subset $C \subset X$ has at least a point of continuity. In particular, if Y is a metric space, then for every $\varepsilon > 0$ and every nonempty subset of $C \subset X$ there is an open set U such that $C \cap U$ is nonempty and the oscillation of f in $C \cap U$ is less than ε. Modifications of the point of continuity property involving geometrical notions have been used in the frame of Banach spaces. Notably, if X is a Banach space with the Radon-Nikodým property and $C \subset X$ is a bounded closed convex set, then the identity map from (C, w) to $(C, \|.\|)$ has the point of continuity property, and moreover, the set C is *dentable*, that is, for every $\varepsilon > 0$ is possible to find an open halfspace H such that $C \cap H$ is nonempty and has diameter less than ε.

Let C be a closed convex bounded set of a Banach space X and let (Y, ρ) be a metric space. We say that a map $f : C \to Y$ is *dentable* if for any nonempty

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convex closed subset $D \subset C$ and $\varepsilon > 0$ is possible to find an open halfspace H intersecting D such that ρ-diam$(f(D \cap H)) < \varepsilon$. For any dentable map f we may consider the following "derivation"

$$[D]'_\varepsilon = \{x \in D : \rho\text{-diam}(f(D \cap H)) > \varepsilon,\ \forall\, H \in \mathbb{H},\ x \in H\}$$

Here $\mathbb{H}$ denotes the set of all the open halfspaces of X. Clearly, $[D]'_\varepsilon$ is what remains of D after removing all the slices of ρ-diameter through f less or equal than ε. Consider the sequence of sets defined by $[C]^0_\varepsilon = C$ and, for every $n \in \mathbb{N}$, inductively by

$$[C]^n_\varepsilon = [[C]^{n-1}_\varepsilon]'_\varepsilon$$

Such a process can be extended to transfinite ordinal numbers in a quite natural way, and for any dentable map the process finishes at the empty set. We are interested in the maps for which this derivation process ends after finitely many steps.

Definition 1.1 *The map $f : C \to Y$ is said to be finitely dentable if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $[C]^n_\varepsilon = \emptyset$.*

We shall denote by $\eta(\varepsilon)$ the greatest integer such that $[C]^n_\varepsilon \neq \emptyset$. If we are dealing with several functions we shall write $[C]^n_{f,\varepsilon}$ and $\eta(f, \varepsilon)$ for the sets and index associated with f.

In the remaining part of this introduction we shall try to motivate the use of the notion of finitely dentable maps by showing how they appear involved in a few results. We hope that the unexplained notation is standard.

A linear operator $T : X \to Y$ between Banach spaces is said uniformly convex if for every $\varepsilon > 0$, there is $\delta > 0$ such that $\|T(x) - T(y)\| < \varepsilon$ whenever $x, y \in S_X$ and $\|x + y\| > 2(1 - \delta)$. An operator is said uniformly convexifying if it becomes uniformly convex for some equivalent norm on X. This class of operators was introduced and studied by Beauzamy [1]. Our next result shows that finitely dentable maps are a non linear version of uniformly convexifying operators.

Theorem 1.2 *A bounded linear operator $T : X \to Y$ between Banach spaces is uniformly convexifying if, and only if, its restriction to B_X is finitely dentable.*

A closed convex bounded set $C \subset X$ is said finitely dentable if the inclusion map into X is so. The unit ball of a superreflexive space is a finitely dentable set. That is an easy consequence of the existence of an equivalent uniformly convex norm on X, but can be also deduced from properties of X which are closer to the definition of superreflexivity, see for instance [8, Lemma 3.1]. Another example is the closed convex hull of the canonical basis in $c_0(\Gamma)$. Indeed, this set is contained in the image of the unit ball of $\ell_2(\Gamma)$ through the inclusion operator. The class of finitely dentable sets of a Banach space lies between the compact and the weakly compact and shares many good properties of those classes. A compact space is said *uniformly Eberlein* if it is homeomorphic to a weakly compact set of a Hilbert space. The next result surveys some properties of convex bounded finitely dentable sets.

Theorem 1.3 *Let C be a closed convex bounded set of a Banach space X which is finitely dentable. Then C is weakly compact and uniformly Eberlein in its weak topology. Moreover, there is a uniformly convex linear operator $T : E \to X$ defined on a reflexive Banach space E, such that $C \subset T(B_E)$.*

This last result addresses us again to the work of Beauzamy [1] obtaining benefits from his results, and maybe some overlapping that we have tried to minimize. Our main motivation for the notion of finite dentability, instead of the *tree properties* used by Beauzamy, is that this notion seems to be more suitable in a nonlinear frame. In particular, we shall also deal with finitely dentable real functions. Haydon, Odell and Rosenthal characterized the functions on a metric compact space that can be approximated uniformly by differences of bounded lower semicontinuous functions in terms of indexes related to the Baire derivation, see Proposition 5.4 for a version of that result. Cepedello [3], see also [2, p. 94], proved that Lipschitz functions on the ball of a uniformly convex Banach space can be approximated by differences of convex Lipschitz functions (*"delta-convex"*), and this property characterizes superreflexivity. Finite dentability allows us to give the following characterization for a single Lipschitz function.

Theorem 1.4 *A Lipschitz function $f : C \to \mathbb{R}$ defined on a closed convex bounded set $C \subset X$ can be uniformly approximated by differences of convex Lipschitz functions on C if, and only if, it is finitely dentable.*

Cepedello's result appears as a corollary using the fact that the unit ball of a superreflexive Banach space is actually finitely dentable.

The paper is organized as follows. In the second section we prove our main result, Theorem 2.2, which characterizes Lipschitz finitely dentable maps by a *renorming*, which is essential for the rest of the paper. The third section is devoted to the general properties of finitely dentable maps. Convex sets which are finitely dentable are studied in fourth section, in relation with the uniformly convexifying operators of Beauzamy [1]. The result about delta-convex approximation for finitely dentable maps is proved in the last section.

2 An adapted renorming

We shall need some arguments which are usual in renorming theory, see for instance [4]. The reader can check easily that for any convex function f the following inequality holds

$$\frac{f(x)^2 + f(y)^2}{2} - f(\frac{x+y}{2})^2 \geq \left(\frac{f(x) - f(y)}{2}\right)^2 \geq 0$$

It implies the following consequence.

Lemma 2.1 *Let p be a seminorm defined on X and let (x_n) and (y_n) be p-bounded sequences in X. Then the two following conditions are equivalent:*

i) $\lim_n (2p(x_n)^2 + 2p(y_n)^2 - p(x_n + y_n)^2) = 0$

ii) $\lim_n (p(y_n) - p(x_n)) = \lim_n (p(\frac{x_n+y_n}{2}) - p(x_n)) = 0$

The following is a key result for the rest of the paper.

Theorem 2.2 *Let C be a closed convex bounded set of a Banach space X and let $f : C \to Y$ be a Lipschitz map into a metric space (Y, ρ). Then the following are equivalent:*

i) The map f is finitely dentable.

ii) There is an equivalent norm $|||.|||$ on X verifying $\lim_n \rho(f(x_n), f(y_n)) = 0$ whenever the sequences $(x_n), (y_n) \subset C$ are such that

$$\lim_n (2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2) = 0$$

iii) There is a convex continuous bounded function $\Phi : C \to \mathbb{R}$ *verifying that for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\rho(f(x), f(y)) < \varepsilon$ *whenever* $x, y \in C$ *are such that*

$$\frac{\Phi(x) + \Phi(y)}{2} - \Phi(\frac{x+y}{2}) < \delta$$

iv) There is a convex continuous bounded function $\Phi : C \to \mathbb{R}$ *verifying that for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for any* $H \in \mathbb{H}$ *and any* $r \geq 0$ *such that* $H \cap \{\Phi \leq r\} = \emptyset$ *then* ρ*-diam*$(f(H \cap \{\Phi \leq r + \delta\})) < \varepsilon$.

Proof. $i) \Rightarrow ii)$ We shall use arguments of Lancien's proof of the Enflo-Pisier theorem [10] as presented by Godefroy in [8, p. 801]. Without loss of generality, we may assume that $C \subset B_X$ and f is 1-Lipschitz.

For $k, n \in \mathbb{N}$, we take $N_k = \eta(2^{-k})$ and $K_k^n = [C]^n_{2^{-k}}$. We shall show that $\rho(f(x), f(y)) < 2^{1-k}$, if the points $x, y \in K_k^n$ are such that the segment $[x, y]$ does not meet K_k^{n+1} (this is a non linear version of the midpoint argument of Lancien). Indeed, consider the sets

$$A = \{z \in [x, y] : \exists H \in \mathbb{H},\ [x, z] \subset H,\ \rho\text{-diam}(f(K_k^n \cap H)) \leq 2^{-k}\}$$

$$B = \{z \in [x, y] : \exists H \in \mathbb{H},\ [z, y] \subset H,\ \rho\text{-diam}(f(K_k^n \cap H)) \leq 2^{-k}\}$$

Clearly we have A and B are nonempty relatively open subsets of $[x, y]$ with $A \cup B = [x, y]$, therefore there is $z \in A \cap B$. By triangle inequality, we get that $\rho(f(x), f(y)) \leq 2^{1-k}$.

Consider the 2-Lipschitz (in fact $\sqrt{2}$-Lipschitz) symmetric convex function F defined on X by the formula

$$F(x)^2 = \sum_{k=1}^{\infty} \sum_{n=1}^{N_k} \frac{2^{-k}}{N_k}\, d(x, K_k^n)^2 + \sum_{k=1}^{\infty} \sum_{n=1}^{N_k} \frac{2^{-k}}{N_k}\, d(x, -K_k^n)^2$$

We claim that if $x, y \in C$ and $\rho(f(x), f(y)) > \varepsilon$, then

$$F(\frac{x+y}{2})^2 \leq \frac{F(x)^2 + F(y)^2}{2} - \frac{\varepsilon^3}{2048\, \eta^3(\varepsilon/8)} \tag{1}$$

Pick k such that $\varepsilon/8 \leq 2^{-k} < \varepsilon/4$, take $\gamma = \varepsilon/4N_k$ and let n be the maximum integer such that $[x, y] \subset K_k^n$. Notice that $n < N_k$ since $\rho(f(x), f(y)) > \varepsilon$.

We shall show that inequality (1) can be deduced from the following: there is $l \in \mathbb{N}$ with $1 \leq l \leq N_k - n$, such that

$$\frac{d(x, K_k^{n+l})^2 + d(y, K_k^{n+l})^2}{2} - d(\frac{x+y}{2}, K_k^{n+l})^2 \geq \frac{\gamma^2}{16} \tag{2}$$

Indeed, by convexity we have

$$\frac{F(x)^2 + F(y)^2}{2} - F(\frac{x+y}{2})^2 \geq \frac{2^{-k}}{N_k}\frac{\gamma^2}{16} \geq (\frac{\varepsilon}{8N_k})(\frac{\varepsilon}{16N_k})^2 \geq \frac{\varepsilon^3}{2048\,\eta^3(\varepsilon/8)}$$

using for the last step that $\eta(\varepsilon/8) \geq N_k$ since $\varepsilon/8 \leq 2^{-k}$. Now we shall show that inequality (2) can be deduced of the following
FACT: For some $l \in \mathbb{N}$ with $1 \leq l \leq N_k - n$, one has

$$\max\{d(x, K_k^{n+l}), d(y, K_k^{n+l})\} - d([x, y], K_k^{n+l}) \geq \gamma \tag{3}$$

Indeed, if $|d(x, K_k^{n+l}) - d(y, K_k^{n+l})| \geq \gamma/2$ inequality (2) is automatically true since

$$\frac{d(x, K_k^{n+l})^2 + d(y, K_k^{n+l})^2}{2} - d(\frac{x+y}{2}, K_k^{n+l})^2 \geq \left(\frac{d(x, K_k^{n+l}) - d(y, K_k^{n+l})}{2}\right)^2$$

So we may assume $|d(x, K_k^{n+l}) - d(y, K_k^{n+l})| < \gamma/2$. In this case, together with inequality (3), we obtain that

$$d([x, y], K_k^{n+l}) < \min\{d(x, K_k^{n+l}), d(y, K_k^{n+l})\} - \frac{\gamma}{2}$$

A simple convexity argument implies the following inequality

$$\frac{d(x, K_k^{n+l}) + d(y, K_k^{n+l})}{2} - d(\frac{x+y}{2}, K_k^{n+l}) \geq \frac{\gamma}{4}$$

Now, we have

$$\frac{d(x, K_k^{n+l})^2 + d(y, K_k^{n+l})^2}{2} \geq \left(\frac{d(x, K_k^{n+l}) + d(y, K_k^{n+l})}{2}\right)^2 \geq$$

$$\left(d(\frac{x+y}{2}, K_k^{n+l}) + \frac{\gamma}{4}\right)^2 \geq d(\frac{x+y}{2}, K_k^{n+l})^2 + \frac{\gamma^2}{16}$$

and therefore inequality (2) holds.

Finally, we shall prove the FACT. Assume that it is not true to get a contradiction. First we shall show by induction on l with $1 \leq l \leq N_k - n$ that

$$\max\{d(x, K_k^{n+l}), d(y, K_k^{n+l})\} < l\gamma \tag{4}$$

That (4) is true for $l = 1$ follows from not-(3) together with $[x, y] \cap K_k^{n+1} \neq \emptyset$. If (4) is true for l, there are $x', y' \in K_k^{n+l}$ such that $\|x - x'\| < l\gamma$ and $\|y - y'\| < l\gamma$. Since f is 1-Lipschitz, we get that $\rho(f(x'), f(y')) > \varepsilon/2$, so it follows that

$$[x', y'] \cap K_k^{n+l+1} \neq \emptyset$$

Take $z' = \lambda x' + (1 - \lambda)y'$ with $\lambda \in [0, 1]$ such that $z' \in K_k^{n+l+1}$ and let $z = \lambda x + (1 - \lambda)y$. Since $\|z - z'\| < l\gamma$, we get that

$$d([x, y], K_k^{n+l+1}) < l\gamma$$

This inequality together with not-(3) implies that (4) holds for $l + 1$, which concludes the induction. Taking $l = N_k - n$ in (4) we obtain

$$\max\{d(x, K_k^{N_k}), d(y, K_k^{N_k})\} < (N_k - n)\gamma \leq \frac{\varepsilon}{4}$$

Using again that f is 1-Lipschitz, we get that there exists points $x', y' \in K_k^{N_k}$ such that $\rho(f(x'), f(y')) > \varepsilon/2$. This would imply that $[x', y'] \cap K_k^{N_k+1} \neq \emptyset$, which is impossible because the last set is empty. This completes the proof of fact above as well as claim (1).

The construction of the norm $|||.|||$ is performed as follows. Consider an enumeration (r_n) of the rational numbers from the interval $(F(0), 4]$ and let g_k be the Minkowski functional of the convex set $\{x \in X : F(x) \leq r_k\}$ which has nonempty interior, and for $x \in X$ define

$$|||x|||^2 = \|x\|^2 + \sum_{k=1}^{\infty} \lambda_k g_k(x)^2$$

where the numbers λ_k are positive and chosen in such a way that the series converges uniformly on bounded sets. Suppose we are given sequences $(x_n), (y_n) \subset C$ verifying that

$$\lim_n (2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2) = 0$$

To prove that $\lim_n \rho(f(x_n), f(y_n)) = 0$ it enough to show that any subsequence of indexes n_k has a further subsequence n_{k_j} such that

$$\lim_j \rho(f(x_{n_{k_j}}), f(y_{n_{k_j}})) = 0.$$

To simplify the writing, we shall not use explicit symbols for subsequences. By Lemma 2.1, passing to a subsequence, we may assume the existence of the following limits and equalities

$$\lim_n |||x_n||| = \lim_n |||y_n||| = \lim_n |||\frac{x_n + y_n}{2}|||$$

$$\lim_n g_k(x_n) = \lim_n g_k(y_n) = \lim_n g_k(\frac{x_n + y_n}{2}) = L_k \tag{5}$$

for every $k \in \mathbb{N}$. Passing to a further subsequence we may assume the existence of the limits

$$\lim_n F(x_n) = \alpha; \ \lim_n F(y_n) = \beta; \ \lim_n F(\frac{x_n + y_n}{2}) = \gamma$$

We claim that $\alpha = \beta = \gamma$. Indeed, we shall show that $\alpha > \beta$ leads to a contradiction and the other inequalities are similar. Take $\delta = \alpha - \beta$ and a rational r such that $\alpha - \delta/3 < r < \alpha$. Let $k \in \mathbb{N}$ such that $r_k = r$ and take $n \in \mathbb{N}$ large enough to have $F(x_n) > r$ and $F(y_n) < \beta + \delta/3$. Let S_k denote the unit sphere of g_k. Since F is 2-Lipschitz, we have $d(y_n, S_k) > \delta/6$. Having in mind that $\|y_n\| \leq 1$, we obtain the upper estimation

$$g_k(y_n) < \frac{6}{6 + \delta}$$

On the other hand, we have $g_k(x_n) \geq 1$. Since $\lim_n g_k(x_n) = \lim_n g_k(y_n)$ we get a contradiction which finishes the proof of the claim.

Once we know that

$$\lim_n F(x_n) = \lim_n F(y_n) = \lim_n F(\frac{x_n + y_n}{2})$$

the inequality (1) above implies that $\lim_n \rho(f(x_n), f(y_n)) = 0$ which finishes the proof of statement *ii*).

ii) $\Rightarrow$ *iii*) Just take $\Phi(x) = |||x|||^2$.

iii) $\Rightarrow$ *iv*) It is trivial.

iv) $\Rightarrow$ *i*) By adding a constant, we may suppose that function Φ is positive. Let $M = \sup\{\Phi(x) : x \in C\}$. It is easy to see that $\eta(\varepsilon) \leq \delta^{-1} M$. ■

The following two corollaries are concerned with the adapted renorming for several functions.

Corollary 2.3 *Suppose we are given a sequence of Lipschitz finitely dentable maps* $f_k : C \to Y_k$*, for* $k \in \mathbb{N}$*, with the same closed convex bounded domain* $C \subset X$*. Then there exists an equivalent norm* $|||.|||$ *on* X *such that* $\lim_n \rho_k(f_k(x_n), f_k(y_n)) = 0$ *for every* $k \in \mathbb{N}$ *whenever the sequences* $(x_n), (y_n) \subset C$ *verifies*

$$\lim_n (2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2) = 0$$

Proof. Let $|||.|||_k$ the equivalent norm on X for the map f_k given by Theorem 2.2. Define the norm $|||.|||$ by the formula

$$|||x|||^2 = \sum_{k=1}^{\infty} \lambda_k \, |||x|||_k{}^2$$

where the numbers (λ_k) are positive and such that the series converges uniformly on bounded sets. A convexity argument implies that the norm $|||.|||$ verifies the desired property. ■

Definition 2.4 *Let* $f_i : C \to Y_i$ *be a family of maps, with* $i \in I$*, defined on a closed convex bounded set* C *with values into metric spaces* (Y_i, ρ_i)*. We say that* $\{f_i : i \in I\}$ *is finitely equi-dentable if for every* $\varepsilon > 0$ *there exists a finite sequence of closed convex sets*

$$\emptyset = C_0 \subset C_1 \subset C_2 \subset \ldots \subset C_N = C$$

such that if S *is a slice of* C_{k+1} *not meeting* C_k*, where* $0 \le k < N$*, then we have* ρ_i*-diam*$(f_i(S)) < \varepsilon$ *for every* $i \in I$*.*

Clearly, if a singleton $\{f\}$ is a finitely equi-dentable family, then f is finitely dentable. To see that the converse is also true apply *iv*) of Theorem 2.2 or the first argument of its proof.

Corollary 2.5 *Let* $f_i : C \to Y_i$ *be an equi-Lipschitz family of maps, with* $i \in I$*, defined on a closed convex bounded set* C *with values into metric spaces. Then the family is finitely equi-dentable if, and only if, there exists an equivalent norm* $|||.|||$ *on* X *such that* $\lim_n \sup_{i \in I} \rho_i(f_k(x_n), f_k(y_n)) = 0$ *whenever the sequences* $(x_n), (y_n) \subset C$ *verifies*

$$\lim_n (2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2) = 0$$

Proof. Consider the metric space $Y = \prod_{i \in I} Y_i$ with the metric $\rho = \sup_{i \in I} \rho_i$. The hypothesis implies that the joint map $(f_i)_{i \in I}$, defined on C with values into Y, is Lipschitz and finitely dentable. It is clear that the norm $|||.|||$ given by Theorem 2.2 verifies the desired condition. For the converse, use a suitable version of *iv*) from Theorem 2.2. ■

3 Properties of finitely dentable maps

In this section we list properties of finitely dentable maps. Some of them follow from Theorem 2.2.

Proposition 3.1 *Let $f : C \to Y$ a map defined on a closed convex bounded set which can be approximated uniformly by finitely dentable maps. Then f is finitely dentable.*

Proof. Given $\varepsilon > 0$, consider a finitely dentable map g such that $\|f - g\|_\infty < \varepsilon/3$. Then it is easy to se that $[D]'_{f,\varepsilon} \subset [D]'_{g,\varepsilon/3}$ for any convex $D \subset C$. This implies that $\eta(f, \varepsilon)$ is finite. ■

Proposition 3.2 *Suppose that the maps $f_k : C \to Y_k$ defined on a closed convex bounded set are Lipschitz and finitely dentable for $k = 1, \ldots, n$. If $h : Y_1 \times \ldots \times Y_n \to Y$ is Lipschitz (with some reasonable product metric), then the composition $h(f_1, \ldots, f_n)$ is finitely dentable.*

Proof. It follows easily from Corollary 2.3. ■

Corollary 3.3 *Let $f_1, f_2 : C \to Y$ be Lipschitz finitely dentable maps from a closed convex bounded set of a Banach space X into a metric space (Y, ρ) and let $* : Y \times Y \to Y$ be an operation which is Lipschitz on bounded sets. Then the map $f = f_1 * f_2$ is finitely dentable.*

Corollary 3.4 *The set of Lipschitz finitely dentable maps from a closed convex bounded set of a Banach space X into a metric space Y inherits the structure of Y for the following ones: complete metric space, normed linear space, normed algebra and normed lattice.*

Proof. It follows from Proposition 3.1 and Corollary 3.3. ■

Proposition 3.5 *If a family $\{f_i : i \in I\}$ of Lipschitz finitely dentable maps is relatively compact in $\mathbf{C}(C, Y)$, then it is finitely equi-dentable.*

Proof. First prove that a finite set of maps is finitely equi-dentable. To do that use the equivalent norm on X given by Corollary 2.3 which satisfies the condition *ii*) of Theorem 2.2 for the joint map to the product space endowed with the supremum metric. Then apply *iv*) of Theorem 2.2 to get the sequence of convex sets. Since $\{f_i : i \in I\}$ is totally bounded, the result follows applying the idea from the proof of Proposition 3.1. ■

Remark 3.6 *It is easy to see that the finitely equi-dentability of a family is preserved by the pointwise closure.*

Some properties proved in this section allow us to produce new finitely dentable maps from previous ones. The following result shows that to know if a map is finitely dentable a reduction to more elementary maps is possible, sometimes.

Proposition 3.7 *Let $f : K \times C \to Y$ be a Lipschitz map, where $K \subset X_1$ is compact convex and $C \subset X_2$ is closed convex bounded, X_1 and X_2 are Banach spaces. Suppose that for every $x \in K$, the map $f_x = f(x, *)$ is finitely dentable as a map on C. Then f is finitely dentable.*

Proof. Since K is separable, there is an equivalent norm $\|.\|_1$ on X_1 which is strictly convex on the linear space spanned by K. Using the compactness of K, it is easy to prove that $\|.\|_1$ satisfies $ii)$ of Theorem 2.2 (in particular, we get that K is finitely dentable). The family of maps $\{f_x : x \in K\}$ defined on C above is compact in $\mathbf{C}(C, Y)$ and thus finitely equi-dentable by Proposition 3.5. Let $\|.\|_2$ be the norm on X_2 given by Corollary 2.5. Define an equivalent norm $|||.|||$ on $X_1 \times X_2$ by $|||(r, x)|||^2 = \|r\|_1^2 + \|x\|_1^2$. We claim that this norm satisfies condition $ii)$ of Theorem 2.2 for f. Indeed, take sequences (r_n, x_n) and (s_n, y_n) such that

$$\lim_n 2|||(r_n, x_n)|||^2 + 2|||(s_n, y_n)|||^2 - |||(r_n + s_n, x_n + y_n)|||^2 = 0$$

By convexity, we have

$$\lim_n 2\|r_n\|_1^2 + 2\|s_n\|_1^2 - \|r_n + s_n\|_1^2 = 0$$

$$\lim_n 2\|x_n\|_2^2 + 2\|y_n\|_2^2 - \|x_n + y_n\|_2^2 = 0$$

We deduce that $\lim_n \|r_n - s_n\| = 0$ and $\lim_n \rho(f(p, x_n), f(p, y_n)) = 0$ for every $p \in K$. We want to prove that $\lim \rho(f(r_n, x_n), f(s_n, y_n)) = 0$. Assume that this is false, so passing to a subsequence $\rho(f(r_n, x_n), f(s_n, y_n)) > \varepsilon$. By compactness, we may assume also that there exists

$$\lim_n r_n = \lim_n s_n = p \in K$$

Since f is Lipschitz, we have $\rho(f(p, x_n), f(p, y_n)) > \varepsilon$ for n large enough, but this is a contradiction with a former equality. ∎

4 Convex finitely dentable sets

In this section we study the properties of the finitely dentable closed convex bounded subsets of Banach spaces. It can be show that finite dentability for symmetric convex sets is equivalent to the finite tree property used by Beauzamy in [1], that we shall not consider here.

Lemma 4.1 *Let $C, D \subset X$ be convex weakly compact sets with $C \subset D$ such that every open slice of D which is disjoint with C has diameter less than $s > 0$. Then every open slice of $D + B[0, r]$ which is disjoint with $C + B[0, r]$ has diameter less than $s + 2r > 0$ for any $r > 0$.*

Proof. Put $C_r = C + B[0, r]$ and $D_r = D + B[0, r]$. Take a slice of the form $S_1 = \{x \in D_r : x^*(x) > a\}$, where $\|x^*\| = 1$, which does not meet C_r. The slice $S_2 = \{x \in D : x^*(x) > a - r\}$ does not meet C because if it is the case then S_1 meets C_r. Take points $x, y \in S_1$ and find points $x', y' \in D$ such that $\|x - x'\| \leq r$ and $\|y - y'\| \leq r$. We claim that $x', y' \in S_2$. Indeed, if $x^*(x') \leq a - r$, then $\|x - x'\| > r$ since $x^*(x) > a$. As $\|x' - y'\| < s$, we deduce that $\|x - y\| < s + 2r$. ■

Lemma 4.2 *Let $C \subset X$ a convex subset such that for every $\varepsilon > 0$ there is finitely dentable closed convex bounded set D such that $C \subset D + B[0, \varepsilon]$. Then C is finitely dentable.*

Proof. Lemma 4.1 implies that every open slice of $[D]^n_{\varepsilon/6} + B[0, \varepsilon/3]$ which does not meet $[D]^{n+1}_{\varepsilon/6} + B[0, \varepsilon/3]$ has diameter less than ε. We deduce that $\eta(C, \varepsilon) \leq \eta(D, \varepsilon/6)$ and therefore C is finitely dentable. ■

Proposition 4.3 *a) Every compact convex set is finitely dentable.*

b) Every finitely dentable closed convex bounded set is weakly compact.

Proof. a) Assume that C is compact. Any equivalent strictly convex norm on the closed span of C satisfies ii) of Theorem 2.2 (this was used in the proof of Proposition 3.7). We shall provide a direct proof. For every $\varepsilon > 0$ we may take a finite dimensional compact convex subset $D \subset C$ such that $C \subset D + B[0, \varepsilon]$. Then C is finitely dentable by Lemma 4.2.
b) Let $C \subset X$ be a finitely dentable closed convex bounded set. Regarding C as a subset of the bidual X^{**}, we shall show that $\overline{C}^{w^*} = C$. Take $x \in \overline{C}^{w^*}$

and $\varepsilon > 0$. Let n be such that $x \in \overline{[C]^n_\varepsilon}^{w^*}$ but $x \notin \overline{[C]^{n+1}_\varepsilon}^{w^*}$. There is a weak* open half space H such that $x \in H$ and $\overline{[C]^{n+1}_\varepsilon}^{w^*} \cap H = \emptyset$. By the midpoint argument of Lancien, the diameter of $[C]^n_\varepsilon \cap H$ does not exceed 2ε. We have

$$\mathrm{diam}(\overline{[C]^n_\varepsilon}^{w^*} \cap H) = \mathrm{diam}(\overline{[C]^n_\varepsilon \cap H}^{w^*}) = \mathrm{diam}([C]^n_\varepsilon \cap H) \leq 2\varepsilon$$

and thus $d(x, C) \leq 2\varepsilon$. Since ε was arbitrary $x \in C$. ■

The following list several facts about finite dentability for sets.

Proposition 4.4 *a) The image of a finitely dentable closed convex bounded set through a bounded linear operator is finitely dentable.*

b) The product of finitely dentable closed convex bounded sets in a finite direct sum of Banach spaces is finitely dentable.

c) The sum and the convex hull of two finitely dentable closed convex bounded sets is finitely dentable. The absolutely convex hull of a finitely dentable closed convex bounded set is also finitely dentable.

Proof. *a*) Let $T : X \to Y$ be a bounded linear operator and $C \subset X$ a finitely dentable convex set. Any slice of $T([C]^n_\varepsilon)$ not meeting the closed convex set $T([C]^{n+1}_\varepsilon)$ has diameter less than $2\varepsilon\|T\|$.
b) Let $C_i \subset X_i$ be finitely dentable closed convex bounded sets for $1 \leq i \leq n$. Consider on $\bigoplus_{i=1}^n X_i$ the equivalent norm $|||.|||$ defined by

$$|||(x_1, \ldots, x_n)|||^2 = \sum_{i=1}^n |||x_i|||_i^2$$

where $|||.|||_i$ is norm on X_i satisfying condition *ii*) of Theorem 2.2 for the set C_i. It is not difficult to show using Lemma 2.1 and Theorem 2.2 again that $\Pi_{i=1}^n C_i$ is finitely dentable.
c) The result for the sum $C_1 + C_2$ follows from parts a) and b). For the convex hull of C_1 and C_2, take points $x_i \in C_i$ and consider the segments $[0, -x_i]$ joining 0 and $-x_i$. Then apply the previous case to the sums $D_i = C_i + [0, -x_i]$ and $D_1 + D_2$ to obtain a finitely dentable closed convex bounded set containing both C_1 and C_2. ■

The interpolation of uniformly convexifying operators was already considered by Beauzamy in [1]. We shall proof the following interpolation result for the sake of completeness.

Theorem 4.5 *Let $C \subset X$ be a finitely dentable closed convex bounded set. Then there exists a reflexive Banach space E and an injective bounded linear operator $T : E \to X$ such that $T(B_E)$ is finitely dentable and contains C.*

Proof. By Proposition 4.4 *c*) there is an absolutely convex weakly compact $K \subset X$ which is finitely dentable and contains C. The interpolation method of Davis-Figiel-Johnson-Pelzynski, see [7, p. 366] provides a reflexive Banach space E and a bounded linear injective operator such that

$$T(B_E) \subset 2^n K + B[0, 2^{-n}]$$

for every $n \in \mathbb{N}$. Lemma 4.2 implies that $T(B_E)$ is finitely dentable. ■

Theorem 1.2 is a consequence of the following.

Proposition 4.6 *For a bounded linear operator $T : X \to Y$ between Banach spaces the following conditions are equivalent:*

i) T is uniformly convexifying.

ii) T restricted to B_X is finitely dentable.

iii) $\overline{T(B_X)}$ is finitely dentable.

Proof. *i*) $\Leftrightarrow$ *ii*) Follows from Theorem 2.2.
i) $\Rightarrow$ *iii*) Without loss of generality we may assume that the norm of X makes T uniformly convex and T has norm one. Given $\varepsilon > 0$, take $\delta > 0$ such that the images by T of the open slices of B_X not meeting $\overline{(1-\delta)B_X}$ have diameter less than ε. It is easy to see that any open slice of $\overline{T(B_X)}$ not meeting $\overline{(1-\delta)\overline{T(B_X)}}$ has diameter less than ε. By iteration we can deduce that $\overline{T(B_X)}$ is finitely dentable.
iii) $\Rightarrow$ *i*) Let $\|.\|_u$ an equivalent norm on Y verifying *ii*) of Theorem 2.2 for the inclusion of $\overline{T(B_X)}$ into Y. Define an equivalent norm $|||.|||$ on X by the formula

$$|||x|||^2 = \|x\|^2 + \|T(x)\|_u{}^2$$

A simple convexity argument shows that $|||.|||$ verifies *ii*) of Theorem 2.2 for the map T on B_X. ■

We shall need these notions. The norm $\|.\|$ of a Banach space is said weak uniformly rotund (WUR) if $w\text{-}\lim_n(x_n - y_n) = 0$ (limit in the weak topology) provided that $\|x_n\| = \|y_n\| = 1$ and $\lim_n \|x_n + y_n\| = 2$. Weak* uniformly rotund norms (W*UR) are defined analogously for dual Banach spaces.

Proof of Theorem 1.3. We already have that C is weakly compact by Proposition 4.3 and the operator $T : E \to X$ is given by Theorem 4.5, which is uniformly convex after a suitable renorming of E by Proposition 4.6. To show that C is uniformly Eberlein , is enough to prove that for B_E since C embeds homeomorphically into B_E. Since T is uniformly convex, we have w-$\lim_n(T(x_n) - T(y_n)) = 0$ whenever $\|x_n\| = \|y_n\| = 1$ and $\lim_n \|x_n + y_n\| = 2$. This implies w-$\lim_n(x_n - y_n) = 0$ since T restricted to the unit ball is a weak homeomorphism. Therefore the norm on E is WUR, and thus W*UR because E is reflexive. A result of Fabian-Godefroy-Zizler [6] states that a dual Banach space with a W*UR norm has weak* uniformly Eberlein dual unit ball. This finishes the proof. ∎

A uniformly convex operator may not factor through a superreflexive Banach space [1], but we at least we have the following.

Corollary 4.7 *A uniformly convex operator factors through a reflexive Banach space with uniformly Eberlein unit ball.*

Considering finite dentability with respect to topologies weaker than the weak topology has no interest since finitely dentable closed convex bounded set are weakly compact. The situation changes if we measure diameters with respect to a metric coarser than the norm. For instance we may consider weak* dentability of subsets in a dual X^* with respect to the uniform convergence on a bounded total subset of X.

Corollary 4.8 *Let X be a Banach space containing a total finitely dentable closed convex bounded set C. Then (B_{X^*}, w^*) is uniformly Eberlein and finitely weak*-dentable for the norm of uniform convergence on C.*

Proof. There is a uniformly convex operator $T : E \to X$ from a reflexive space with dense range. The adjoint operator $T^* : X^* \to E^*$ is injective and uniformly convexifying by a result of Beauzamy [1]. We may construct an equivalent W*UR norm on X^* as in the proof of Theorem 1.3, therefore (B_{X^*}, w^*) is uniformly Eberlein by [6]. Observe B_{X^*} is finitely weak*-dentable for the norm of uniform convergence on $T(B_{E^*})$ which is stronger than the uniform convergence on C. ∎

We shall finish the section explaining the lack of interest of finitely dentable closed non-convex bounded sets.

Example 4.9 *There exists a reflexive Banach space X and a finitely dentable weakly compact subset $K \subset X$ such that the closed convex hull of K is not finitely dentable.*

Proof. Take a reflexive space X such that B_{X^*} is not uniformly Eberlein. Let $\{x_i, x_i^*\}_{i\in I} \subset X \times X^*$ a Markushevich basis of X with $\|x_i\| = 1$. Then $K = \{x_i : i \in I\} \cup \{0\}$ is weakly compact and finitely dentable. Indeed, $[K]'_\varepsilon = \{0\}$ for any $0 < \varepsilon < 1$, since $\{x_i\} = K \cap H_i$ where $H_i = \{x \in X : x_i^*(x) > 1/2\}$, therefore $[K]^2_\varepsilon = \emptyset$. The closed convex hull of K is not finitely dentable because if it is not the case, then the previous corollary would imply that B_{X^*} is uniformly Eberlein. ∎

5 Differences of convex functions

In this section we shall apply the notion of finite dentability to know when a function defined on a closed convex bounded set can be approximated by differences of Lipschitz convex functions. Criterions to know if a given function is actually a difference of two good convex functions can be found in [5].

Proposition 5.1 *The difference of two bounded convex lower semicontinuous functions defined on a closed convex set is finitely dentable.*

Proof. Firstly we shall assume that f is convex lower semicontinuous. Take $m = \inf_C f$, $M = \sup_C f$ and fix $\varepsilon > 0$. It is easy to prove that the following inclusion holds

$$[C]^n_\varepsilon \subset f^{-1}([m, M - n\varepsilon])$$

which implies the finite dentability of f.

Suppose now that $f = f_1 - f_2$ where f_1 and f_2 are bounded convex lower semicontinuous functions. Without loss of generality we may assume that they are also positive. Consider the bounded convex lower semicontinuous function $g(x) = f_1(x)^2 + f_2(x)^2$. By convexity we have the following inequalities

$$|f(x) - f(y)|^2 \leq (|f_1(x) - f_1(y)| + |f_2(x) - f_2(y)|)^2$$
$$\leq 2\left((f_1(x) - f_1(y))^2 + (f_2(x) - f_2(y))^2\right)$$

$$\leq 8\left(\frac{f_1(x)^2+f_1(y)^2}{2}-f_1(\frac{x+y}{2})^2\right)+8\left(\frac{f_2(x)^2+f_2(y)^2}{2}-f_2(\frac{x+y}{2})^2\right)$$
$$=8\left(\frac{g(x)+g(y)}{2}-g(\frac{x+y}{2})\right)$$

The finite dentability of g and this relation imply that f is finitely dentable.■

Proof of Theorem 1.4. Differences of Lipschitz convex functions are finitely dentable by Proposition 5.1, and by Proposition 3.1 we can pass to the uniform closure.

Assume that f is Lipschitz and finitely dentable. Let $|\!|\!|.|\!|\!|$ the norm given by *ii*) of Theorem 2.2. We will follow Cepedello's construction [3], but the fact that f is bounded will simplify the proof. Define a sequence of functions

$$f_n(x)=\inf_{y\in C}\{f(y)+n(2|\!|\!|x|\!|\!|^2+2|\!|\!|y|\!|\!|^2-|\!|\!|x+y|\!|\!|^2)\}$$

Notice that f_n can be decomposed as a difference of convex Lipschitz functions

$$f_n(x)=2n|\!|\!|x|\!|\!|^2-\sup_{y\in C}\{n|\!|\!|x+y|\!|\!|^2-2n|\!|\!|y|\!|\!|^2-f(y)\}$$

We also have that the sequence (f_n) is increasing and $f_n(x)\leq f(x)$. If $y\in C$ is such that

$$f(y)+n(2|\!|\!|x|\!|\!|^2+2|\!|\!|y|\!|\!|^2-|\!|\!|x+y|\!|\!|^2)\leq f(x) \tag{6}$$

we can deduce

$$0\leq 2|\!|\!|x|\!|\!|^2+2|\!|\!|y|\!|\!|^2-|\!|\!|x+y|\!|\!|^2\leq n^{-1}(f(x)-f(y))\leq n^{-1}\mathrm{diam}(f(C))$$

Given $\varepsilon>0$, if $n\in\mathbb{N}$ is large enough we have $|f(x)-f(y)|<\varepsilon$ for any $y\in C$ satisfying (6). Thus $f_n(x)\geq f(x)-\varepsilon$, which shows that (f_n) converges to f uniformly on C. ■

Remark 5.2 *Notice that Proposition 5.1 and Proposition 3.1 do not use the Lipschitz property. Therefore we actually get that a Lipschitz function f which can be uniformly approximated by differences of bounded convex lower semicontinuous functions is finitely dentable, and thus f can be uniformly approximated by differences of Lipschitz convex functions.*

The next corollary extends Cepedello's result [3]

Corollary 5.3 *Let C be a finitely dentable closed convex bounded set. Then any uniformly continuous function defined on C can be approximated uniformly by differences of convex Lipschitz functions.*

Proof. The result is a direct consequence of Theorem 1.4 for Lipschitz functions. If f is uniformly continuous, then the problem can be reduced to the Lipschitz case using, for instance, the sequence

$$f_n(x) = \inf\{f(y) + n\|x - y\| : y \in C\}$$

of Moreau-Yoshida which converges uniformly to f. ■

We shall briefly sketch the argument to obtain Cepedello's converse result within the frame of finitely dentable functions, that is, if X is not superreflexive there exists a function defined on B_X that cannot be approximated uniformly by differences of bounded convex continuous functions. Indeed, if X is not superreflexive, then there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there is a dyadic tree $\{x_s : s \in \{0,1\}^{\leq n}\}$ of height n inside B_X. This tree can be taken ε-discrete, see [2, p. 412] for instance. Let f_n be the distance to the union of the even levels $\{x_s : |s| \in 2\mathbb{N}\}$. It is easy to see that $\eta(f, \varepsilon) \geq n - 1$. Consider now a 1/3-discrete sequence (B_n) of balls of radius 1/3 inside B_X. By similarity, we may take a $\varepsilon/3$-discrete dyadic tree of height n inside B_n. The same construction that before will provide a non finitely dentable function f, which is not uniformly approximated by differences of bounded convex continuous functions by Theorem 1.4 and Remark 5.2.

To finish we shall give the topological counterpart of Theorem 1.4. Let Z be a topological space and Y a metric space. Given a map $f : Z \to Y$, for any closed subset $D \subset Z$ and $\varepsilon > 0$ we shall consider the following derivation

$$(D)'_\varepsilon = \{x \in D : \rho\text{-diam}(f(D \cap U)) > \varepsilon, \ \forall U \in \mathcal{V}(x)\}$$

Consider the sequence starting by $(Z)^0_\varepsilon = Z$ and, inductively, $(Z)^n_\varepsilon = ((Z)^{n-1}_\varepsilon)'_\varepsilon$ for every $n \in \mathbb{N}$. We say that the map f is finitely fragmentable if for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $(Z)^n_\varepsilon = \emptyset$.

The following is a reformulation of a result due to Haydon, Odell and Rosenthal [9].

Theorem 5.4 *A bounded function $f : Z \to \mathbb{R}$ can be approximated uniformly by differences of bounded lower semicontinuous functions if, and only if, it is finitely fragmentable.*

Proof (Sketch). It is not difficult to see that a bounded lower semicontinuous function is finitely fragmentable and that property is stable by differences and uniform limits.
Let f be bounded and finitely fragmentable. Let $\varepsilon > 0$ and F_ε a finite subset of reals such that $d(f(x), F_\varepsilon) < \varepsilon$ for every $x \in Z$. It is easy to find a finite cover $(U_{k,j})$ of $(Z)^k_\varepsilon \setminus (Z)^{k+1}_\varepsilon$ made up of relatively open subsets (and therefore they belong to the algebra $\mathcal{A}$ of differences of open subsets of Z) such that $d(f(U_{k,j}), \lambda_{k,j}) < \varepsilon$ for some $\lambda_{k,j} \in F_\varepsilon$. The sets $A_{k,j} = U_{k,j} \setminus \bigcup_{i<j} U_{k,i}$ belong to $\mathcal{A}$, and thus its characteristic function $\chi_{A_{k,j}}$ is a difference of lower semicontinuous functions. The construction above allows us to build a linear combination $g = \sum \lambda_{k,j}\chi_{A_{k,j}}$ with $\lambda_{k,j} \in F_\varepsilon$ such that $\|f - g\|_\infty < \varepsilon$. We get that g can be expressed as a difference of bounded lower semicontinuous functions. ■

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