Continuity at the extreme points

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October, 2007

Dedicated to Prof. Isaac Namioka on the occasion of his 80th birthday

Abstract

We prove that a bounded convex lower semicontinuous function defined on a convex compact set K is continuous at a dense subset of extreme points. If there is a bounded strictly convex lower semicontinuous function on K, then the set of extreme points contains a dense completely metrizable subset.

1 Introduction

The celebrated Krein-Milman theorem, see [2, 25.12] for instance, establishes the existence of extreme points in convex compact sets. On the other hand, it is well known that some kinds of functions on a compact space have points of continuity, such as the semicontinuous or the Baire-one functions. It is known that an affine Baire-one function defined on a convex compact set K of a locally convex space is continuous at a dense set of $\operatorname{ext} K$, see [5, Lemma II.2] where a more general result is proved. Here we shall prove a similar result for convex functions under usual hypothesis.

Theorem 1.1 *Let K be a convex compact subset of a locally convex space and let $f : K \to \mathbb{R}$ be a bounded convex lower semicontinuous function. Then $\operatorname{ext} K$ contains a dense subset of continuity points of f.*

The proof of this theorem depends on the result of Choquet establishing that $\operatorname{ext} K$ is a Baire space, see [2, 27.9]. Hervé proved in [7] that the existence of a continuous strictly convex function on K implies its metrizability, giving as a consequence that $\operatorname{ext} K$ is a Polish space. We apply Theorem 1.1 to prove the following related result.

Theorem 1.2 *Let K be a convex compact subset of a locally convex space. If there exists a bounded strictly convex lower semicontinuous function $f : K \to \mathbb{R}$, then $\operatorname{ext} K$ contains a dense subset which is $\mathcal{G}_\delta$ in K and completely metrizable.*

These hypotheses also imply that K contains a dense $\mathcal{G}_\delta$ set which is completely metrizable, by results of Ribarska using the notion of fragmentability, see [10, 11]. Theorem 1.2 extends a result from [6]. It is clear that Theorems 1.1 and 1.2 remain true changing convex lower semicontinuous to concave upper semicontinuous.

2 Proofs and comments

The geometrical arguments used to prove Theorem 1.1 come from Namioka's study [8] of topological properties of the set of extreme points. Later, Bourgain applied these ideas to dentability and the Radon-Nikodým property in Banach spaces, see chapter 3 of [1]. A *slice* of a given set K is the intersection of K with an open half space. We shall use continuously the so called Choquet's Lemma, see for instance [2, 25.13] or [4, p. 77]: if K is convex compact any $x \in \operatorname{ext} K$ has a neighbourhood basis made up of slices.

Proof of Theorem 1.1. First recall that a l.s.c. function on a Baire space has a dense set of continuity points. Since $\operatorname{ext} K$ is a Baire space [2, 27.9], it contains a dense subset of continuity points of $f|_{\operatorname{ext} K}$. We shall prove that actually that set is made of continuity points of f.

Adding a constant we may assume that f is non negative. Let $M > 0$ be an upper bound for f. Take any $x \in \operatorname{ext} K$ where $f|_{\operatorname{ext} K}$ is continuous. To show that f is continuous at x it is enough to check upper semicontinuity. Fix $0 < \varepsilon < M^{-1}$ and take an open half space H such that

$$x \in H \cap \operatorname{ext} K \subset \{y \in K : f(y) \leq f(x) + \varepsilon\}$$

which is possible by continuity. Define

$$C_1 = \{y \in K : f(y) \leq f(x) + \varepsilon\} \cap \overline{H}$$

$$C_2 = K \setminus H$$

which are closed convex subsets of K with $\operatorname{ext} K \subset C_1 \cup C_2$ by construction. Therefore, the convex hull of $C_1 \cup C_2$ is K. Consider the set

$$C = \{(1-\lambda)x_1 + \lambda x_2 : x_1 \in C_1;\, x_2 \in C_2;\, \lambda \in [\frac{\varepsilon}{M}, 1]\}$$

which is closed and convex and $x \notin C$ by extremality. Then $K \setminus C$ is an open neighbourhood of x. If $y \in K \setminus C$, then $y = (1-\lambda)x_1 + \lambda x_2$ where $x_1 \in C_1$, $x_2 \in C_2$ and $0 < \lambda < \frac{\varepsilon}{M}$. By convexity, we have

$$f(y) \leq (1-\lambda)f(x_1) + \lambda f(x_2) < f(x) + \varepsilon + \frac{\varepsilon}{M}M = f(x) + 2\varepsilon$$

which completes the proof of the upper semicontinuity of f at x. ■

Notice that the isolated extreme points must be continuity points of f.

Proof of Theorem 1.2. It is convenient to introduce the "symmetric"

$$\rho(x,y) = \frac{f(x)+f(y)}{2} - f(\frac{x+y}{2})$$

which has the property that $\rho(x,y) = 0$ implies $x = y$ by strict convexity.
Claim: Let $U \subset K$ be an open set which is dense in the relative topology of $\operatorname{ext} K$ and $\varepsilon > 0$. Then there exists a family $\mathfrak{S}$ of open slices of K with the following properties:

a) For every $S \in \mathfrak{S}$, $\overline{S} \subset U$ and $x, y \in S$ implies $\rho(x,y) < \varepsilon$.

b) The closures of the members of $\mathfrak{S}$ are disjoint on $\operatorname{ext} K$.

c) The union V of the members of $\mathfrak{S}$ is dense in the relative topology of $\operatorname{ext} K$.

To prove the claim just take a maximal family with properties *a*) and *b*). We shall prove that condition *c*) is fulfilled. Indeed, if it is not the case, by Theorem 1.1 we may find $x \in U \cap \operatorname{ext} K$ where f is continuous such that $x \notin \overline{V \cap \operatorname{ext} K}$. Take S a slice containing x, with $\overline{S} \subset U$ disjoint from $\overline{V \cap \operatorname{ext} K}$ and such that the oscillation of f on S is less than $\varepsilon/2$. An easy computation gives that $\rho(y,z) < \varepsilon$ for any $y, z \in S$. Then $\mathfrak{S} \cup \{S\}$ violates the maximality.

Starting with $\mathfrak{S}_0 = \{K\}$ define inductively families $\mathfrak{S}_n$ satisfying the claim with U the union of $\mathfrak{S}_{n-1}$ and $\varepsilon = 1/n$. Let U_n be the union of $\mathfrak{S}_n$ and take $Z = \bigcap_{n=1}^{\infty} U_n$. We claim that $Z \subset \operatorname{ext} K$. Indeed, if $x \in Z$ then there are slices $S_n \in \mathfrak{S}_n$ containing x. If x is not extreme, put $x = \frac{y+z}{2}$ nontrivially. Then $S_n \cap \{y,z\} \neq \emptyset$, and therefore we have either $\rho(x,y) < 1/n$ or $\rho(x,z) < 1/n$ giving a contradiction. Notice that the sequence (S_n) for x is unique and $\overline{S_n} \subset S_{n+1}$. By compactness, (S_n) is a local base at x for the topology. The metrizability of Z follows from Bing's theorem [3, 4.4.8], since every family $\mathfrak{S}_n$ is discrete relatively on Z. The metric can be chosen complete because Z is Čech-complete as a $\mathcal{G}_\delta$ subset of a compact space [3, 4.3.26]. The density of Z in $\operatorname{ext} K$ is clear from the construction. ■

To finish we shall consider an application to Banach spaces. Let X be a normed space and τ a locally convex topology on X. A point $x \in K$ is said to be τ-denting if it is contained in τ-open slices of K of arbitrarily small norm diameter. The norm of X is said to be τ-Kadec if on the unit sphere the norm topology and τ coincide. The following result is a version of [4, Exercise 8.88] which was a main motivation for this paper.

Example 2.1 *Let X have a τ-Kadec norm and let $K \subset X$ be convex and τ-compact. Then the $x \in K$ is τ-denting if, and only if, $x \in \operatorname{ext} K$ and the restriction of the norm to K is τ-continuous at x.*

Proof. One of the implications is obvious. Given $\varepsilon > 0$, the Kadec property implies that there is a τ-open neighbourhood U of x and $\delta > 0$ such that if $|\|y\| - \|x\|| < \delta$ then $\|y - x\| < \varepsilon/2$. If the norm is τ-continuous at x, then it is possible to find a τ-open neighbourhood V of x, such that $y \in V$ implies $|\|y\| - \|x\|| < \delta$. Therefore the norm diameter of $U \cap V$ is less than ε. If x is extreme, just take a slice S such that $x \in S \subset U \cap V$. ■

Since a τ-Kadec norm is τ-l.s.c. see [4, Exercise 8.86], from the above and Theorem 1.1 one obtains the weak*-dentability of bounded subsets in dual spaces with a weak*-Kadec norm [9] and the dentability of weakly compact sets, via the renorming theorem of Troyanski [12]. In all these cases, the set given by Theorem 1.2 can be replaced by the set of τ-denting points which is metrized by the norm.

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