On weak* uniformly Kadec-Klee renormings

M. Raja

December, 2007

Abstract

Let X be a Banach space with Szlenk index ω, then its dual space X^* has an equivalent weak* uniformly Kadec-Klee norm with modulus of power type. That extends results of Knaust, Odell and Schlumprecht [8] and solves a problem of Huff [6].

1 Introduction

The norm of a Banach space X is said to be *uniformly Kadec-Klee* (UKK) if for every $\varepsilon > 0$ there is $\theta(\varepsilon) \in (0,1)$ such that every $x \in B_X$ with $\|x\| > 1-\theta(\varepsilon)$ has a weak open neighborhood U with $\mathrm{diam}(B_X \cap U) < \varepsilon$. The function θ is called *modulus*. Dealing with dual Banach spaces, we say that the norm is weak* uniformly Kadec-Klee (UKK*) if the weak topology is replaced by the weak* above. These notions were originally introduced by Huff [6] using sequences, but we are using a more restrictive version due to Lancien [10]. Our choice has some advantages. For instance, the UKK* property, as defined above, is dual to the *asymptotically uniformly smoothness*, studied in [7] in relation with the differentiability of Lipschitz mappings. In the case of reflexive Banach spaces, Lancien's UKK property coincides with Huff's definition and also with the notion of *nearly uniformly convex* norm [6, Theorem 1], which has applications in fixed point theory [1]. Finally, with the above definition, our main result provides a characterization of the UKK* renormability in dual Banach spaces.

Let X be an Asplund Banach space, see [2], and X^* its dual. For any bounded subset $A \subset X^*$ we define a set derivation

$$\langle A\rangle'_\varepsilon = \{x^* \in A : \forall\, U \; w^*\text{-neighbourhood of } x^*, \mathrm{diam}(A \cap U) \geq \varepsilon\}$$

which is a proper subset of A if it is nonempty. By iteration, the sets $\langle A\rangle^\gamma_\varepsilon$ are defined for any ordinal γ, taking intersection in the case of limit ordinals. The Szlenk indices are ordinal numbers defined by

$$Sz(X)_\varepsilon = \inf\{\gamma : \langle B_{X^*}\rangle^\gamma_\varepsilon = \emptyset\}$$

and $Sz(X) = \sup_{\varepsilon>0} Sz(X)_\varepsilon$. It is not difficult to see that $Sz(X)$ is an isomorphic invariant of X. See [9] for an account of properties of the Szlenk indices.

Notice that the norm of a dual Banach space X^* is UKK* if for every $\varepsilon > 0$ there is $\theta(\varepsilon) \in (0, 1)$ such that

$$\langle B_{X^*} \rangle'_{\varepsilon} \subset (1 - \theta(\varepsilon))B_{X^*}$$

By iteration of that set inclusion it is easy to deduce that $Sz(X)_{\varepsilon}$ is finite for every $\varepsilon > 0$. Therefore, if X^* has an equivalent UKK* norm, then $Sz(X) \leq \omega$. A natural question is to know if the converse is true [6]. An affirmative answer was obtained by Knaust, Odell and Schlumprecht [8] in the case of separable Banach spaces. Moreover, their renorming verifies $\theta(\varepsilon) = c\varepsilon^p$ for some $c, p > 0$, analogously to Pisier results about superreflexive spaces [11]. Godefroy, Kalton and Lancien gave in [5] optimal results linking the exponent of the modulus and the growth of the Szlenk indices and proved that the condition $Sz(X) \leq \omega$ is invariant by uniform homeomorphisms. Let us also mention that [12, 3, 10] contain results about UKK* renorming in certain classes of Banach spaces.

Our main result links the Szlenk index of a Banach space and the UKK* renormability of its dual in the general case.

Theorem 1.1 *Let X be a Banach space with $Sz(X) \leq \omega$. Then there is an equivalent norm on X such that the dual norm on X^* is UKK* with modulus of power type $\theta(\varepsilon) = c\varepsilon^p$.*

This answers a question of Huff [6] about nearly uniformly convex renorming of reflexive spaces. In particular, using duality [12, Theorem 2.4] a reflexive Banach space X is *nearly uniformly smooth* renormable if, and only if, $Sz(X) \leq \omega$. For a general Banach space, the condition $Sz(X) \leq \omega$ implies that X is asymptotically uniformly smoothable, see [7] for the definition and properties.

The arguments to prove Theorem 1.1 are very different of the proofs in the separable case [8, 5]. Here the UKK* norm is obtained as a sum of Minkowski functionals of equivalent balls obtained by a suitable slicing of the original unit ball. The process of the proof can be easily understood with pictures if the Kuratowski measure of non compactness is replaced by the diameter.

2 Slow slicing in finitely many steps

In order to simplify the reading, along this section X will be a dual Banach space and the predual will be denoted F and identified as a subspace of X^*. Topological notions (open, closed, closure, compact, limit) are always referred to the weak* topology. The family of open halfspaces, $\{x \in X : f(x) > a\}$ with $f \in F$, is denoted $\mathcal{H}$. A slice of $A \subset X$ is a subset of the form $A \cap H$. Dealing with a bounded set A and $f \in F$, the following notation will be useful: $\sup\{f, A\} = \sup\{f(x) : x \in A\}$ and

$$S(A, f, \xi) = \{x \in A : f(x) > \sup\{f, A\} - \xi\}$$

Kuratowski's measure of non-compactness is denoted α. Recall that $\alpha(A) < \varepsilon$ means that A is covered by finitely many sets of diameter less than ε. Fixing a closed convex set $B \subset X$, the "slow slicing" set derivation is defined for any bounded subset $A \subset X$ as

$$^{B}[A]'_\varepsilon = \{x \in A : \forall\, S(A, f, \xi) \ni x \,\&\, S(A, f, 2\xi) \cap B = \emptyset \Rightarrow \alpha(S(A, f, 2\xi)) \geq \varepsilon\}$$

For any ordinal $n \in \mathbb{N}$, the sets $^{B}[A]^n_\varepsilon$ are defined in the obvious way. If $B = \emptyset$ we simply write $[A]'_\varepsilon$. Using compactness, it is easy to see that $\alpha(A \cap H) < \varepsilon$ for any $H \in \mathcal{H}$ with $[A]'_\varepsilon \cap \overline{H} = \emptyset$. Notice that $\alpha(A) < \varepsilon$ whenever A is compact and $\langle A\rangle'_\varepsilon = \emptyset$. The falsity of the last two facts for the weak topology are the main handicaps to extend our results to non dual Banach spaces.

Lemma 2.1 *Suppose that $[C]'_\varepsilon \subset D$, then $[C + Q]'_{\varepsilon+\alpha(Q)} \subset D + Q$.*

Proof. The reader can easily check that if $S(C + Q, f, \xi) \cap (D + Q) = \emptyset$ then $S(C, f, \xi) \cap D = \emptyset$, and

$$S(C + Q, f, 2\xi) \subset S(C, f, 2\xi) + Q$$

implying the statement. ■

The following technical lemma use ideas from [4], where the authors gave a version of the so called Bourgain-Namioka Lemma, see [2, Theorem 3.4.1], for the Kuratowski measure of non compactness.

Lemma 2.2 *Given $A, B \subset X$ closed convex with $\alpha(A) \leq \varepsilon \leq 1$ and $\alpha(B) \leq 1$, $f \in F$ with $\sup\{f, A\} > \sup\{f, B\}$ and $\delta \in (0, 2)$. Then the sequence (A_n) defined by $A_1 = \overline{conv}(A \cup B)$ and $A_{n+1} = \overline{conv}((^{B}[A_n]'_{\varepsilon+\delta} \cap A) \cup B)$ verifies*

$$\sup\{f, A_n\} - \sup\{f, B\} \leq (1 - \frac{\delta}{4})^{n-1}(\sup\{f, A\} - \sup\{f, B\})$$

Proof. It is enough to prove

$$\sup\{f, A_2\} - \sup\{f, B\} \leq (1 - \frac{\delta}{4})(\sup\{f, A_1\} - \sup\{f, B\})$$

Take

$$D = \{(1 - \lambda)y + \lambda z : y \in A, z \in B, \lambda \in [\frac{\delta}{2}, 1]\}$$

If $x \in A_1 \setminus D$ then $x = (1 - \lambda)y + \lambda z$ with $y \in E$, $z \in B$ and $\lambda \in [0, \frac{\delta}{2}]$. Since $x - y = \lambda(z - y)$, we have

$$A_1 \setminus D \subset A + \lambda(B - A)$$

and so

$$\alpha(A_1 \setminus D) \leq \alpha(A) + 2\lambda \leq \varepsilon + \delta$$

We have

$$\sup\{f, D\} \leq (1 - \frac{\delta}{2})\sup\{f, A\} + \frac{\delta}{2}\sup\{f, B\}$$

If $\xi = 2^{-1}(\sup\{f, A\} - \sup\{f, D\})$ then

$$S(A_1, f, 2\xi) \cap B = \emptyset \ \text{ and } \ \alpha(S(A_1, f, 2\xi)) < \varepsilon + \delta.$$

Therefore ${}^B[A_1]'_{\varepsilon+\delta} \cap S(A_1, f, \xi) = \emptyset$, that is

$$\sup\{f, {}^B[A_1]'_{\varepsilon+\delta}\} \leq \sup\{f, A_1\} - \xi$$

$$\leq \frac{1}{2}\sup\{f, A\} + \frac{1}{2}\sup\{f, D\} \leq +(1 - \frac{\delta}{4})\sup\{f, A\} + \frac{\delta}{4}\sup\{f, B\}$$

and thus

$$\sup\{f, {}^B[A_1]'_{\varepsilon+\delta}\} - \sup\{f, B\} \leq (1 - \frac{\delta}{4})(\sup\{f, A\} - \sup\{f, B\})$$

and the proof is finished. ∎

We say that a pair $D \subset C$ of closed convex subsets of X is (n, ε)-*admissible* if $\langle \overline{D \cap H} \rangle_\varepsilon^n = \emptyset$ for every $H \in \mathcal{H}$ with $D \cap \overline{H} = \emptyset$. In all that follows $0 < \varepsilon \leq 1$.

Lemma 2.3 *There exists $N(n, \delta) \in \mathbb{N}$ for $n \in \mathbb{N}$ and $\delta \in (0, 1)$ such that*

$$\sup\{f, {}^D[C]_{\varepsilon+\delta}^{N(n,\delta)}\} - \sup\{f, D\} \leq \frac{1}{2}(\sup\{f, C\} - \sup\{f, D\})$$

whenever $D \subset C$ is an (n, ε)-admissible pair with $\alpha(C) \leq 1$ and every $f \in F$ with $\sup\{f, C\} \geq \sup\{f, B\}$.

Proof. We shall need the following fact: the inequality in the thesis implies that

$${}^D[C]_{\varepsilon+\delta}^{mN(n,\delta)} \subset D + 2^{-m}Q$$

for every $m \in \mathbb{N}$ if $C \subset D + Q$. It is enough to check it for $m = 1$. Indeed, assume it is not true and find $f \in F$ such that

$$\sup\{f, {}^D[C]_{\varepsilon+\delta}^{N(n,\delta)}\} > \sup\{f, D + 2^{-1}Q\} = \sup\{f, D\} + 2^{-1}\sup\{f, Q\}$$

on the other hand

$$\sup\{f, {}^D[C]_{\varepsilon+\delta}^{N(n,\delta)}\} \leq \sup\{f, D\} + 2^{-1}(\sup\{f, C\} - \sup\{f, D\})$$

$$= 2^{-1}(\sup\{f, C\} + \sup\{f, D\}) \leq 2^{-1}(\sup\{f, D\} + \sup\{f, Q\} + \sup\{f, D\})$$

$$= \sup\{f, D\} + 2^{-1}\sup\{f, Q\}$$

which is a contradiction.

The proof will be by induction on n for all the (n, ε)-admissible pairs of closed convex sets $D \subset C$ with $\alpha(C) \leq 1$ and every $\delta \in (0, 1)$. For $n = 1$ we may take $N(1, \delta) = 2$ for any δ. Assume that the result is true for n and let $D \subset C$ an $(n + 1, \varepsilon)$-admissible pair and $f \in F$ with $\sup\{f, C\} > \sup\{f, D\}$.

Fix $p \in \mathbb{N}$ such that $(1 - \frac{\delta}{8})^{p-1} < \frac{1}{8}$ and $q \in \mathbb{N}$ such that $2^{-q} < \frac{\delta}{4p}$.

Given $f \in F$ such that $\sup\{f, C\} > \sup\{f, D\}$, take $Q = C - \{x_0\}$ where $x_0 \in D$ and $f(x_0) = \sup\{f, B\}$. That implies $C \subset B + Q$ and $\sup\{f, Q\} = \sup\{f, C\} - \sup\{f, B\}$. Take $a \in \mathbb{R}$ such that

$$\sup\{f, D\} < a < \sup\{f, D\} + 8^{-1} \sup\{f, Q\}$$

and $H = \{x : f(x) > a\}$. We may assume without loss of generality that $\langle \overline{C \cap H} \rangle_\varepsilon^n \neq \emptyset$. Since $\langle \overline{C \cap H} \rangle_\varepsilon^{n+1} = \emptyset$, by compactness we have $\alpha(\overline{C \cap H}) < \varepsilon$. Let $(A_k)_{k=1}^p$ the sets given by Lemma 2.2 for $A = \overline{conv}(\langle \overline{C \cap H} \rangle_\varepsilon^n)$, $B = C \setminus H$ and eating with α-size $\varepsilon + \frac{\delta}{2}$. We have

$$\sup\{f, A_p\} < \sup\{f, B\} + 8^{-1} \sup\{f, Q\} < \sup\{f, D\} + 4^{-1} \sup\{f, Q\}$$

The pair of sets $A_{k+1} \subset {}^B[A_k]'_{\varepsilon+\delta/2}$ is (n, ε)-admissible. Indeed, if $H \cap A_{k+1} = \emptyset$ then

$$\langle H \cap {}^B[A_k]'_{\varepsilon+\delta/2} \rangle_\varepsilon^n = \emptyset$$

since $H \cap {}^B[A_k]'_{\varepsilon+\delta/2} \cap A = \emptyset$ by definition of the sets A_k. Using the induction hypothesis and the fact above we have

$$^D[A_k]_{\varepsilon+\delta/2}^{1+qN(n,\delta/2)} \subset {}^{A_{k+1}}\left[{}^B[A_k]'_{\varepsilon+\delta/2}\right]_{\varepsilon+\delta/2}^{qN(n,\delta/2)} \subset A_{k+1} + 2^{-q} Q$$

Put $C_0 = C$ and $C_k = A_k + k2^{-q}Q$. Then

$$^D[C_k]_{\varepsilon+\delta}^{1+qN(n,\delta/2)} \subset {}^D[C_k]_{\varepsilon+\delta/2+k2^{-q}}^{1+qN(n,\delta/2)} \subset C_{k+1}$$

We have

$$^D[C]_{\varepsilon+\delta}^{p+pqN(n,\delta/2)} \subset A_p + 4^{-1} Q$$

If we take $N(n+1, \delta) = p + pqN(n, \delta/2)$ then

$$\sup\{f : {}^D[C]_{\varepsilon+\delta}^{N(n+1,\delta)}\} < \sup\{f, D\} + 2^{-1} \sup\{f, Q\}$$

which finishes the proof of the lemma. ■

The Szlenk index is submultiplicative [10]. The next result shows that is possible to define a submultiplicative dentability index.

Proposition 2.4 *Suppose that $Sz(F) \leq \omega$, then for every $\varepsilon \in (0, 1)$ there is a least $N(\varepsilon) \in \mathbb{N}$ with the property*

$$S(C, f, \xi) \cap {}^B[C]_\varepsilon^{N(\varepsilon)} = \emptyset$$

whenever $B \subset C$ are closed convex subsets of X, $f \in F$, $B \cap S(C, f, 2\xi) = \emptyset$ and $\alpha(S(C, f, 2\xi)) \leq 1$. Moreover, $N(\varepsilon)$ is a submultiplicative function, that is, $N(\varepsilon_1 \varepsilon_2) \leq N(\varepsilon_1) N(\varepsilon_2)$.

Proof. Fix $\varepsilon_1 \in (0,1)$. First we shall show the existence of $N(\varepsilon_1)$. Indeed, if $n = Sz(F)_{\frac{\varepsilon_1}{3}}$, then $\langle A\rangle^n_{\frac{2\varepsilon_1}{3}} = \emptyset$ whenever A is closed with $\alpha(A) \leq 1$. Taking $D = \{x \in C : f(x) = 1\}$ we have that the pair $D \subset \overline{S(C,f,2\xi)}$ is $(n, \frac{2\varepsilon_1}{3})$-admissible. Putting $\varepsilon = \frac{2\varepsilon_1}{3}$ and $\delta = \frac{\varepsilon_1}{3}$ in Lemma 2.3 we have

$$^D[\overline{S(C,f,2\xi)}]^m_{\varepsilon_1} \cap S(C,f,\xi) = \emptyset$$

if $m = N(n, \frac{2\varepsilon_1}{3})$ since $\alpha(S(C,f,2\xi)) \leq 1$. Notice that

$$^B[C]^m_{\varepsilon_1} \cap S(C,f,2\xi) \subset {}^D[\overline{S(C,f,2\xi)}]^m_{\varepsilon_1}$$

implying $^B[C]^m_{\varepsilon_1} \cap S(C,f,\xi) = \emptyset$ and so the existence of $N(\varepsilon_1)$.
Take $\varepsilon_2 \in (0,1)$. In order to show that $N(\varepsilon_1\varepsilon_2) \leq N(\varepsilon_1)N(\varepsilon_2)$ it is enough to prove that $^B[C]^{N(\varepsilon_2)}_{\varepsilon_1\varepsilon_2} \subset {}^B[C]'_{\varepsilon_1}$. If $x \in C \setminus {}^B[C]'_{\varepsilon_1}$, then for some slice $x \in S(C,f,\xi)$ and $\alpha(S(C,f,2\xi)) \leq \varepsilon_1$. By scaling, it is clear that $^B[C]^{N(\varepsilon_2)}_{\varepsilon_1\varepsilon_2} \cap S(C,f,\xi) = \emptyset$. Therefore $x \notin {}^B[C]^{N(\varepsilon_2)}_{\varepsilon_1\varepsilon_2}$ and the proof is complete. ∎

3 Construction of the UKK* norm

We say that a net (x_ϖ) is ε-separated if $\|x_{\varpi_1} - x_{\varpi_2}\| > \varepsilon$ for $\varpi_1 \prec \varpi_2$. The next result applied to the unit ball of a dual Banach space provides a characterization of the UKK* property using nets. Huff's definition [6] of the UKK* property consists in replacing nets by sequences.

Lemma 3.1 *Given $A \subset X$ and $\delta > 2\varepsilon > 0$ then*

$$\langle A\rangle'_\delta \subset \{x \in A : \exists (x_\varpi) \subset A,\ \varepsilon\text{-separated},\ x = \lim_\varpi x_\varpi\} \subset \langle A\rangle'_\varepsilon$$

Proof. The second inclusion is easy, so we shall give the arguments for the second one. Fix $x \in \langle A\rangle'_{2\varepsilon}$ and $\mathcal{B}$ a local basis at x of the topology. The set Ω will be the finite subsets of $\mathcal{B}$ directed by inclusion. For $\varpi \in \Omega$ take $U_\varpi = V_1 \cap \ldots \cap V_n$ if $\varpi = \{V_1, \ldots, V_n\}$. It is obvious that the net (x_ϖ) converges to x for any choice $x_\varpi \in U_\varpi$. We shall build by induction on $\#(\varpi)$ an ε-separated net (x_ϖ) with the property $\|x_\varpi - x\| > \varepsilon$. If $\#(\varpi) = 1$, as $\operatorname{diam}(A \cap U_\varpi) \geq \delta$, we may find $x_\varpi \in U_\varpi$ with $\|x_\varpi - x\| > \varepsilon$. Assume now that $x_{\varpi'}$ is built if $\#(\varpi') < \#(\varpi)$ and consider

$$U = U_\varpi \setminus \bigcup_{\varpi' \prec \varpi} B[x_{\varpi'}, \varepsilon]$$

which is a neighborhood of x. Since $\operatorname{diam}(A \cap U) \geq \delta$, it is possible to find $x_\varpi \in A \cap U$ with $\|x_\varpi - x\| > \varepsilon$. That finishes the proof of the first set inclusion. ∎

Consider the following set derivation

$$(A)'_\varepsilon = \{x \in A : \forall\, H \in \mathcal{H},\ x \in H,\ \alpha(A \cap H) \geq \varepsilon\}$$

The reader can check without difficulty that

$$\langle B_X\rangle'_{2\varepsilon} \subset (B_X)'_\varepsilon \subset \overline{conv}(\langle B_X\rangle'_\varepsilon)$$

As a consequence, the norm of X is UKK* if, and only if, $(B_X)'_\varepsilon \subset (1-\theta(\varepsilon))B_X$ with some $\theta(\varepsilon) > 0$. The convex Szlenk indices are introduced in [5] by

$$Cz(F)_\varepsilon = \inf\{\gamma : (B_X)^\gamma_\varepsilon = \emptyset\}$$

and $Cz(F) = \sup_{\varepsilon>0} Cz(F)_\varepsilon$, where F is the predual of X.

Proposition 3.2 *Assume that $Sz(F) \le \omega$. Then $Cz(F)_\varepsilon < c\varepsilon^{-p}$ for some positive constants $c, p > 0$, in particular $Cz(F) \le \omega$.*

Proof. For every convex set C we have $(C)'_\varepsilon \subset [C]'_\varepsilon$. Since $\alpha(B_X) = 2$, we obtain that $(B_X)^{N(\varepsilon/2)}_\varepsilon = \emptyset$. It is well know that the submultiplicativity implies $N(\varepsilon) \le k\varepsilon^{-p}$ for some $p > 0$ and $k > 0$, following the power bound for $Cz(F)_\varepsilon$.■

For a Banach space with $Sz(F) \le \omega$, Lancien deduced in [10] from the submultiplicativity of the Szlenk index that $Sz(F)_\varepsilon < c\varepsilon^{-p}$ for some positive constants. In [5] the authors prove that $Sz(F)$ and $Cz(F)$ have the same power type among the separable Banach spaces.

Proof of Theorem 1.1. Let X be a dual Banach space with predual F. Taking $N_n = N(2^{-n-1})$, we know by Proposition 2.4 that

$$\frac{1}{3}B_X \subset {}^{\frac{1}{3}B_X}[B_X]^{N_n}_{2^{-n}} \subset \frac{2}{3}B_X$$

Take $C_{n,m} = {}^{\frac{1}{3}B_X}[B_X]^m_{2^{-n}}$ for $1 \le m \le N_n$ and $C_{n,0} = B_X$. Let $f_{n,m}$ be the Minkowski functional of $C_{n,m}$. We have $\|x\| \le f_{n,m}(x) \le 3\|x\|$. Consider the equivalent dual norm defined by

$$|||x||| = \|x\| + \frac{1}{15}\sum_{n=1}^{\infty}\frac{n^{-2}}{N_m+1}\sum_{m=0}^{N_m} f_{n,m}(x)$$

Clearly we have $\|x\| \le |||x||| \le \frac{4}{3}\|x\|$. Let $B_{|||.|||}$ denote the unit ball of the norm $|||.|||$. We shall prove that $|||.|||$ is UKK* with modulus of power type. By Lemma 3.1 we can use nets for that aim. Given $\varepsilon > 0$ take $n \in \mathbb{N}$ with $2^{-n} < \frac{\varepsilon}{3} \le 2^{-n+1}$. Let $(x_\varpi) \subset B_{|||.|||}$ an ε-separated net weak* converging to x. Without loss of generality we may assume that $|||x_\varpi||| = 1$ and $\lim_\varpi f_{n,m}(x_\varpi)$ exists for every $0 \le m \le N_n$. Fix m and consider

$$y_\varpi = f_{n,m}(x_\varpi)^{-1}x_\varpi$$

we have that (y_ϖ) is 2^{-n}-separated for ϖ large enough and so

$$\lim_\varpi y_\varpi = \lim_\varpi f_{n,m}(x_\varpi)^{-1}\, x \in C_{n,m+1}$$

therefore

$$f_{n,m+1}(x) \le \lim_\varpi f_{n,m}(x_\varpi)$$

Observe that

$$\sum_{m=0}^{N_m} f_{n,m}(x) \leq f_{n,0}(x) + \lim_{\varpi} \sum_{m=0}^{N_m-1} f_{n,m}(x_\varpi) \leq \lim_{\varpi} \sum_{m=0}^{N_m} f_{n,m}(x_\varpi) - \frac{1}{8}$$

since $f_{n,0}(x) = \|x\| \leq 1$ and $f_{n,N_n}(x_\varpi) \geq \frac{3}{4}\frac{3}{2}$. That implies

$$|\!|\!|x|\!|\!| \leq 1 - \frac{n^{-2}}{120(N_n+1)} \leq 1 - \frac{(1-\log_2(\varepsilon/3))^{-2}}{120(N(\varepsilon/6)+1)}$$

If $N(\varepsilon) \leq k\varepsilon^{-q}$, then for any $p > q$ there is some $c > 0$ such that

$$|\!|\!|x|\!|\!| \leq 1 - c\varepsilon^p$$

therefore we may take $\theta(\varepsilon) = c\varepsilon^p$ as we wanted. ■

The exponent in the modulus of power type verifies that $p \in [1, +\infty)$. Indeed, assume that an infinite-dimensional dual space X has a norm with $\theta(\varepsilon) = c\varepsilon^p$ where $0 < p < 1$. Since $B_X \setminus (B_X)'_\varepsilon$ contains a ball B of diameter $\theta(\varepsilon)$, then $c\varepsilon^p \leq 2\alpha(B) < 2\varepsilon$ which is not true for $\varepsilon > 0$ small enough.

The method of "slow slicing" and the subsequent construction of the norm can be used to prove the Enflo-Pisier theorem about uniformly convex renorming of superreflexive spaces [11]. That will appear in [14] among some other results.

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Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, SPAIN
E-mail: matias@um.es