# Compact spaces of Szlenk index $\omega$

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*Dedicated to Gabriel Vera on the occasion of his retirement*

**Abstract**

The Szlenk index has found many applications in the isomorphic theory of Banach spaces. Its definition is based in some kind of interplay between a weak topology and the norm metric with not much care on the linear structure. There is no obstacle to consider the notion of Szlenk index in more general settings. In this paper we study the compact spaces of Szlenk index $\omega$ at most with respect to an associated metric. We include new applications to Banach spaces of the these methods, where the estimations of the growth speed of the finite Szlenk indices play a fundamental role.

## 1 Introduction

Consider $X$ a topological space together with a pseudometric $d$ defined on it which may be not related to the topology of $K$. We say that $X$ is fragmented by $d$ if for every nonempty subset $A \subset X$ and every $\varepsilon > 0$ there is $U \subset X$ open such that $A \cap U \neq \emptyset$ and $\operatorname{diam}(A \cap U) < \varepsilon$, where 'diam' is the diameter measured with respect to $d$. For any subset $A \subset X$ of a topological space with an associated pseudometric $d$ we define a set derivation

$$\langle A \rangle'_\varepsilon = \{x \in A : \forall U \text{ neighbourhood of } x, \operatorname{diam}(A \cap U) \geq \varepsilon\}.$$

By iteration, the sets $\langle A \rangle^\gamma_\varepsilon$ are defined for any ordinal $\gamma$, taking intersection in the case of limit ordinals. The Szlenk indices of the space $X$ with respect to $d$ are ordinal numbers defined by

$$Sz(X, \varepsilon) = \inf\{\gamma : \langle X \rangle^\gamma_\varepsilon = \emptyset\}$$

and $Sz(X) = \sup_{\varepsilon>0} Sz(X, \varepsilon)$. If $X$ is fragmented by $d$, the Szlenk indices always exist. If it not the case, for some $\varepsilon > 0$ there is an ordinal $\gamma$ such that $\langle X \rangle^\gamma_\varepsilon = \langle X \rangle^{\gamma+1}_\varepsilon \neq \emptyset$. Then we put $Sz(X, \varepsilon) = \infty$ and $Sz(X) = \infty$ with the agreement that any ordinal number is less than $\infty$.

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At this moment we have to point out that the original definition of Szlenk index is intended for Banach spaces. The Szlenk index of an Asplund Banach space $X$ is defined dually as the Szlenk index (in the above sense) of its dual ball $B_{X^*}$ endowed with the weak$^*$ topology and the norm metric. It follows that the Szlenk index so defined for the space $X$ is an isomorphic invariant of $X$. See [21] for the original sequential definition and [12, 6] for an account of properties of the Szlenk indices on Banach spaces.

This paper is concerned with compact spaces $K$ such that $Sz(K) \leq \omega$ for a finer metric. Notice that in such a case, for every $\varepsilon > 0$, the indices $Sz(K, \varepsilon)$ must be finite. The study of the variation of $Sz(K, \varepsilon)$, as a function of $\varepsilon$, is one the aims in this paper. Our self limitation to compact spaces is motivated by the fact that compactness is an essential hypothesis in most of the main results, and simplifies many of the arguments in the proofs. Although some of the results can be stated with more general hypothesis, our examples are mainly compact spaces and weak$^*$ compact subsets of dual Banach spaces.

The class of compact spaces of Szlenk index at most $\omega$ includes trivially the metrizable compacta if we take any compatible metric $d$. Examples far away from metrizability are provided by the scattered compacta with finite Cantor-Bendixon index together with the discrete metric. Geometrically, the nicest examples of compacta with Szlenk index at most $\omega$ with respect to the norm metric are the balls of superreflexive Banach spaces endowed with the weak topology. This a consequence of the uniformly convex renorming, and in the particular case of $\ell_p$ spaces we have that $Sz(B_{\ell_p}, \varepsilon)$ asymptotically behaves like $\varepsilon^{-p}$, see Example 4.10. Notice that the power type of the uniform convexity modulus of $\ell_p$ is 2 for every $p \in (1, 2]$, implying that the Szlenk index is more sensitive than the *dentability index*, see [12] for this definition.

From the Szlenk index point of view, uniformly convex spaces are a particular case of dual Banach spaces with a *weak$^*$ uniformly Kadec-Klee norm.* The norm of a dual Banach space $X^*$ is UKK$^*$ if for every $\varepsilon > 0$ there is $\theta(\varepsilon) \in (0, 1)$ such that

$$\langle B_{X^*} \rangle'_\varepsilon \subset (1 - \theta(\varepsilon)) B_{X^*},$$

where the distance associated to the derivation is the norm distance. Notice that using iteration on the set inclusion it is easy to deduce that $Sz(B_{X^*}, \varepsilon)$ is finite for every $\varepsilon > 0$ and so $Sz(B_{X^*}) \leq \omega$. In our paper [19] we have shown the converse, that is, if $Sz(B_{X^*}) \leq \omega$ then there exists an equivalent UKK$^*$ dual norm on $X^*$ (previously it was done in [9] with separability assumptions). Here we are able to give the following improvement

**Theorem 1.1** *Let $K \subset X^*$ be a weak$^*$ compact of Szlenk index $\omega$ at most. Then there exists $B \subset X^*$ weak$^*$ compact convex symmetric with $K \subset B$ such that for every $\varepsilon > 0$ there is $\theta(\varepsilon) \in (0, 1)$ such that*

$$\langle B \rangle'_\varepsilon \subset (1 - \theta(\varepsilon)) B.$$

*Moreover, $\theta(\varepsilon)$ can be taken of the form $a\,\varepsilon^p Sz(K, b\,\varepsilon)^{-1}$ for any $p > 1$ and some constants $a, b > 0$.*

A straightforward consequence is that the class of weak$^*$ compact subsets of Szlenk index at most $\omega$ is stable by convex hulls. Further, given $K$ a compact space together with a metric $d$ such that $Sz(K) \leq \omega$, we shall prove that the Radon probabilities on $K$ (with a suitable extension of the metric) is also of Szlenk index at most $\omega$, see Corollary 4.6.

We shall begin our study considering the compact spaces of Szlenk index at most $\omega$ in an abstract topological context, Section 2. If the metric is skipped, then a compact space of Szlenk index $\omega$ at most is simply a descriptive compact space, see Theorem 2.8. For that reason, we shall pay special attention to the estimation of the finite Szlenk indices of new compact spaces obtained by basic operations from compact spaces of Szlenk index $\omega$ at most. In Section 3 the compacts are placed in a locally convex vector space and the metric is induced by a norm. In this setting we shall study the pass to the closed convex hull of the Szlenk indices, obtaining Theorem 1.1 as a consequence. Along the last section we shall give some applications to Banach spaces. The possibility of obtaining good estimations from below of the Szlenk index of the unit ball of a Banach space is related to its *asymptotic uniform smoothness*. All these techniques are illustrated with some examples around the classical $\ell_p$ spaces.

# 2 Basic properties

This section deals with the abstract topological setting. All the compact spaces are Hausdorff and the associated pseudometric is understood in the auxiliary results. The first result of this characterizes the property $Sz(K, \varepsilon) < \omega$ without any mention to the order imposed by the derivation process.

**Theorem 2.1** *Given $K$ a compact space together with an associated pseudometric $d$ and $\varepsilon > 0$, the following are equivalent:*

*i)* $\langle K \rangle_\varepsilon^n = \emptyset$.

*ii) There exists closed subsets $A_j \subset K$ with $1 \leq j \leq n$ such that for every $x \in K$, there is $j$ and $U \subset K$ open such that $x \in A_j \cap U$ and $diam(A_j \cap U) < \varepsilon$.*

*If $d$ is moreover lower semicontinuous, the following is also equivalent:*

*iii) There exists a partition $K = \bigcup_{j=1}^n D_j$ such that for every $j$ and $x \in D_j$, there is and $U \subset K$ open such that $x \in U$ and $diam(D_j \cap U) < \varepsilon$.*

**Proof.** If *i*) holds the reader can verify with no difficulty that *ii*) is satisfied with the sets $A_j = \langle K \rangle_\varepsilon^{j-1}$ and *iii*) is satisfied with the sets $D_j = \langle K \rangle_\varepsilon^{j-1} \setminus \langle K \rangle_\varepsilon^j$. Assume *ii*) is true and define sets

$$B_i = \{x \in K : \#\{j : x \in A_j\} \geq i\}.$$

It is clear that every $B_i$ is closed, $B_1 = K$ and $B_i = \emptyset$ if $i > n$. We claim that $\langle B_i\rangle'_\varepsilon \subset B_{i+1}$. Indeed, if $x \in B_i \setminus B_{i+1}$ there is a set of indices $S_0$ with $\#(S_0) = i$ such that $x \in \bigcap_{j\in S_0} A_j$ and for some $j_o \in S_0$ there is $U$ open with $x \in A_{j_o} \cap U$ and $\mathrm{diam}(A_{j_o} \cap U) < \varepsilon$. It is easy to check that

$$V = U \setminus \bigcup\{\bigcap_{j\in S} A_j : S \neq S_0; \#(S) = i\}$$

is a neighborhood of $x$. The fact that $\mathrm{diam}(B_i \cap V) < \varepsilon$ follows from

$$B_i \cap V \subset (\bigcap_{j\in S_0} A_j) \cap U \subset A_{j_o} \cap U.$$

After the claim is proved, clearly we have $\langle K\rangle^n_\varepsilon = \emptyset$.
If the pseudometric $d$ is lower semicontinuous, then closures keep the diameter. Assuming that *iii*) holds, it is easy to check that $A_j = \overline{D_j}$ satisfies condition *ii*).■

**Corollary 2.2** *A scattered compact $K$ verifies $K^{(\omega)} = \emptyset$ if and only if it is a finite union of relatively discrete subsets.*

A slight modification of the derivation process has some advantages when establishing quantitative results. We shall use the following version of *Kuratowski's measure of noncompactness*

$$\alpha(A) = \inf\{\max_{1\le i\le n} diam(A_i) : n \in \mathbb{N}, A_i \subset K \text{ closed } A \subset \bigcup_{i=1}^{n} A_i\},$$

the corresponding set derivation

$${}^{\mathfrak{K}}\langle K\rangle'_\varepsilon = \{x \in A : \forall U \text{ neighbourhood of } x, \alpha(A \cap U) \geq \varepsilon\}$$

and ordinal indices $Sk(K,\varepsilon)$ and $Sk(K)$ defined after iteration in the obvious way. Next lemma shows that the derivation using Kuratowski's measure is equivalent to the one defined in the introduction and to the 'net'-derivation, which is a modification of the 'sequence'-derivation used in the original definition of the Szlenk index.

**Lemma 2.3** *For every $A \subset K$ closed, $\varepsilon > 0$ and $\delta > 2\varepsilon$ we have*

$$\langle A\rangle'_\delta \subset \{x \in A : \exists (x_\varpi) \subset A,\, d(x_\varpi, x) > \varepsilon,\, x = \lim_\varpi x_\varpi\} \subset {}^{\mathfrak{K}}\langle A\rangle'_\varepsilon \subset \langle A\rangle'_\varepsilon$$

*and therefore $Sz(K,\delta) \leq Sk(K,\varepsilon) \leq Sz(K,\varepsilon)$.*

**Proof.** We shall prove the inclusions starting from the left side. If $x \in \langle A\rangle'_\delta$ then $x \in \overline{A \setminus B[x,\varepsilon]}$. Indeed, if not then $A \setminus \overline{A \setminus B[x,\varepsilon]}$ would be a neighborhood of $x$ of diameter at most $2\varepsilon$ which contradicts the choice of $x$. Therefore is possible to take $(x_\varpi) \subset A \setminus B[x,\varepsilon]$ converging to $x$. For the second inclusion, if $x \notin {}^{\mathfrak{K}}\langle A\rangle'_\varepsilon$ then there is a neighborhood of $x$ covered by finitely many compacts $K_i$ of diameter less than $\varepsilon$. If a net $(x_\varpi)$ is converging to $x$, eventually it will be in the neighborhood and for some $i$ the set $\{\varpi : x_\varpi \in K_i\}$ will be cofinal. Therefore $x \in K_i$ and then $d(x_\varpi, x) \leq \varepsilon$. The third and last inclusion just follows from regularity of the topology. ■

This result shows one of the advantages of the derivation with Kuratowski's measure of non compactness.

**Lemma 2.4** *Let $A_1, \ldots, A_n \subset K$ be closed sets, then ${}^{\mathfrak{K}}\langle \bigcup_{i=1}^n A_i \rangle'_\varepsilon = \bigcup_{i=1}^n {}^{\mathfrak{K}}\langle A_i \rangle'_\varepsilon$. If moreover $A_i \cap A_j \subset \langle A_i \rangle'_\varepsilon$ for every $i \neq j$, then $\langle \bigcup_{i=1}^n A_i \rangle'_\varepsilon = \bigcup_{i=1}^n \langle A_i \rangle'_\varepsilon$.*

**Proof.** The first part is left to the reader. For the second one, notice that one of the set inclusions is obvious. For the other observe that

$$x \in \bigcup_{i=1}^n A_i \setminus \bigcup_{i=1}^n \langle A_i \rangle'_\varepsilon$$

implies that $x \in A_i \setminus \bigcup_{j \neq i} A_j$ for some $1 \leq i \leq n$. Otherwise $x \in A_i \cap A_j \subset \langle A_i \rangle'_\varepsilon$, which is a contradiction. Take $U \ni x$ open such that $diam(A_i \cap U) < \varepsilon$, define $V = U \setminus \bigcup_{j \neq i} A_j$, and observe

$$x \in (\bigcup_{j=1}^n A_j) \cap V \subset A_i \cap U.$$

Therefore $x \in \bigcup_{j=1}^n A_j \setminus \langle \bigcup_{j=1}^n A_j \rangle'_\varepsilon$ and so $\langle \bigcup_{i=1}^n A_i \rangle'_\varepsilon \subset \bigcup_{i=1}^n \langle A_i \rangle'_\varepsilon$. ■

The class of subsets of Szlenk index $\omega$ at most of a given compact together with an associated metric is obviously stable by closed subsets and by finite unions by the previous lemma. We leave to the reader the easy proof of the next result.

**Proposition 2.5** *Let $K$ be a compact space together with an associated pseudo-metric $d$ and let $A, A_1, \ldots, A_n \subset K$ be closed subsets having Szlenk index $\omega$ at most. Then for any $\varepsilon > 0$:*

*a) If $B \subset A$ is closed, then $Sz(B, \varepsilon) \leq Sz(A, \varepsilon)$.*

*b) $Sk(\bigcup_{i=1}^n A_i, \varepsilon) = \max\{Sk(A_i, \varepsilon) : i = 1, \ldots, n)\}$.*

Another typical operation in topology is the cartesian product. The following is implicitly contained in [15].

**Proposition 2.6** *Let $(K_1, d_1)$ and $(K_2, d_2)$ be compact spaces of Szlenk index $\omega$ at most. Then $(K_1 \times K_2, d)$ where $d = \max\{d_1, d_2\}$ is of Szlenk index $\omega$ at most. Moreover*

$$Sz(K_1 \times K_2, \varepsilon) = Sz(K_1, \varepsilon) + Sz(K_2, \varepsilon) - 1.$$

**Proof.** It is not difficult to show that $K_1 \times K_2$ is of Szlenk index $\omega$ at most by means of the characterization given in Theorem 2.1 but for an exact calculation we need to describe the derivation process. It is easy to show that

$$\langle K_1 \times K_2 \rangle'_\varepsilon = (\langle K_1 \rangle'_\varepsilon \times K_2) \cup (K_1 \times \langle K_2 \rangle'_\varepsilon).$$

Assume $i_1 + j_1 = i_2 + j_2 = n$, $i_1 < i_2$ and $j_1 > j_2$. The following chain of set inclusions

$$\begin{aligned}(\langle K_1\rangle_\varepsilon^{i_1} \times \langle K_2\rangle_\varepsilon^{j_1}) \cap (\langle K_1\rangle_\varepsilon^{i_2} \times \langle K_2\rangle_\varepsilon^{j_2}) &= \langle K_1\rangle_\varepsilon^{i_2} \times \langle K_2\rangle_\varepsilon^{j_1} \\ &\subset (\langle K_1\rangle_\varepsilon^{i_1+1} \times \langle K_2\rangle_\varepsilon^{j_1}) \cap (\langle K_1\rangle_\varepsilon^{i_1} \times \langle K_2\rangle_\varepsilon^{j_2+1}) \\ &\subset \langle\langle K_1\rangle_\varepsilon^{i_1} \times \langle K_2\rangle_\varepsilon^{j_1}\rangle'_\varepsilon \cap \langle\langle K_1\rangle_\varepsilon^{i_2} \times \langle K_2\rangle_\varepsilon^{j_2}\rangle'_\varepsilon\end{aligned}$$

implies that the hypothesis in Lemma 2.4 is fulfilled. Now, there is not obstacle to obtain by recurrence

$$\langle K_1 \times K_2\rangle_\varepsilon^n = \bigcup_{i+j=n} \langle K_1\rangle_\varepsilon^i \times \langle K_1\rangle_\varepsilon^j.$$

The estimation of the Szlenk index follows straight as a consequence. ■

**Example 2.7** *Given a left continuous decreasing function $\phi : (0,1] \to \mathbb{N}$, there exists a zero dimensional metrizable compact $K$ together with a lower semicontinuous metric such that $Sz(K,\varepsilon) = \phi(\varepsilon)$ for all $\varepsilon > 0$.*

**Proof.** We may assume that $\lim_{\varepsilon\to 0^+} \phi(\varepsilon) = \infty$, otherwise just stop the following construction at the suitable step. Consider together with a strictly decreasing sequence $(\varepsilon_n) \subset (0,1]$ with $\varepsilon_1 = 1$ and a strictly increasing sequence $(k_n) \subset \mathbb{N}$ such that $\phi(\varepsilon) = k_n$ if $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$. Define a new sequence by $a_1 = k_1$ and $a_n = k_n - k_{n-1} + 1$. Let $K_n = [1, \omega^{a_n - 1}]$ together with the associated metric $d_n = \varepsilon_n d$ where $d$ is the discrete metric. Then $Sz(K_n, \varepsilon) = 1$ in case of $\varepsilon > \varepsilon_n$ and $Sz(K_n, \varepsilon) = a_n$ if $\varepsilon \leq \varepsilon_n$. Consider $\prod_{i=1}^n K_i$ endowed with the product topology and the maximum metric. We claim that $Sz(\prod_{i=1}^n K_i, \varepsilon) = \phi(\varepsilon)$ if $\varepsilon_{n+1} < \varepsilon \leq 1$. The proof will be by induction. For $n = 1$ is clear since $Sz(K_1, \varepsilon) = k_1$, so assume that the hypothesis is true for some $n$. Having in mind

$$\prod_{i=1}^{n+1} K_i = (\prod_{i=1}^{n} K_i) \times K_{n+1},$$

we obtain for $\varepsilon_{n+1} < \varepsilon \leq 1$ using Lemma 2.6 that

$$Sz(\prod_{i=1}^{n+1} K_i, \varepsilon) = \phi(\varepsilon) + Sz(K_{n+1}, \varepsilon) - 1 = \phi(\varepsilon).$$

Now assume that $\varepsilon_{n+2} < \varepsilon \leq \varepsilon_{n+1}$, then we obtain again by Lemma 2.6

$$Sz(\prod_{i=1}^{n+1} K_i, \varepsilon) = k_n + a_{n+1} - 1 = k_{n+1} = \phi(\varepsilon).$$

Finally consider $K = \prod_{i=1}^\infty K_n$ with the maximum metric and just notice that given an arbitrary $\varepsilon > 0$ then $Sz(K, \varepsilon) = Sz(\prod_{i=1}^n K_i, \varepsilon) = \phi(\varepsilon)$ if $n \in \mathbb{N}$ is chosen such that $\varepsilon_{n+1} < \varepsilon$. ■

A compact space $K$ is said to be *descriptive* if its topology has a $\sigma$-isolated network. Let us remind the terminology. Let $\{H_i : i \in I\}$ be a family of subsets of a topological space $(Z, \tau)$. The family is said to be *isolated* if it is discrete in its union endowed with the relative topology, or in other words, if for every $i \in I$ we have

$$H_i \cap \overline{\bigcup_{j \in I \setminus \{i\}} H_j} = \emptyset.$$

If there is a decomposition $I = \bigcup_{n=1}^{\infty} I_n$ such that every family $\{H_i : i \in I_n\}$ is isolated, then the family $\{H_i : i \in I\}$ is said to be $\sigma$-isolated. A family $\mathfrak{N}$ of subsets of $Z$ is said to be a *network* if every open set is a union of members of $\mathfrak{N}$. To see the topological properties of the family of descriptive compacta and its relation to renorming see [14, 16].

**Theorem 2.8** *A compact space is of Szlenk index at most $\omega$ with respect to some finer metric if and only if it is descriptive.*

**Proof.** Let $\tau$ be the topology of $K$ and let $d$ a finer metric such that $(K, d)$ is of Szlenk index $\omega$ at most. Taking the sets $(A_j)$ given by Theorem 2.1 for all $\varepsilon = 1/n$ and arranging into a unique sequence $(A_n)$ that clearly, enjoys the following property that we called $P(\tau, d)$: for every $x \in K$ and $\varepsilon > 0$ there is $U$ open and $n \in \mathbb{N}$ such that $x \in A_n \cap U$ and $\operatorname{diam}(A_n \cap U) < \varepsilon$. Using [14, Theorem 2.2], $d$ has network $\mathfrak{N}$ which is $\sigma$-isolated with respect to $\tau$. Clearly $\mathfrak{N}$ is a $\sigma$-isolated network for $\tau$.
Assume now that $K$ is descriptive. Using again [14, Theorem 2.2] there exists a finer metric $\rho$, closed sets $A_n$ and families $\{U_i : i \in I_n\}$ of open sets, such that the families $\{A_n \cap U_i : i \in I_n\}$ are disjoint and $\{A_n \cap U_i : n \in \mathbb{N}, i \in I_n\}$ is a network for $\rho$, and therefore a network for $\tau$. Define a pseudometric $d_n$ on $K$ by $d_n(x, y) = 0$ if either

$$\{x, y\} \subset K \setminus A_n;$$

$$\{x, y\} \subset A_n \cap U_i \ \text{ for some } \ i \in I_n;$$

$$\text{or } \ \{x, y\} \subset A_n \setminus \bigcup_{i \in I_n} U_i$$

and $d_n(x, y) = 1/n$ in any other case. Considering the Szlenk derivation with respect to $d_n$ for $\varepsilon < 1/n$, it is clear that $\langle K \rangle'_\varepsilon \subset A_n$, $\langle K \rangle''_\varepsilon \subset A_n \setminus \bigcup_{i \in I_n} U_i$ and $\langle K \rangle'''_\varepsilon = \emptyset$. Define $d(x, y) = \max\{d_n(x, y) : n \in \mathbb{N}\}$. It is not difficult to check that $d$ is a finer metric on $K$. On the other hand $(K, d)$ is isometric to the diagonal of $\prod_{k=1}^{\infty}(K, d_k)$. Given $\varepsilon > 0$, for $n \in \mathbb{N}$ such that $1/n < \varepsilon$, notice for the Szlenk index computed with respect to $d$ that

$$Sz\left(\prod_{k=1}^{\infty}(K, d_k), \varepsilon\right) = Sz\left(\prod_{k=1}^{n}(K, d_k), \varepsilon\right) \leq 2n + 1$$

by iterated use of Proposition 2.6, and so $(K, d)$ is of Szlenk index $\omega$ at most. ■

Next result is concerned with the stability of the Szlenk indices by continuous images.

**Proposition 2.9** *Consider two pairs $(K_1, d_1)$ and $(K_2, d_2)$ of compacta together with associated metrics such that there exits a surjection of $K_1$ onto $K_2$ which is continuous and uniformly continuous between the metrics with modulus of uniform continuity $\tau$. Then $Sk(K_2, \tau(\varepsilon)) \leq Sk(K_1, \varepsilon)$ for any $\varepsilon > 0$.*

**Proof.** Let $f : K_1 \to K_2$ be the surjection. It enough to show that

$$^{\mathfrak{K}}\langle f(H)\rangle'_{\tau(\varepsilon)} \subset f(\ ^{\mathfrak{K}}\langle H\rangle'_{\varepsilon})$$

for every closed subset $H \subset K_1$. Indeed, if $x \in f(H) \setminus f(\ ^{\mathfrak{K}}\langle H\rangle'_{\varepsilon})$, then $f^{-1}(x)$ is compact subset of $H$ disjoint with $^{\mathfrak{K}}\langle H\rangle'_{\varepsilon}$. The set $f^{-1}(x)$ can be covered with finitely many open sets $U$ such that $\alpha(H \cap U) < \varepsilon$. Let $V$ be the union of all such open sets, so $V \cap\ ^{\mathfrak{K}}\langle H\rangle'_{\varepsilon} = \emptyset$ and $\alpha(H \cap V) < \varepsilon$. Taking the open set $W = K_2 \setminus f(H \setminus V)$ we have $x \in f(H) \cap W \subset f(H \cap V)$ which implies that $x \notin\ ^{\mathfrak{K}}\langle f(H)\rangle'_{\tau(\varepsilon)}$ since $\alpha(f(H \cap V)) < \tau(\varepsilon)$. ■

The character of being of Szlenk index $\omega$ at most is preserved by continuous surjections which are uniformly continuous with respect to the metrics, but the finite indices may be very different.

**Example 2.10** *Let $\phi_1, \phi_2 : (0, 1] \to \mathbb{N}$ be two surjective decreasing left continuous functions. Let $K_1$ and $K_2$ be the compact spaces associated to these functions provided by the method of Example 2.7. Then $K_1$ and $K_2$ are homeomorphic and uniformly homeomorphic with respect to the associated metrics.*

**Proof.** Both compact are of the form $\prod_{n=1}^{\infty}[1, \omega]$ with a metric of described by $d((x_n), (y_n)) = \varepsilon_m$ if $m \in \mathbb{N}$ is the least of the coordinates where $x_m \neq y_m$, being $(\varepsilon_n)$ a strictly decreasing sequence in $(0, 1]$ with limit 0. The identity map between two metrics of that form is always a uniform homeomorphism. ■

Asking Lipschitz property with respect to the metrics, the fragmentability speed of the image is bounded by the speed of the domain in a more usable form.

**Corollary 2.11** *The compact spaces of Szlenk index at most $\omega$ are preserved by continuous images which are Lipschitz with respect to the metrics. Moreover if $K_2$ is a continuous and Lipschitz image of $K_1$ then*

$$Sk(K_2, \varepsilon) \leq Sk(K_1, \varepsilon/\lambda)$$

*where $\lambda > 0$ is the Lipschitz constant.*

The former result fails for non compact spaces. Just observe that every separable Banach space is a quotient of $\ell_1$.

# 3 Convex hulls

Along this section we shall place our compact $K$ inside a locally convex space $X$, where there is defined a certain norm $\|\cdot\|$. The topological dual space of $X$ will be denoted $X^*$. All the topological notions will always be referred to the topology of $X$, unless explicit mention to a different topology. Furthermore, we shall assume the following technical conditions:

a) $K$ is $\|\cdot\|$-bounded.

b) the closed convex hull in $X$ of $K$ is compact.

c) $\|\cdot\|$ generates a finer topology on bounded sets.

These conditions together with the $\|\cdot\|$-fragmentability of $K$ are not completely independent. For instance, the reader may easily prove that a $\|\cdot\|$-fragmentable convex compact must be $\|\cdot\|$-bounded. On the other hand, the proof of Lemma 3.1 implies the compactness of the closed convex hull of a $\|\cdot\|$-fragmentable compact if $(X, \|\cdot\|)$ is complete. Notice that $\|\cdot\|$ does not need to be lower semicontinuous.

The topological case of a compact together with a finer metric can be reduced to this setting by means of the following 'canonical' construction. Let $L(K,d)$ denote the space of Lipschitz functions on $K$ with respect to the metric $d$ endowed with the norm

$$\|f\|_L = \max\{\|f\|_\infty, L(f)\}$$

where $L(f)$ is the optimal Lipschitz constant for $f$. If $K$ is fragmented by $d$, then Lipschitz functions are universally measurable. Therefore we may define a norm on $C(K)^*$ by the formula

$$\|\mu\|_L^* = \sup\{\int_K f\,d\mu : f \in L(K,d), \|f\|_L \leq 1\}.$$

This norm extends the metric $d$ and generates a topology finer than the weak$^*$ on bounded subsets since any $f \in C(K)$, as $d$-continuous function, can be uniformly approached by Lipschitz functions. If the metric $d$ is lower semicontinuous, then the set $W = B_{L(K,d)} \cap C(K)$ is rich enough to recover $d$ from it. Therefore, if $d$ is lower semicontinuous, we may change $\|\cdot\|_L^*$ by the norm on $C(K)^*$ calculated by taking the supremum on $W$, which has the advantage of being weak$^*$-lower semicontinuous.

The following result appears in [13] for a lower semicontinuous norm.

**Lemma 3.1** *If $K \subset X$ is a $\|\cdot\|$-fragmentable compact, then*

$$\overline{conv}(K) = \overline{conv}^{\|\cdot\|}(K).$$

**Proof.** Given $x \in \overline{\text{conv}}(K)$ there is a Radon probability $\mu$ on $K$ representing $x$. Using fragmentability, for every $\varepsilon > 0$ there is a transfinite sequence $(D_\alpha^\varepsilon)_{\alpha<\gamma}$ of nonempty disjoint measurable sets of $\|\cdot\|$-diameter less than $\varepsilon$ where $\mu$ is additive, that means $\mu(D_\alpha^\varepsilon) = 0$ except for countably many indices, where the measure is concentrated. Pick points $x_\alpha^\varepsilon \in D_\alpha^\varepsilon$. Taking $\varepsilon = 1/n$, define a function $f_n : K \to X$ by $f_n(x) = x_\alpha^{1/n}$ if $x \in D_\alpha^{1/n}$. Since $f_n$ is $\mu$-measurable, bounded and has essentially separable range, it is $\mu$-Bochner integrable in a completion of $(X, \|\cdot\|)$. By construction $f_n$ converges uniformly to the identity $\mathcal{I}$ of $K$, that turns to be $\mu$-Bochner integrable as well. Therefore $x$ is represented by $\mu$ also in Bochner sense and we have

$$x = \int_K \mathcal{I}\, d\mu = \|\cdot\| - \lim_n \int_K f_n\, d\mu \in \overline{\text{conv}}^{\|\cdot\|}(K)$$

since $\int_K f_n\, d\mu = \sum_{\alpha<\gamma} \mu(D_\alpha^{1/n}) x_\alpha$ is a $\|\cdot\|$-convergent series. ■

The following result tells us the good behavior by convex hulls of Kuratowski's measure of non-compactness. The proof avoids the difficulty of dealing with non lower semicontinuous metrics.

**Lemma 3.2** *Let $K_1, \ldots, K_n$ be convex compact sets, $K = conv(K_1 \cup \ldots \cup K_n)$, $\Delta = \{(\lambda_i) \in [0,1]^n : \sum_{i=1}^n \lambda_i = 1\}$ and $A \subset K$ closed. Then*

$$\alpha(A) \leq \sup\{\sum_{i=1}^n \lambda_i\, diam(K_i) : (\lambda_i) \in \Delta,\ A \cap \sum_{i=1}^n \lambda_i\, K_i \neq \emptyset\}.$$

**Proof.** For every point $x \in A$ and every $\varepsilon > 0$ we shall find an open neighborhood $U$ of $x$ such that

$$\alpha(K \cap U) < \max\{\sum_{i=1}^n \lambda_i \operatorname{diam}(K_i) : (\lambda_i) \in \Delta,\ x \in \sum_{i=1}^n \lambda_i\, K_i\} + 2\varepsilon.$$

Indeed, take $M_i = \sup_{x \in K_i} \|x\|$ and consider $\sum_{i=1}^n M_i \lambda_i$ as a function of $(\lambda_i)$. By compactness of $\Delta$, we can find finitely many closed subsets $\Delta_k \subset \Delta$ with $\Delta = \bigcup_{k=1}^m \Delta_k$ such that the oscillation of $\sum_{i=1}^n M_i \lambda_i$ is less than $\varepsilon$ on each $\Delta_k$. Take $d_k = \max\{\sum_{i=1}^n \lambda_i \operatorname{diam}(K_i) : (\lambda_i) \in \Delta_k\}$. With all these choices, it is easy to check that the compact sets

$$C_k = \{\sum_{i=1}^n \lambda_i x_i : (\lambda_i) \in \Delta_k,\ x_i \in K_i\}$$

verify $\operatorname{diam}(C_k) < d_k + \varepsilon$ and $K = \bigcup_{k=1}^m C_k$. For $x \in A$, we can take $U$ as the complement of the compact set $\bigcup_{x \notin C_k} C_k$ and then we have $K \cap U \subset \bigcup_{x \in C_k} C_k$ and thus $\alpha(K \cap U) < \max\{d_k : x \in C_k\} + \varepsilon$. The proof is finished by compactness of $A$ and the arbitrary choosing of $\varepsilon$. ■

The following result uses ideas from [19] to build a somehow UKK-function, but here is just an intermediate step towards a much better result, Theorem 3.8. A similar function in a dual Banach space $X^*$ with $Sz(B_{X^*}) \leq \omega$ was built by Lancien [10, 11] in a very different way.

**Lemma 3.3** *Let $K$ be a symmetric compact with $Sz(K, \varepsilon) < \omega$ for some $\varepsilon > 0$. Then given $\varepsilon^* > 2\varepsilon$, there is an homogeneous lower semicontinuous function $F : X \to [0, +\infty]$, such that the radial set $B = \{x \in X : F(x) \leq 1\}$ verifies $K \subset B \subset 2\overline{conv}(K)$ and $\langle B \rangle'_{\varepsilon^*} \subset (1-\eta)B$ for some $0 < \eta < 1$.*

**Proof.** Take $n = Sz(K,\varepsilon)$ and fix $r \in (0,1)$ such that $\frac{4\varepsilon}{r+r^2} < \varepsilon^*$. Define the symmetric radial compact sets

$$H = \{\lambda x : \lambda \in [0,1], x \in K\} \quad \text{and}$$

$$H_k = \{\lambda x : \lambda \in [0,1], x \in \langle K\rangle_\varepsilon^k\} \cup rH.$$

It is not difficult to see that $\lim_\omega x_\omega \in H_{k+1}$ whenever $(x_\omega) \subset H_k$ is a converging $\varepsilon$-separated net. Let $f$ be the Minkowski functional of $H$ and for every $0 \le k \le n$ let $f_k$ be the Minkowski functional of $H_k$. We have $f = f_0 \le f_k \le r^{-1} f$. Define

$$F(x) = \frac{1}{2} f(x) + \frac{r}{2(n+1)} \sum_{k=0}^{n} f_k(x).$$

The function $F$ is lower semicontinuous and $\frac{1+r}{2} f \le F \le f$. If $B = \{x \in X : F(x) \le 1\}$, then the former inequality implies that $B \subset 2H \subset 2\overline{\text{conv}}(K)$. By Lemma 2.3, if $x \in \langle B\rangle'_{\varepsilon^*}$ then there is net $(x_\omega) \subset B$ converging to $x$ and such that $\|x_\omega - x\| > \frac{2\varepsilon}{r+r^2}$. Without loss of generality we may assume that $F(x_\omega) = 1$ for every $\omega$. An easy computation gives

$$1 \le f_k(x_\omega) \le \frac{2}{r+r^2}$$

for all $k$ (for $k = 0$ or $k = n$ estimations are much better). We may assume that $\lim_\omega f_k(x_\omega)$ exists without loss of generality. For $\omega$ large enough $f_k(x_\omega)^{-1} x_\omega$ is $\varepsilon$-separated from its limit and contained in $H_k$. Therefore its limit $\lim_\omega f_k(x_\omega)^{-1} x$ belongs to $H_{k+1}$, implying

$$f_{k+1}(x) \le \lim_\omega f_k(x_\omega).$$

Now we have

$$\sum_{k=0}^{n} f_k(x) \le f_0(x) + \lim_\omega \sum_{k=0}^{n} f_k(x_\omega) - \lim_\omega f_n(x_\omega) \le \lim_\omega \sum_{k=0}^{n} f_k(x_\omega) - \frac{1-r}{r+r^2}$$

since $f_0(x) = f(x) \le \frac{2}{1+r}$ and $f_n(x_\omega) = r^{-1} f(x_\omega) \ge \frac{1}{r}$. We deduce that

$$F(x) \le 1 - \frac{1-r}{2(r+1)(n+1)} = 1 - \eta$$

and so $x \in (1-\eta)B$ as we wanted. ■

**Proposition 3.4** *For every symmetric compact $K \subset X$ with $Sz(K,\varepsilon) < \omega$ there exists an homogeneous lower semicontinuous function $F : X \to [0,+\infty]$ verifying that the radial set $B_F = \{x \in X : F(x) \le 1\}$ is compact, $K \subset B_F \subset 2\overline{conv}(K)$ and $\langle B_F\rangle'_{3\varepsilon} \subset (1-\eta)B_F$, where $\eta = \frac{1}{30} Sz(K,\varepsilon)^{-1}$.*

**Proof.** Taking $\varepsilon^* = 3\varepsilon$ in the previous Lemma, then observe in the proof that we may fix the value of $r \in (0,1)$ independently of $\varepsilon$. The function $\eta$ is given by

$$\eta(\varepsilon) = \frac{1-r}{2(r+1)(Sz(K,\varepsilon)+1)}$$

which satisfies the statement for any $\varepsilon > 0$. The value $\frac{1}{30}$ is obtained for $r = \frac{4}{5}$ after some rough estimations. ■

At this point we need to introduce the *convex Szlenk index*. Given $K \subset X$ convex compact, for every $\varepsilon > 0$ take

$$[K]'_\varepsilon = \{x \in K : \forall H \text{ open halfspace containing } x, \alpha(K \cap H) \geq \varepsilon\}$$

and define $[K]^\alpha_\varepsilon$ inductively. The index $Cz(K, \varepsilon)$ is the least ordinal $\alpha$ (if there exists) such that $[K]^\alpha_\varepsilon = \emptyset$.

**Lemma 3.5** *Let $K \subset X$ be a fragmentable compact with diam$(K) \leq M$. Assume that for some $\varepsilon \in (0, M)$, some compact $C \supset \langle K \rangle'_\varepsilon$ and $f \in X^*$ is such that* $\sup\{f, K\} > \sup\{f, C\}$*, then*

$$\sup\{f, [\overline{conv}(K)]'_{2\varepsilon}\} - \sup\{f, C\} \leq (1 - \frac{\varepsilon}{M})(\sup\{f, K\} - \sup\{f, C\}).$$

**Proof.** Take $b = \sup\{f, K\}$ and $\sup\{f, C\} < a < b$. We have $\{x \in K : f(x) \geq a\}$ can be covered by relatively open sets of diameter less than $\varepsilon$. Using compactness, we easily obtain that $\{x \in K : f(x) \geq a\} = \bigcup_{i=1}^n K_i$ where the sets $K_i$ with $i = 1, \dots, n$ are compact and have diameter less than $\varepsilon$. Consider the convex sets $C_i = \overline{\text{conv}}(K_i)$ with $i = 1, \dots, n$ and $C_0 = \{x \in \overline{\text{conv}}(K) : f(x) \leq a\}$. Lemma 3.1 implies that $\text{diam}(C_i) \leq \varepsilon$ for $i = 1, \dots, n$ and $\text{diam}(C_0) \leq M$. Notice $\overline{\text{conv}}(K) = \text{conv}(\bigcup_{i=0}^n C_i)$. Take $c = a + (1 - \varepsilon/M)(b - a)$. We claim that if $x = \sum_{i=0}^n \lambda_i x_i$ with $x_i \in C_i$, $\lambda_i \geq 0$ and $\sum_{i=0}^n \lambda_i = 1$, verifies $f(x) \geq c$ then $\lambda_0 \leq \varepsilon/M$. Indeed, assume the contrary, then $\sum_{i=1}^n \lambda_i < 1 - \varepsilon/M$ and so

$$f(x) \leq (\sum_{i=1}^n \lambda_i)b + \lambda_0 a = a + (\sum_{i=1}^n \lambda_i)(b - a) < a + (1 - \varepsilon/M)(b - a) = c$$

a contradiction. Therefore, if for some numbers $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ we have

$$(\sum_{i=0}^n \lambda_i C_i) \cap \{x \in \overline{\text{conv}}(K) : f(x) \geq c\} \neq \emptyset$$

then necessarily it must be $\sum_{i=0}^n \lambda_i \text{diam}(C_i) \leq (\sum_{i=1}^n \lambda_i)\varepsilon + \lambda_0 M < 2\varepsilon$. Using Lemma 3.2 we obtain that

$$\alpha(\{x \in \overline{\text{conv}}(K) : f(x) \geq c\}) \leq 2\varepsilon$$

and thus

$$\sup\{f(x) : [\overline{\text{conv}}(K)]'_{2\varepsilon}\} - a \leq c - a = (1 - \frac{\varepsilon}{M})(b - a).$$

Since $a$ can be taken arbitrarily close to $\sup\{f, C\}$ we obtain the statement. ■

**Proposition 3.6** *Given* $\varepsilon \in (0, \frac{2}{3})$, *for every symmetric compact* $K \subset X$ *with* $Sz(K, \varepsilon) < \omega$ *and* $diam(K) \leq 1$, *there exists a homogeneous lower semicontinuous convex function* $F : X \to [0, +\infty]$ *verifying that the symmetric convex set*

$$B_F = \{x \in X : F(x) \leq 1\}$$

*is compact,* $\overline{conv}(K) \subset B_F \subset 2\overline{conv}(K)$ *and* $\langle B_F \rangle'_{6\varepsilon} \subset (1 - \theta)B_F$, *where* $\theta = \frac{1}{20}\,\varepsilon\, Sz(K, \varepsilon)^{-1}$.

**Proof.** Fix $\varepsilon > 0$ and let $B$ the radial compact set given by Proposition 3.4. The function $F$ is just the Minkowski functional of $\overline{\text{conv}}(B)$. Notice that diameter of $B$ is less than 2. Moreover, we have

$$\langle B \rangle'_{3\varepsilon} \subset (1 - \eta)B$$

where $\eta = \frac{1}{30}\, Sz(K, \varepsilon)^{-1}$. Taking $C = (1 - \eta)B$ in Lemma 3.5 and an arbitrary $f \in X^*$, we have

$$\sup\{f, [\overline{\text{conv}}(B)]'_{6\varepsilon}\} - \sup\{f, C\} \leq (1 - \frac{3\varepsilon}{2})(\sup\{f, B\} - \sup\{f, C\}).$$

Having in mind $\sup\{f, C\} = (1 - \eta) \sup\{f, B\}$, we obtain after a short calculation

$$\sup\{f, [\overline{\text{conv}}(B)]'_{6\varepsilon}\} \leq (1 - \frac{3\varepsilon\eta}{2}) \sup\{f, B\}.$$

Therefore

$$[\overline{\text{conv}}(B)]'_{6\varepsilon} \subset (1 - \frac{3\varepsilon\eta}{2})\, \overline{\text{conv}}(B)$$

so finishing the proof. ■

**Corollary 3.7** *Let* $K \subset X$ *be a compact of Szlenk index* $\omega$ *at most, then* $\overline{conv}(K)$ *and* $\overline{aconv}(K)$ *are also of Szlenk index* $\omega$ *at most, and moreover*

$$Cz(\overline{conv}(K), \varepsilon) \leq a\, \varepsilon^{-1} Sz(K, b\, \varepsilon)$$

*for some constants* $a, b > 0$ *and every* $\varepsilon > 0$.

**Theorem 3.8** *Let* $K$ *be a symmetric compact of Szlenk index* $\omega$ *at most and diameter less than* 1. *There is an homogeneous lower semicontinuous convex function* $F : X \to [0, +\infty]$ *and* $\Theta(\varepsilon) \in (0, 1)$ *such that the symmetric convex set*

$$B_F = \{x \in X : F(x) \leq 1\}$$

*is compact and verifies* $\overline{conv}(K) \subset B_F \subset 2\overline{conv}(K)$ *and* $\langle B_F \rangle'_{25\varepsilon} \subset (1 - \Theta(\varepsilon))B_F$. *Moreover, it is possible to take* $\Theta$ *such that for every* $p > 1$ *then*

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-p}\, \Theta(\varepsilon)\, Sz(K, \varepsilon) = +\infty.$$

**Proof.** Let $f$ be the Minkowski's functional of $\overline{\text{conv}}(K)$. Let $F_n$ be the functions given by Proposition 3.6 for $\varepsilon = 2^{-n}$. Then we have $\frac{1}{2}f \leq F_n \leq f$. Take

$$F(x) = 6\pi^{-2} \sum_{n=1}^{\infty} n^{-2} F_n(x).$$

Let $(x_\varpi) \subset B_F$ a net converging to $x$ such that $\|x_\varpi - x\| > 12\varepsilon$. Without loss of generality we may suppose that $F(x_\varpi) = 1$. Fix $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon \leq 2^{1-n}$. Again, without loss of generality we may suppose that $F_n(x_\varpi)$ is convergent and thus

$$F_n(x) \leq (1 - \theta(2^{1-n})) \lim_{\varpi} F_n(x_\varpi)$$

where $\theta$ is the function introduced in Proposition 3.6. Summing all the terms and using lower semicontinuity we get

$$F(x) \leq \liminf_{\varpi} F(x_\varpi) - \frac{6\,\theta(2^{1-n})}{\pi^2 n^2} \lim_{\varpi} F_n(x_\varpi)$$

and using the bounds $F_n \geq \frac{1}{2}f \geq \frac{1}{2}F$ we obtain

$$F(x) \leq 1 - \frac{3\,\theta(2^{1-n})}{\pi^2 n^2}$$

which implies $\langle B_F \rangle'_{25\varepsilon} \subset (1 - \Theta(\varepsilon))B_F$ where

$$\Theta(\varepsilon) = \frac{3\,\theta(2^{1-n})}{\pi^2 n^2} = \frac{3 \cdot 2^{1-n}}{20\,\pi^2 n^2 Sz(K, 2^{1-n})} \geq \frac{c\,\varepsilon}{(1 - \log_2(\varepsilon))^2 Sz(K, \varepsilon)}$$

for a suitable constant $c > 0$, following the asymptotic behavior of $\Theta$. ∎

**Proof of Theorem 1.1.** Follows straight from Theorem 3.8 starting with the symmetric set $K \cup (-K)$. Notice that this operation introduces a factor 2 since the union may be not disjoint. ∎

We shall finish this section by showing the existence of an upper bound for the modulus function $\theta(\varepsilon)$ obtained in previous results.

**Proposition 3.9** *If $\theta : (0,1) \to (0,1)$ is such that $\langle K \rangle'_\varepsilon \subset (1 - \theta(\varepsilon))K$ for some symmetric convex compact $K \subset X$ and for every $\varepsilon$, then $\theta(\varepsilon) \leq \alpha(K)^{-1}\varepsilon$.*

**Proof.** Given $f \in X^*$ with $\sup\{f, K\} = 1$ and $r \geq 1$, observe that taking an homothety from a point in the slice we obtain that

$$\alpha(\{x \in K : f(x) > 1 - r\,\theta(\varepsilon)\}) \leq r\,\alpha(\{x \in K : f(x) > 1 - \theta(\varepsilon)\}) \leq r\,\varepsilon.$$

The proof is finished just taking $r = \theta(\varepsilon)^{-1}$. ∎

# 4 Applications and more examples

Let us recall that the norm of a dual Banach space $X^*$ is UKK* if for every $\varepsilon > 0$ there is $\theta(\varepsilon) \in (0,1)$ such that

$$\langle B_{X^*} \rangle'_\varepsilon \subset (1 - \theta(\varepsilon)) B_{X^*}.$$

The following is the main result in [19] that appears now as a consequence of the results about convex hulls of compacta of Szlenk index $\omega$ at most.

**Corollary 4.1** *Let $X$ be a Banach space such that $Sz(B_{X^*}) \leq \omega$. Then there is an equivalent norm on $X$ such that the dual norm on $X^*$ is UKK* with modulus of power type $\theta(\varepsilon) = c\,\varepsilon^p$.*

**Proof.** If the ball $B_{X^*}$ of a dual Banach space together with the norm metric is of Szlenk index $\omega$ then the finite indices $Sz(B_{X^*}, \varepsilon)$ are submultiplicative, so there exist $C, p > 0$ such that $Sz(B_{X^*}, \varepsilon) \leq C\varepsilon^{-p}$, see [12] for instance. Then the convex set given by Theorem 1.1 is the ball of an equivalent UKK* dual norm with modulus of power type. ■

Every compact space $K$ together with a lower semicontinuous metric $d$ imbeds homeomorphically as a weak* compact in a dual Banach space such that the metric induced by the norm coincides with $d$, see [7, Theorem 2.1]. The next two results are concerned with the imbedding of compact spaces of Szlenk index $\omega$ into dual Banach spaces.

**Corollary 4.2** *If $K$ imbeds as $w^*$-compact of a dual UKK* Banach space with the induced metric, then*

$$Sz(K, \varepsilon) \leq a\,\varepsilon^{-p}$$

*for some constants $a > 0$, $p \geq 1$ and every $\varepsilon > 0$.*

**Corollary 4.3** *There exists compact a space of Szlenk index $\omega$ with respect to a lower semicontinuous metric which embed into no dual UKK* Banach space.*

**Proof.** By Example 2.7 there is a compact space $K$ together with a lower semicontinuous metric such that $Sz(K, \varepsilon) > 2^{1/\varepsilon}$. ■

Recall that the $d$-Lipschitz functions on $K$ are denoted $L(K, d)$. The following is a transfer type result.

**Theorem 4.4** *Let $(K, d)$ have Szlenk index $\omega$ at most. If $Y$ is a Banach space which embeds isomorphically into $C(K)$ as a subset of $L(K, d)$, then $Sz(B_{Y^*}) \leq \omega$. Moreover*

$$Sz(B_{Y^*}, \varepsilon) \leq a\,\varepsilon^{-1} Sz(K, b\,\varepsilon)$$

*for some constants $a, b > 0$ and every $\varepsilon > 0$.*

**Proof.** Let $J : Y \to C(K)$ be the embedding and $J^* : C(K)^* \to Y^*$ its adjoint. By the Baire category theorem it is easy to show that there is a common Lipschitz bound $\lambda$ for all the functions of $J(B_Y)$. Clearly, $J^*(K)$ is a weak$^*$ compact and norming subset of $Y^*$. We shall show that also it is a Lipschitz image of $K$. Indeed, if $y \in B_Y$ and $x_1, x_2 \in K$ then

$$|J^*(x_1)(y) - J^*(x_2)(y)| = |J(y)(x_1) - J(y)(x_2)| \leq \lambda\, d(x_1, x_2).$$

Taking supremum on $y \in B_Y$ we get $\|J^*(x_1) - J^*(x_2)\| \leq \lambda\, d(x_1, x_2)$. Then $Sk(J^*(K), \varepsilon) \leq Sk(K, \varepsilon/\lambda)$ by Corollary 2.11. We finish applying Corollary 3.7 since $\overline{\text{aconv}}(J^*(K))$ contains a ball of $Y^*$. ■

**Corollary 4.5** *A Banach space $Y$ admits an equivalent norm such that the dual norm is UKK$^*$ if and only there is a compact space $K$ together with a metric $d$ of Szlenk index $\omega$ at most such that $Y$ imbeds as a closed subspace of $C(K)$ made up of $d$-Lipschitz functions.*

The following results are concerned with the properties of the space $C(K)^*$ with $Sz(K) \leq \omega$. Let us recall the definition of the norm defined on $C(K)^*$ by

$$\|\mu\|_L^* = \sup\{\int_K f\, d\mu : f \in L(K, d), \|f\|_L \leq 1\},$$

where $\|\cdot\|_L$ is the Lipschitz norm of $L(K, d)$. The results of the former section have the following implications.

**Corollary 4.6** *Let $K$ be a compact space together with a finer metric $d$. If $K$ is of Szlenk index $\omega$ at most, then $(B_{C(K)^*}, \|\cdot\|_L^*)$ is also of Szlenk index $\omega$ at most. Moreover*

$$Cz(B_{C(K)^*}, \varepsilon) \leq a\, \varepsilon^{-1} Sz(K, b\, \varepsilon)$$

*for some constants $a, b > 0$ and every $\varepsilon > 0$.*

**Proof.** It follows from Corollary 3.7. ■

**Corollary 4.7** *Let $K$ a compact space together with a finer metric $d$ having Szlenk index $\omega$ at most. Then there is an equivalent dual norm $|\!|\!|.|\!|\!|$ on $C(K)^*$ with the following property: for every $\varepsilon > 0$ there is $\theta > 0$ such that*

$$|\!|\!|x|\!|\!| \leq 1 - \theta$$

*whenever $x = w^*\text{-}\lim_{\varpi} x_{\varpi}$ with $|\!|\!|x_{\varpi}|\!|\!| \leq 1$ and $\|x_{\varpi} - x\|_L^* > \varepsilon$ for all $\varpi$.*

**Proof.** In this case the symmetric convex set given by Theorem 3.8 is the unit ball of the desired equivalent dual norm using Lemma 2.3. ■

If $K$ is a scattered compact then the norm $\|.\|_L^*$ coincides with the standard norm of $C(K)^*$ retrieving this result of Lancien [11].

**Corollary 4.8** *Let $K$ be a compact space. The $C(K)$ admits an equivalent UKK$^*$ dual norm if and only if $K$ is scattered with $K^{(\omega)} = \emptyset$.*

The UKK$^*$ renorming of a dual Banach space $X^*$ has a counterpart on its predual $X$. It will be convenient to introduce some more notation. Consider a Banach space $X$. For $Y \subset X$ a subspace, we shall denote $B_Y = B_X \cap Y$ and $B_Y(x,r) = x + (B(0,r) \cap Y)$. Following [8], the *modulus of asymptotic uniform smoothness* of $X$ is defined for $\varepsilon > 0$ by

$$\overline{\rho}_X(\varepsilon) = \sup_{\|x\|=1} \inf_{\mathcal{D}(X/Y)<\infty} \sup_{y \in B_Y[x,\varepsilon]} \|y\| - 1$$

where $\mathcal{D}$ stands for dimension, so $Y$ runs on the finite codimensional subspaces of $X$. The space $X$ is said *asymptotically uniformly smooth* if $\lim_{\varepsilon\to 0} \varepsilon^{-1}\overline{\rho}_X(\varepsilon) = 0$. It is possible to show that $X$ is asymptotically uniformly smooth if and only if $X^*$ is UKK$^*$, and moreover $\overline{\rho}_X(\varepsilon)$ are related quantitatively to the UKK$^*$ modulus $\theta(\varepsilon)$ by Young's duality, see [4, 8] for the details.

We wish to relate the modulus of asymptotic smoothness with the Szlenk index for the unit ball in the same Banach space endowed with the weak topology and the norm metric. In the following we shall not require any kind of compactness for the ball. The *essential inner radius* $\rho(A)$ of a set $A \subset X$ is the supremum of the numbers $r > 0$ such that $B_Y(x,r) \subset A$ for some $x \in A$ and $Y \subset X$ a finite codimensional subspace. Consider the following set derivation

$$\lceil A \rfloor'_\varepsilon = \{x \in A : \forall\, U \text{ } w\text{-neighbourhood of } x, \rho(A \cap U) \geq \varepsilon\}$$

and extend it by iteration. The associated ordinal index is the *goal Szlenk index* and it will be denoted $Gz(A,\varepsilon)$. The name is motivated by comparison with soccer game, because we are trying that balls, up to some diameter, do not enter into the set. The reader can easily check that $\rho(B_{c_0} \cap U) = 1$ for every weakly open $U$ meeting the unit sphere of $c_0$. Therefore $\lceil B_{c_0} \rfloor'_\varepsilon = B_{c_0}$ and so $Gz(B_{c_0}) = \infty$.

The knowledge of modulus of asymptotic smoothness of a Banach space $X$ allows us to bound $Sz(B_X,\varepsilon)$ and $Gz(B_X,\varepsilon)$ from below, where $B_X$ is endowed with the weak topology.

**Proposition 4.9** *If $\varepsilon \in (0, 1/6)$, then $(2\overline{\rho}_X(3\varepsilon))^{-1} \leq Gz(B_X,\varepsilon) \leq Sz(B_X, 2\varepsilon)$.*

**Proof.** The inequality between $Sz$ and $Gz$ follows from the obvious set inclusion $\lceil A \rfloor'_\varepsilon \subset \langle A \rangle'_{2\varepsilon}$ by iteration. Now, the definition of $\overline{\rho}_X(\varepsilon)$ and the fact that any weakly open set contains finite codimensional affine subspaces imply that

$$B_X \subset \lceil B[0, 1 + \overline{\rho}_X(\varepsilon)] \rfloor'_\varepsilon.$$

Fix $\varepsilon < 1/6$ and take $\rho = \overline{\rho}_X(3\varepsilon)$, thus $\rho < 1/2$. If $r \in [1/2, 1]$, then by homogeneity we have

$$\lceil B[0,r] \rfloor'_\varepsilon \supset \lceil B[0,r] \rfloor'_{\frac{3r\varepsilon}{1+\rho}} \supset B[0, \frac{r}{1+\rho}] \supset B[0, r - \overline{\rho}_X(3\varepsilon)].$$

The statement follows from the fact that the inclusion above can be iterated as long as the radius of ball on the right side is greater that $1/2$. ∎

It is easy to see that $Gz(B_X,\varepsilon) \geq \varepsilon^{-1}$ for any Banach space $X$, but this is too trivial to be useful. Under suitable equivalent norms, the moduli of Kadec-Klee uniformity and uniform smoothness provide good estimations of the Szlenk index. A typical case is that of $\ell_p(\Gamma)$ spaces, for which, after some rough standard computations, it is possible to prove the following result (the conjugate exponent of $p$ is denoted by $p'$).

**Example 4.10** *For every infinite set $\Gamma$ and $1 \leq p < +\infty$ then*

$$2^{-p}\varepsilon^{-p} \leq Gz(B_{\ell_p(\Gamma)},\varepsilon) \leq Sz(B_{\ell_p(\Gamma)},\varepsilon) \leq Cz(B_{\ell_p(\Gamma)},\varepsilon) \leq 3^p p\, \varepsilon^{-p}.$$

A similar result for general $L^p$ spaces is not true as $L^p[0,1]$ contains copies of $\ell^2$. In the case of Orlicz sequence spaces, the computation of the Szlenk indices has been done by L. Borel-Mathurin in [3].

Next result shows that inverting the growth speed of the indices $Sz$ and $Gz$ by a linear operator may cause norm compactness.

**Theorem 4.11** *Let $T : X \to Y$ be a bounded linear operator. If for every $n \in \mathbb{N}$ there exists $\varepsilon > 0$ such that $Sz(\overline{T(B_X)},\varepsilon) < Gz(B_X, n\varepsilon)$, then $T$ is compact.*

**Proof.** For a given $n \in \mathbb{N}$, take $\varepsilon > 0$ such that

$$N = Sz(\overline{T(B_X)},\varepsilon) < Gz(B_X,(n+1)\varepsilon).$$

Consider the weakly closed sets $A_j = B_X \cap T^{-1}(\langle\overline{T(B_X)}\rangle_\varepsilon^j)$ with $0 \leq j < N$. Then for some $j$ there exists $x \in A_j \setminus A_{j+1}$ such that every weakly neighborhood $U$ of $x$ verifies $\rho(A_j \cap U) \geq (n+1)\varepsilon$. Since $T(x) \in \langle\overline{T(B_X)}\rangle_\varepsilon^j \setminus \langle\overline{T(B_X)}\rangle_\varepsilon^{j+1}$, by continuity we may fix $U$ such that $\operatorname{diam}(\overline{T(A_j \cap U)}) < \varepsilon$. Without loss of generality, we may assume $B_Z[x, n\varepsilon] \subset A_j \cap U$, where $Z \subset X$ is a finite codimensional subspace. Therefore $\operatorname{diam}(T(B_Z[x,n\varepsilon])) < \varepsilon$ and by scaling we have $\operatorname{diam}(T(B_Z)) < n^{-1}$. Using the Bartle-Graves selection of the quotient map from $X$ onto $X/Z$, see [2, Proposition 1.19], we have $B_X \subset 3B_Z + F$ where $F \subset X$ is a norm compact subset. Therefore

$$T(B_X) \subset T(F) + 3T(B_Z) \subset T(F) + B_Y[0, 3/n].$$

Since $n \in \mathbb{N}$ was arbitrary, we deduce that $\overline{T(B_X)}$ is norm compact. ■

With these ideas and arguing like in [8, Proposition 2.3] it is possible to prove that if $Gz(B_X,\varepsilon) > Sz(B_Y,\delta)$ for some $0 < \delta < 2\varepsilon \leq 1$, then every bounded linear operator from $X$ into $Y$ is compact. Let us mention here that the classical Pitt's theorem, see [5] for instance, can be deduced easily as a consequence of Theorem 4.11 together with the estimations of Example 4.10.

Next results are devoted to some nonlinear applications of the Szlenk indices to $\ell_p$ spaces. Consider the Mazur bijective mapping $\psi_{p,q} : B_{\ell_p(\Gamma)} \to B_{\ell_q(\Gamma)}$ defined by

$$\psi_{p,q}(\,(x_\gamma)_{\gamma\in\Gamma}\,) = (\operatorname{sign}(x_\gamma)|x_\gamma|^{p/q})_{\gamma\in\Gamma}$$

where $p, q \geq 1$. The Mazur mapping is always a homeomorphism between the pointwise topologies and a uniform homeomorphism between the metrics, see [2, Theorem 9.1]. Proposition 2.9 guaranties that all these balls should have the same Szlenk index $\omega$, already calculated in Example 4.10. The growth speed of Szlenk indices can distinguish between them in the Lipschitz classification.

**Example 4.12** *The Mazur mapping $\psi_{p,q} : B_{\ell_p(\Gamma)} \to B_{\ell_q(\Gamma)}$ is Lipschitz if and only if $p \geq q$.*

**Proof.** If $p > q$ then $\psi_{p,q}$ is Gâteaux differentiable everywhere and the norm of its differential is bounded by $p/q$ at the points of $B_{\ell_p(\Gamma)}$, see the details in the proof of [2, Theorem 9.1]. On the other hand, the existence of a continuous and Lipschitz surjection $\psi : B_{\ell_p(\Gamma)} \to B_{\ell_q(\Gamma)}$ for $p < q$ would imply by Corollary 2.11 and Example 4.10 that $\varepsilon^{-q} \leq c\,\varepsilon^{-p}$ for some constant $c > 0$ and every $\varepsilon > 0$ which is clearly impossible. ■

**Example 4.13** *There exists a metrizable compact $K$ of Szlenk index $\omega$ with respect to a lower semicontinuous metric such that for every $p \in [1, +\infty)$ there is a continuous and Lipschitz surjection $\Pi_p : K \to B_{\ell_p}$.*

**Proof.** Take $K_n = B_{\ell_n}$ with the metric $d_n(x, y) = n^{-1}\|x - y\|_n$ and $K = \prod_{n=1}^{\infty} K_n$ together with the supremum metric. It is easy to see that

$$Sz(K, \varepsilon) = Sz(\prod_{i=1}^{n} K_i, \varepsilon) < \omega,$$

where $n > 2\varepsilon^{-1}$. Given $p \in [1, +\infty)$, take $n > p$ and compose the projection on the $n$-th coordinate with the Mazur mapping $\psi_{n,p}$. ■

We proved in [17] that if $K$ is a compact which is fragmented by a lower semicontinuous metric $d$, then $C(K)$ cannot contain a copy of $\ell_1$ made up of $d$-Lipschitz functions. In the next result we study the possibility of finding copies of other $\ell_p$ spaces made up of metrically Lipschitz functions in certain $C(K)$ spaces.

**Proposition 4.14** *Let $p \geq 1$ and endow $B_{\ell_p}$ with the pointwise topology, which makes it compact, and consider also the norm metric on it. Given $q \geq 1$, then*

*a) $C(B_{\ell_p})$ contains a copy of $\ell_q$ made up of Lipschitz functions if $q \geq \frac{p}{p-1}$,*

*b) $C(B_{\ell_p})$ contains no copy of $\ell_q$ made up of Lipschitz functions if $q < \frac{p+1}{p}$.*

**Proof.** If $q \geq \frac{p}{p-1}$ then for the conjugate exponent $q' \leq p$. The natural embedding of $\ell_q$ into $C(B_{\ell_{q'}})$ together with the Mazur map $M_{p,q'}$ will provide the Lipschitz copy. On the other hand, if $X$ is a subspace of $C(B_{\ell_p})$, then $Sz(B_{X^*}, \varepsilon) \leq a\,\varepsilon^{-p-1}$ by Theorem 4.4 and so $\ell_q$ does not imbed as Lipschitz functions if $q' < p + 1$. ■

In the case of $p = 2$, we get that $C(B_{\ell_2})$ contains Lipschitz copies of $\ell_q$ for $q \geq 2$ but it cannot for $q < 3/2$. We do not know what happens in the gap between these values. Observe that the space $C(K)$, with $K$ the compact built in Example 4.13, contains isometric copies of the spaces $\ell_p$ for every $p \in (1, +\infty)$ made up of metrically Lipschitz functions, but it cannot contain such a copy of $\ell_1$ by [17].

**Final Remarks.**
**(1)** The main source of inspiration for this work was the Doctoral Thesis of Gilles Lancien [10] realized under the supervision of G. Godefroy. In particular, the results concerning the quantitative use of the Szlenk and dentability indices when they are finite.
**(2)** Godefroy, Kalton and Lancien showed in [4] that the exponent of the modulus in the UKK$^*$ renorming can be taken arbitrarily close to the exponent bounding the Szlenk index. Unfortunately our method of proving Corollary 4.1 here leaves a gap of 1 between both exponents. On the other hand, the asymptotic estimation of $Sz(C(K)^*, \varepsilon)$ given in Corollary 4.6 is optimal for $K$ scattered with $K^{(\omega)} = \emptyset$. Indeed, if $K$ is the one point compactification of an infinite discrete set $\Gamma$, Corollary 4.6 gives that $Sz(B_{C(K)^*}, \varepsilon) < C\,\varepsilon^{-1}$ for some $C > 0$. The exponent $-1$ cannot be improved by Proposition 3.9.
**(3)** If $K$ is a scattered compact with $K^{(\omega)} = \emptyset$, then it is possible to improve Corollary 4.8 saying that the modulus can be taken of the form $\theta(\varepsilon) = c\,\varepsilon^{-1}$. That is called a Lipschitz-UKK$^*$ norm. A careful reading of the proofs of Section 3 allows to see that the function $F$ given by Proposition 3.6 for a fixed $\varepsilon < 1/3$ is such a norm. But doing that is not worth at all since, in that case, it is easier to write an explicit formula, see [6, p. 84].
**(4)** Some of the results in [14, 16], such as the stability of the descriptive compacta by taking convex hulls, can be deduced from the results of this paper. Notice that Corollary 4.7 implies that for $K$ descriptive there exists a dual norm on $C(K)^*$ with a $w^*$-metrizable sphere, but that fact is implicit in the construction done for the main result in [16]. The construction of a weak$^*$ rotund renorming on such $C(K)^*$, see [16], from the results for compact spaces of Szlenk index $\omega$ needs some extra work. See [20] for results on another kind of uniform renorming of $C(K)^*$ with $K$ descriptive.
**(5)** In our paper [18] we studied the finitely dentable bounded convex (F.D. for short) sets in Banach spaces. A F.D. set is a particular case of weakly compact of Szlenk index $\omega$ at most with respect to the norm metric. We address to [18] for the definition and the properties that will be used in the following construction. Consider $X = (\bigoplus_{n=2}^{\infty} \ell_n)_2$ and $Y = (\bigoplus_{n=2}^{\infty} \ell_n)_\infty$ and $T : X \to Y$ defined by $T((x_n)) = (n^{-1}x_n)$. Then $T(B_X)$ is a F.D. weakly compact convex subset of $Y$. Indeed, as in Example 2.7, for every $\varepsilon > 0$ just the coordinates with $n < 2/\varepsilon$ are relevant and the finite product of F.D. set is again F.D. by [18, Proposition 4.4]. By construction, $\lim_{\varepsilon \to 0^+} \varepsilon^p\, Sz(T(B_X), \varepsilon) = +\infty$ for every $p > 0$, and therefore $T$ cannot factorize through any reflexive UKK Banach space. In particular, $T$ is a uniformly convexifying operator [1] factorizing through no superreflexive space. Beauzamy gave an example of such an operator using Orlicz spaces [1, p. 122].
**(6)** Notice that Theorem 4.11 is valid for infinite ordinal indices. In order to explore the potential of this compactness criterion it is necessary to make a detailed study of the index $Gz$, which is quite different of $Sz$ in behavior. Moreover, it is

advisable to compute the indices of other classes of Banach spaces. These tasks are out of the scope of this paper. We believe that the example of classical $\ell_p$ spaces is enough as a motivation for the reader about the utility of the Szlenk indices techniques in several situations.

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