On asymptotically uniformly smooth Banach spaces

M. Raja

October, 2012

Abstract

The class of asymptotically uniformly smoothable Banach spaces has been considered in connection with several problems of Nonlinear Functional Analysis, as the differentiability of Lipschitz functions, the uniform classification of Banach spaces or the fixed point property. The known characterizations for those spaces come from p-estimates when finite dimensional block decompositions are available or via duality by means of the Szlenk index. In this paper we found a geometrical characterization for the existence of asymptotically uniformly smooth renorming using an ordinal type index defined for subsets of the space. Among the applications, we prove that the modulus of asymptotic uniform smoothness is the same for all the non-asymptotically uniformly smoothable Banach spaces.

1 Introduction

Consider a real Banach space X. Following [4], the *modulus of asymptotic uniform smoothness* of X is defined for $\varepsilon > 0$ by

$$\overline{\rho}_X(\varepsilon) = \sup_{\|x\|=1} \inf_{\mathcal{D}(X/Y)<\infty} \sup_{y \in Y,\, \|y\| \leq \varepsilon} (\|x+y\| - 1),$$

where $\mathcal{D}$ stands for dimension, so Y runs over the finite codimensional subspaces of X. The space X is said to be *asymptotically uniformly smooth* if $\lim_{\varepsilon \to 0} \varepsilon^{-1} \overline{\rho}_X(\varepsilon) = 0$. The *modulus of asymptotic uniform convexity* of X is defined for $\varepsilon > 0$ by

$$\overline{\delta}_X(\varepsilon) = \inf_{\|x\|=1} \sup_{\mathcal{D}(X/Y)<\infty} \inf_{y \in Y,\, \|y\| \geq \varepsilon} (\|x+y\| - 1).$$

The space is said to be *asymptotically uniformly convex* if $\overline{\delta}_X(\varepsilon) > 0$ for every $\varepsilon > 0$. Both moduli were introduced by Milman in [9] under different names. The related notion of *nearly uniformly smooth* Banach space was introduced by Prus in [12] and its corresponding modulus defined by Domínguez-Benavides in [1]. In the case of X being a dual Banach space, if Y runs over the finite codimensional weak* closed subspaces of X, the obtained modulus is denoted by $\overline{\delta}^*_X(\varepsilon)$, and the space is called *weak* asymptotically uniformly convex* if $\overline{\delta}^*_X(\varepsilon) > 0$ for all $\varepsilon > 0$. It is possible to show that X is asymptotically uniformly smooth if and only if X^* is weak* asymptotically uniformly convex, and moreover $\overline{\rho}^*_X(\cdot)$ is related quantitatively to $\overline{\delta}_X(\cdot)$ by Young's duality, see [2, 4] for the details. Analogously, we may define in a dual Banach space the modulus $\overline{\rho}^*_X(.)$. The second section of [4] is the best source of information and references about asymptotically uniformly smooth and asymptotically uniformly convex Banach spaces.

In words of the authors of [4], the papers [5] and [2] contain *implicitly* the deepest information about the moduli of asymptotic smoothness and asymptotic convexity. This is due to the fact that the weak* asymptotically uniformly convexity for a dual Banach space coincides with the so called *weak* uniformly Kadets-Klee* property, abbreviated UKK* (for X non-separable it is necessary to use Lancien's definition [7] instead of the original one by Huff [3]). The papers [5, 2] are concerned, among other things, with UKK* renorming in relation to the Szlenk index. In [5] it is shown that the UKK* modulus, which is essentially equivalent to the function $\overline{\delta}^*_X(\cdot)$, can be taken of power type. A different proof of this result can be found in [2] giving the sharpest estimation of the power exponent in terms of the Szlenk index. See also [13] for a more geometrical approach to the UKK* renorming. By duality, an asymptotically uniformly smoothable Banach space X admits an equivalent norm such that $\overline{\rho}_X(\varepsilon) \leq k\varepsilon^p$ for some $k > 0$ and $p > 1$. A more recent paper [10] contains interesting information about asymptotically uniformly smooth renorming of separable Banach spaces.

Our purpose is to provide a non-dual isomorphic characterization of the existence of an asymptotically uniformly smooth equivalent norm. This is done by means of an ordinal index Gz similar to the Szlenk index, denoted Sz following [6], but defined in terms of subsets of the space. The construction of the smooth norm is completely geometrical and allows to relate the index Gz to $\overline{\rho}_X(\cdot)$ in such a way that the power type appears naturally. To define this index, some preliminaries are necessary. For $Y \subset X$ a subspace, we shall denote $B_Y = B_X \cap Y$ and $B_Y(x, r) = x + rB_Y$ (all the balls we consider will be closed). The *essential inner radius* $\varrho(A)$ of a set $A \subset X$ is the supremum of the numbers $r \geq 0$ such that $B_Y(x, r) \subset A$ for some $x \in A$ and $Y \subset X$ is a finite codimensional subspace. For instance, consider the space ℓ_p with $p \in [1, +\infty)$. If U is a weak neighborhood of some $x \in B_{\ell_p}$ then we have $\varrho(B_{\ell_p} \cap U) \geq (1 - \|x\|_p^p)^{1/p}$ (just take finite codimensional balls with center x). In order to define an ordinal index, consider the following set derivation

$$[A]'_\varepsilon = \{x \in A : \forall\, U \; w\text{-neighbourhood of } x, \varrho(A \cap U) \geq \varepsilon\}$$

and extend for $n \in \mathbb{N}$ by iteration taking $[A]^n_\varepsilon = [[A]^{n-1}_\varepsilon]'_\varepsilon$. The *goal Szlenk index*, denoted $Gz(A, \varepsilon)$ is defined as

$$Gz(A, \varepsilon) = \min\{n \in \mathbb{N} : [A]^n_\varepsilon = \emptyset\}$$

if such a number exists. If it is not the case, the set derivation can be defined for transfinite ordinals in a very natural way in order to assign a value to $Gz(A, \varepsilon)$.

If $[A]^\alpha_\varepsilon = [A]^{\alpha+1}_\varepsilon \neq \emptyset$ for some ordinal α, then the derivation process does not arrive at $\emptyset$ and we take $Gz(A,\varepsilon) = \infty$, which is beyond the ordinals. Since we are mainly interested in numerical lower bounds for the goal index, any non-finite value of $Gz(A,\varepsilon)$ will be taken as $+\infty$ in terms of real arithmetics. We shall also consider the index $Gz(A) = \sup_{\varepsilon>0} Gz(A,\varepsilon)$. The goal index was introduced in [14] as a tool to bound below the Szlenk index. In fact, $Gz(A,\varepsilon) \leq Sz(A, 2\varepsilon)$ for every $\varepsilon > 0$, where the Szlenk index of a subset $Sz(A,\varepsilon)$ is defined as above just using the usual norm diameter instead of the essential inner radius "ϱ". In a dual Banach space we may consider the analogous notions in terms of the weak* topology. In the paper we shall point out the results that are true, with suitable changes, in their dual version. The relation of this derivation to asymptotic uniform smoothness is given by the following.

Proposition 1.1 *Let X be a Banach space. Then X is asymptotically uniformly smooth if and only if $\lim_{\varepsilon\to 0^+} \varepsilon^{-1}\zeta_X(\varepsilon) = 0$ where*

$$\zeta_X(\varepsilon) = 1 - \sup\{r : rB_X \subset [B_X]'_\varepsilon\}.$$

Notice that the supremum above is actually a maximum since $[B_X]'_\varepsilon$ is closed. The function $\zeta_X(\varepsilon)$ is equivalent to $\overline{\rho}_X(\varepsilon)$, but it is more suitable for our computations. In a dual Banach space we may consider $\zeta^*_X(\varepsilon)$ defined in the obvious way. The index $Gz(B_X,\varepsilon)$ is a monotone function of $\varepsilon > 0$ taking natural values. Moreover, we shall see that always $Gz(B_X,\varepsilon) > \varepsilon^{-1}$, but a difference bigger than one between both numbers implies the existence of an asymptotically uniformly smooth equivalent norm. The characterizations of the renorming are summarized in the following.

Theorem 1.2 *For a Banach space X the following statements are equivalent:*

i) X has an equivalent asymptotically uniformly smooth norm;

ii) X has an equivalent norm such that its modulus of asymptotic uniform smoothness is of power type, that is, $\overline{\rho}_X(\varepsilon) = O(\varepsilon^p)$ for some $p > 1$;

iii) X has an equivalent norm such that $\overline{\rho}_X(\varepsilon) < \varepsilon$ for some $\varepsilon > 0$;

iv) there exists $c > 0$, $p > 1$ such that $Gz(B_X,\varepsilon) > c\,\varepsilon^{-p}$ for every $\varepsilon \in (0,1)$;

v) there exists $\varepsilon \in (0,1)$ such that $Gz(B_X,\varepsilon) > \varepsilon^{-1} + 1$.

We recall that the equivalence of *i*) and *ii*) was established for separable spaces via duality in [5] for the first time. It was proven in [4] that statement *iii*) implies that X is Asplund. The relation between $p > 1$ in statement *iv*) and the power type of the asymptotic uniform smoothness modulus is almost optimal, in the same sense that the results of [2] concerning to the Szlenk index are. However, we have not found a direct argument (without renorming) to relate the goal index of a Banach space X to its Szlenk index, computed in X^*. A consequence of statement *iii*) above is the following.

Corollary 1.3 *If X is a Banach space which has no equivalent asymptotically uniformly equivalent norm, then $\overline{\rho}_X(\varepsilon) = \varepsilon$ for every $\varepsilon > 0$.*

Notice that the equality $\overline{\rho}_X(\varepsilon) = \varepsilon$ holds under any equivalent renorming of X.

The most asymptotically uniformly smooth Banach space is c_0, since it is easy to see that $\overline{\rho}_X(\varepsilon) = 0$ for every $\varepsilon < 1$. On the other hand, Milman proved that X contains a copy of c_0 provided that $\overline{\rho}_X(\varepsilon) = 0$ for some $\varepsilon > 0$. The next result gives a quantitative estimation of "how much c_0 is contained" inside an asymptotically uniformly smooth Banach space.

Theorem 1.4 *If X has an equivalent asymptotically uniformly smooth norm of power type $p > 1$, then there exists $k > 0$ such that for every $n \in \mathbb{N}$ the following holds*

$$\inf\{d_{BM}(Y, \ell_\infty^n) : Y \subset X, \mathcal{D}(Y) = n\} \leq k\, n^{1/p},$$

where d_{BM} stands for the Banach-Mazur distance between isomorphic spaces.

The organization of the paper is as follows. In section 2 we present the properties of the goal derivation that we will need for the rest of the paper. Section 3 is devoted to the main renorming result, whose proof is split into several lemmas. Finally, the results involving copies of the space c_0 and its finite representability are studied in section 4.

2 Properties of the goal derivation

We shall begin this section with some elementary properties of the set derivation introduced above and its associated index. In the following X will be an infinite dimensional real Banach space and $Y \subset X$ will always denote a subspace. An obvious argument that will be used continuously is the following: if $U \subset X$ is weakly open and $x \in U$, then there exists $Y \subset X$ of finite codimension such that $x + Y \subset U$. Most of the results of this section can be stated in a dual Banach space with minor changes in terms of the weak* topology.

Lemma 2.1 *Let $A \subset X$ be a closed subset and $\varepsilon > 0$, then*

$$\{x \in A : \; d(X \setminus A, x) \geq \varepsilon\} \subset [A]'_\varepsilon.$$

In particular, for any $r \geq \varepsilon$, then $(r - \varepsilon)B_X \subset [rB_X]'_\varepsilon$.

Proof. If $x \in A$ is such that $d(X \setminus A, x) \geq \varepsilon$, then for any finite codimensional $Y \subset X$ and any $\varepsilon' < \varepsilon$, we have $B_Y(x, \varepsilon') \subset A$ and so $x \in [A]'_\varepsilon$. ∎

Corollary 2.2 *If X is a Banach space, then $Gz(B_X, \varepsilon) > \varepsilon^{-1}$ and $\xi_X(\varepsilon) \leq \varepsilon$ for every $\varepsilon > 0$.*

Lemma 2.3 *For every $A \subset X$ closed convex and $\varepsilon > 0$, the set $[A]'_\varepsilon$ is closed convex and for every $\delta > 0$ and $x \in [A]'_\varepsilon$ there is $y \in B_X(x, \delta)$ and $Y \subset X$ of finite codimension such that $B_Y(y, \varepsilon - \delta) \subset A$.*

Proof. Consider for any $r > 0$ the set

$$A_r = \{x \in A : \exists Y \subset X,\, \mathcal{D}(X/Y) < \infty,\, B_Y(x,r) \subset A\}$$

which is convex. Indeed, suppose we are given $x_1, x_2 \in A_r$ and $\lambda \in (0,1)$, for some finite codimensional subspaces $Y_i \in X$ $i = 1, 2$ we have $B_{Y_i}(x_i, r) \subset A$. Take $Y = Y_1 \cap Y_2$ which is of finite codimension, then

$$B_Y(\lambda x_1 + (1-\lambda)x_2, r) \subset \lambda B_Y(x_1, r) + (1-\lambda)B_Y(x_2, r) \subset A.$$

Notice that the norm and weak closures of A_r coincide by Mazur's Theorem. We claim that

$$[A]'_\varepsilon = \bigcap_{r<\varepsilon} \overline{A_r}.$$

Indeed, if $x \in A \setminus \bigcap_{r<\varepsilon} \overline{A_r}$ then $x \notin \overline{A_r}$ for some $r < \varepsilon$ and so there exists a weakly open $U \ni x$ such that $\varrho(A \cap U) \leq r < \varepsilon$. This shows that $[A]'_\varepsilon \subset \bigcap_{r<\varepsilon} \overline{A_r}$. On the other hand, suppose $x \in \bigcap_{r<\varepsilon} \overline{A_r}$. Take any weakly open $U \ni x$ and any $r < \varepsilon$. Since $x \in \overline{A_r}$, by construction we have $\varrho(U \cap A) \geq r$ and therefore $\varrho(U \cap A) \geq \varepsilon$. The last statement is a consequence of this representation of $[A]'_\varepsilon$. ■

Remark 2.4 *In the case of the weak* topology, the analogous result will provide weak* density of the centers of big finite codimensional balls, which is enough to prove the weak* versions of the remaining results of the section, with the exception of Proposition 2.11.*

We shall need the following rules for the "arithmetics" of sets.

Lemma 2.5 *Let $A, B \subset X$ and $\varepsilon, \eta \geq 0$, then*

$$\eta[A]'_\varepsilon = [\eta A]'_{\eta\varepsilon},$$

$$[A]'_\varepsilon + B \subset [A+B]'_\varepsilon,$$

$$[A]'_\varepsilon + [B]'_\eta \subset [A+B]'_{\varepsilon+\eta}.$$

Proof. The first equality is simply obtained by scaling. The second set inclusion follows from the third one just noticing that $B = [B]'_0$. Finally, take $a \in [A]'_\varepsilon$ and $b \in [B]'_\eta$ and let U be any weakly open neighborhood of $a+b$. Applying Lemma 2.3, for any $\varepsilon' < \varepsilon$ and $\eta' < \eta$ we can find $Y \subset X$ of finite codimension, $x \in [A]'_\varepsilon$ and $y \in [B]'_\eta$ such that $x + y \in U$, $B_Y(x, \varepsilon') \subset A$ and $B_Y(y, \eta') \subset B$. It follows that $B_Z(x+y, \varepsilon' + \eta') \subset (A+B) \cap U$ where $Z \subset Y$ is a suitable finite codimensional subspace. Therefore $\varrho((A+B) \cap U) \geq \varepsilon' + \eta'$, and so $\varrho((A+B) \cap U) \geq \varepsilon + \eta$, implying that $a + b \in [A+B]'_{\varepsilon+\eta}$. ■

The following properties are known for $\overline{\rho}_X(\varepsilon)$, see [4, Proposition 2.3].

Proposition 2.6 *The function ζ is convex and the map $\varepsilon \to \varepsilon^{-1}\zeta_X(\varepsilon)$ is not decreasing.*

Proof. Given $\varepsilon, \eta \geq 0$ and $\lambda \in [0,1]$ consider the following chain of set inclusions

$$\begin{aligned}(\,\lambda(1-\zeta_X(\varepsilon)) + (1-\lambda)(1-\zeta_X(\eta))\,)\,B_X &\subset \lambda[B_X]'_\varepsilon + (1-\lambda)[B_X]'_\eta \\ &\subset [\lambda B_X]'_{\lambda\varepsilon} + [(1-\lambda)B_X]'_{(1-\lambda)\eta} \subset [B_X]'_{\lambda\varepsilon+(1-\lambda)\eta}.\end{aligned}$$

Therefore, we have

$$\lambda(1-\zeta_X(\varepsilon)) + (1-\lambda)(1-\zeta_X(\eta)) \leq 1 - \zeta_X(\lambda\varepsilon + (1-\lambda)\eta),$$

which implies the convexity of ζ_X. For the second part, if $0 < \varepsilon < \eta$, take $\lambda = \varepsilon/\eta$. Applying the convexity to $\varepsilon = (1-\lambda)0 + \lambda\eta$ we have $\zeta_X(\varepsilon) \leq (\varepsilon/\eta)\zeta_X(\eta)$. ■

Proposition 2.7 *If $Y \subset X$ is of finite codimension, $A \subset X$ closed convex symmetric and $\varepsilon > 0$, then*

$$[Y \cap A]'_\varepsilon = Y \cap [A]'_\varepsilon.$$

Proof. It is clear that we may suppose that Y has codimension 1 and so $Y = \ker(x^*)$ with $x^* \in X^*$. The inclusion $[Y \cap A]'_\varepsilon \subset Y \cap [A]'_\varepsilon$ is obvious. Suppose that it is strict and take $y \in Y \cap [A]'_\varepsilon \setminus [Y \cap A]'_\varepsilon$. We can find U, a convex weakly open neighborhood of y, such that $\varrho(Y \cap A \cap U) < \varepsilon$. On the other hand, fixing $\varepsilon' < \eta < \varepsilon$, there exist $a \in A$ and $Z \subset X$ of finite codimension such that $B_Z(a,\eta) \subset A \cap U$. Without loss of generality we may assume $x^*(a) > 0$. The segment joining y with $-a$ is contained in $[A]'_\eta$ and meets U at points of the set $\{x \in A : x^*(x) < 0\}$. This implies, with the help of Lemma 2.3, the existence of $b \in U$ with $x^*(b) < 0$ and $B_W(b,\varepsilon') \subset A \cap U$ for some $W \subset Z$ of finite codimension. A suitable convex combination of a and b provides $c \in Y$ such that $B_W(c,\varepsilon') \subset A \cap U$. Since $\varepsilon' < \varepsilon$ was arbitrary, this implies $\varrho(Y \cap A \cap U) \geq \varepsilon$, which is a contradiction. ■

Corollary 2.8 *For every $Y \subset X$ of finite codimension and $\varepsilon > 0$,*

$$Gz(B_Y,\varepsilon) = Gz(B_X,\varepsilon).$$

We say that the functions $f, g : (0,\infty) \to \mathbb{R}$ are *equivalent* if there exist constants $a, b > 0$ such that $a^{-1}f(b^{-1}x) \leq g(x) \leq af(bx)$ for every $x > 0$. It is almost obvious that $Gz(B_X,\varepsilon)$ and $Gz(B_Y,\varepsilon)$ are equivalent if X and Y are isomorphic Banach spaces.

Corollary 2.9 *Let $A \subset X$ be a nonempty bounded set. Then either $Gz(A,\varepsilon) = 1$ for every $\varepsilon > 0$, or $Gz(A,\varepsilon)$ is equivalent to $Gz(B_X,\varepsilon)$.*

For infinite ordinal values of the goal index we have the following.

Corollary 2.10 *Let $A \subset X$ be a nonempty bounded set. Suppose that $Gz(A) > \omega$, then $Gz(A) \geq \omega^\omega$.*

Proof. By Corollary 2.9 we may take $A = B_X$. There exists $\varepsilon \in (0,1)$, $r \in [\varepsilon, 1]$ and $Y \subset X$ of finite codimension such that $B_Y(0,r) \subset [B_X]^\omega_\varepsilon$. A scaling argument implies that $B_Y(0,r^2) \subset [B_X]^{\omega^2}_{r\varepsilon}$, and recursively we have $B_Y(0,r^n) \subset [B_X]^{\omega^n}_{r^{n-1}\varepsilon}$ for $n \in \mathbb{N}$. This implies $Gz(B_X) \geq \omega^\omega$, as desired. ■

Proposition 2.11 *Let $A \subset X$ be closed convex symmetric with nonempty interior and $\varepsilon > 0$. Then for every $x \in [A]'_\varepsilon$ and $\lambda > 1$, there is $Y \subset X$ of finite codimension such that $B_Y(x,\varepsilon) \subset \lambda A$.*

Proof. Fix $\delta > 0$ such that $A + B_X(0, 2\delta) \subset \lambda A$ and take $\varepsilon - \delta < \varepsilon' < \varepsilon$. By Lemma 2.3 we can find $z \in A$ with $\|z - x\| < \delta$ and $Y \subset X$ of finite codimension such that $B_Y(z, \varepsilon') \subset A$. Therefore

$$B_Y(x,\varepsilon) \subset B_Y(z,\varepsilon') + B_X(0,2\delta) \subset A + B_X(0,2\delta) \subset \lambda A,$$

which finishes the proof. ∎

3 Asymptotically uniformly smooth renorming

We shall begin the section by showing the equivalence of $\overline{\rho}_X(\varepsilon)$ and $\zeta_X(\varepsilon)$.

Proposition 3.1 *For $\varepsilon > 0$ we have*

$$\zeta_X(\varepsilon) \geq \overline{\rho}_X(\varepsilon) \geq \zeta_X(\frac{\varepsilon}{1+\overline{\rho}_X(\varepsilon)}).$$

Proof. Given $\varepsilon > 0$, $\lambda > 1$ and $x \in X$ with $\|x\| = 1$, then $x' = (1 - \zeta_X(\varepsilon))x$ is the center of some ball $B_Y(x',\varepsilon) \subset \lambda B_X$ as an application of Proposition 2.11, where $Y \subset X$ is of finite codimension. Since $B_Y(x,\varepsilon) = \zeta_X(\varepsilon)x + B_Y(x',\varepsilon)$, we deduce that $B_Y(x,\varepsilon) \subset (\lambda + \zeta_X(\varepsilon))B_X$, and so $\overline{\rho}_X(\varepsilon) \leq \lambda + \zeta_X(\varepsilon) - 1$ because x was arbitrary. As $\lambda > 1$ was arbitrary too, we arrive to $\overline{\rho}_X(\varepsilon) \leq \zeta_X(\varepsilon)$. On the other hand, the definition of the modulus of asymptotic uniform smoothness implies that

$$B_X \subset [(1+\overline{\rho}_X(\varepsilon))B_X]'_\varepsilon = (1+\overline{\rho}_X(\varepsilon))[B_X]'_{\frac{\varepsilon}{1+\overline{\rho}_X(\varepsilon)}},$$

hence

$$B_X \subset (1+\overline{\rho}_X(\varepsilon))(1-\zeta_X(\frac{\varepsilon}{1+\overline{\rho}_X(\varepsilon)}))\,B_X,$$

and thus $(1+\overline{\rho}_X(\varepsilon))(1-\zeta_X(\frac{\varepsilon}{1+\overline{\rho}_X(\varepsilon)})) \geq 1$, which implies the result. ∎

Remark 3.2 *An analogous result for the weak* version of this results is not true. Consider $\ell_\infty = \ell_1^*$, then it is easy to check that $\overline{\rho}^*_{\ell_\infty}(\varepsilon) = \varepsilon$ while $\zeta^*_{\ell_\infty}(\varepsilon) = 0$ for every $\varepsilon \in (0,1)$. Moreover, notice that a uniformly smoothable dual Banach space is necessarily reflexive.*

Proof of Proposition 1.1. By the previous proposition, it is easy to see that $\lim_{n\to 0} \varepsilon^{-1}\zeta_X(\varepsilon) = 0$ if and only if $\lim_{n\to 0} \varepsilon^{-1}\overline{\rho}_X(\varepsilon) = 0$. ∎

The proof of Theorem 1.2 will be split into several lemmas.

Lemma 3.3 *For any Banach space X and $0 < \varepsilon < 1/2$, then*

$$Gz(B_X,\varepsilon) \geq \zeta_X(2\varepsilon)^{-1}.$$

Proof. With the help of Lemma 2.5, for $n \in \mathbb{N}$ we have the following set inclusions

$$B_X(0, 1 + (n-1)\zeta_X(2\varepsilon)) = B_X(0, 1 - \zeta_X(2\varepsilon)) + B_X(0, n\zeta_X(2\varepsilon))$$

$$\subset [B_X]'_{2\varepsilon} + B_X(0, n\zeta_X(2\varepsilon)) \subset [B_X + B_X(0, n\zeta_X(2\varepsilon))]'_{2\varepsilon}$$

$$= [B_X(0, 1 + n\zeta_X(2\varepsilon))]'_{2\varepsilon}.$$

As a consequence, we have $[B_X(0,2)]^n_{2\varepsilon} \neq \emptyset$ if n is such that $n\zeta_X(2\varepsilon) < 1$. Taking n as the integer part of $\zeta_X(2\varepsilon)^{-1}$, we have $Gz(2B_X, 2\varepsilon) \geq n + 1$ and the final statement is obtained by scaling. ■

The next result shows how the power type appears in a very natural way.

Lemma 3.4 *Suppose that $Gz(B_X, \varepsilon_0) > \varepsilon_0^{-1} + 1$ for some $\varepsilon_0 \in (0,1)$. Then there exist $c > 0$ and $p > 1$ such that $Gz(B_X, \varepsilon) > c\,\varepsilon^{-p}$ for all $\varepsilon \in (0,1)$.*

Proof. Since $Gz(B_X, \varepsilon)$ has countably many discontinuities we may find $0 < \eta < \varepsilon_0$ such that $Gz(B_X, \varepsilon)$ is continuous at η and $Gz(B_X, \eta) > \eta^{-1} + 1$. Take $N = Gz(B_X, \eta) - 1$. We claim that $[A]'_\varepsilon \subset [A]^N_{\eta\varepsilon}$ for any $A \subset B_X$ and any $\varepsilon \in (0,1)$. Indeed, if it is not the case, take $U = X \setminus [A]^N_{\eta\varepsilon}$ and $\varepsilon' < \varepsilon$ such that

$$Gz(B_X, \frac{\eta\varepsilon}{\varepsilon'}) = Gz(B_X, \eta).$$

We have $U \cap [A]'_\varepsilon \neq \emptyset$, and so there is a ball $B_Y(x, \varepsilon') \subset U \cap A$ for some finite codimensional $Y \subset X$. We deduce that $[B_Y(x, \varepsilon')]^N_{\eta\varepsilon} = \emptyset$ and by scaling

$$[B_Y]^N_{\frac{\eta\varepsilon}{\varepsilon'}} = \emptyset,$$

which contradicts the choice of N after Proposition 2.7.
Once the fact is proven, by recurrence we have $[A]^k_\varepsilon \subset [A]^{kN}_{\eta\varepsilon}$. In particular, for $\varepsilon = \eta$ and $k = N$ we have $[B_X]^N_\eta \subset [B_X]^{N^2}_{\eta^2}$, which implies $[B_X]^{N^2}_{\eta^2} \neq \emptyset$ and so $Gz(B_X, \eta^2) > N^2$. Choosing ε equal to integer powers of η and the same recurrence arguments provides that $[B_X]^{N^k}_{\eta^k} \neq \emptyset$ for $k \in \mathbb{N}$ and therefore $Gz(B_X, \eta^k) > N^k$. Given any $\varepsilon \in (0,1)$ take $k \in \mathbb{N}$ such that $\eta^{k+1} < \varepsilon \leq \eta^k$. We have

$$Gz(B_X, \varepsilon) \geq Gz(B_X, \eta^k) > N^k > \frac{N^{\log_\eta \varepsilon}}{N} = \frac{\varepsilon^{\log_\eta N}}{N} = \frac{\varepsilon^{-p}}{N}$$

where $p = -\log_\eta N > 1$. This completes the proof of the lemma. ■

Remark 3.5 *Notice that the proof of Lemma 3.4 gives an estimation of the power type of $Gz(B_X, \varepsilon)$ that will also be an estimation of the power type of the modulus of asymptotic uniform smoothness of the equivalent norm provided by Lemma 3.8 below.*

Remark 3.6 *Suppose that $Gz(B_X, \varepsilon)$ has power type p and $Sz(B_X, \varepsilon)$ has power type q, that is, $Sz(B_X, \varepsilon) \leq c\,\varepsilon^{-q}$ for some $c > 0$ and any $\varepsilon > 0$. The inequality $Gz(B_X, \varepsilon) \leq Sz(B_X, \varepsilon)$ implies that $p \leq q$.*

Lemma 3.7 *Assume that $[B_X]^m_\varepsilon \neq \emptyset$ for $m > 1$. Then there is symmetric convex set B such that $B_X \subset B \subset 2B_X$ and*

$$(1 - \frac{1}{m})\, B \subset [B]'_\varepsilon.$$

Proof. We shall prove that the set

$$B = B_X + \frac{1}{m}\,(B_X + [B_X]'_\varepsilon + \ldots + [B_X]^{m-1}_\varepsilon)$$

does the work. We have the following set inclusions

$$(1 - \frac{1}{m})\, B \subset (1 - \frac{1}{m})\, B_X + \frac{1}{m}\,(B_X + [B_X]'_\varepsilon + \ldots + [B_X]^{m-1}_\varepsilon) =$$

$$B_X + \frac{1}{m}\,([B_X]'_\varepsilon + \ldots + [B_X]^{m-1}_\varepsilon) \subset [B]'_\varepsilon,$$

where the last one is a consequence of Lemma 2.5. ■

The following result shows that the power type of the asymptotically uniformly smooth renorming is almost optimal.

Lemma 3.8 *Let $c > 0$ and $p > 1$ such that $Gz(B_X, \varepsilon) \geq c\varepsilon^{-p}$. Then there exist an equivalent norm $|\!|\!|.|\!|\!|$ on X, such that $\lim_{\varepsilon\to 0} \varepsilon^{-q}\zeta_{|\!|\!|.|\!|\!|}(\varepsilon) = 0$ for any $1 < q < p$.*

Proof. Without loss of generality we may assume that $Gz(B_X, \varepsilon) - c\varepsilon^{-p} > 2$. For $n \in \mathbb{N}$ consider the set B_n given by Lemma 3.7 for $\varepsilon = n^2 2^{-n}$ and $m > c\varepsilon^{-p}$. Consider the set $B = \sum_{n=1}^{\infty} n^{-2} B_n$ and notice that

$$[\frac{1}{n^2} B_n]'_{2^{-n}} = \frac{1}{n^2}\,[B_n]'_{n^2 2^{-n}} \supset (1 - \frac{n^{2p}}{c\,2^{pn}})\,\frac{1}{n^2}\, B_n.$$

The application of Lemma 2.5 gives us

$$(1 - \frac{n^{2p}}{c\,2^{pn}}) B \subset [B]'_{2^{-n}}.$$

Given $\varepsilon > 0$, take $n \in \mathbb{N}$ such that $2^{-n-1} < \varepsilon \leq 2^{-n}$. Then we have

$$[B]'_\varepsilon \supset [B]'_{2^{-n}} \supset (1 - \frac{2^p}{c}(-\log_2 \varepsilon)^{2p}\varepsilon^p)\, B.$$

Therefore, we deduce that

$$\zeta_{|\!|\!|.|\!|\!|}(\frac{\varepsilon}{4}) \leq \frac{2^p}{c}(-\log_2 \varepsilon)^{2p}\varepsilon^p,$$

where the factor $1/4$ inside $\zeta_{|\!|\!|.|\!|\!|}$ corrects the fact that distances are now computed with the new norm $|\!|\!|.|\!|\!|$. Finally we arrive to $\lim_{\varepsilon\to 0} \varepsilon^{-q}\zeta_{|\!|\!|.|\!|\!|}(\varepsilon) = 0$ for any $1 < q < p$. ■

Proof of Theorem 1.2. It is clear that *ii)* $\Rightarrow$ *i)* and *i)* $\Rightarrow$ *iii)*. Suppose that *i)* holds. Then by Proposition 3.1 we have $\lim_{\varepsilon\to 0}\zeta_X(2\varepsilon)^{-1}\varepsilon = +\infty$. Thus, there is $0 < \varepsilon < 1$ such that

$$Gz(B_X, \varepsilon) \geq \zeta_X(2\varepsilon)^{-1} > 2\varepsilon^{-1} > \varepsilon^{-1} + 1$$

by application of Lemma 3.3. Therefore we have *i)* $\Rightarrow$ *v)*. Now, *v)* $\Rightarrow$ *iv)* is established in Lemma 3.4. Finally, *iv)* $\Rightarrow$ *ii)* is consequence of Lemma 3.8 and Proposition 3.1. Once we have proven the equivalences *i)* $\Leftrightarrow$ *ii)* $\Leftrightarrow$ *iv)* $\Leftrightarrow$ *v)*, the equivalence with *iii)* will be established in Lemma 3.9 below. ∎

The following is an improvement of [4, Proposition 2.4], since an asymptotically uniformly smooth Banach space is Asplund.

Lemma 3.9 *Let X be a Banach space. Assume that exists $\tau > 0$ such that $\overline{\rho}_X(\tau) < \tau$. Then X is asymptotically uniformly smoothable.*

Proof. We can take a number $\delta < \tau$ such that $B_X(0, r) \subset [B_X(0, r+\delta)]'_\tau$ for any $r \geq 1$. Therefore, for every $n \in \mathbb{N}$ we have $[B_X(0, 1+n\delta)]^n_\tau \neq \emptyset$. By scaling, we deduce that $Gz(B_X, \frac{\tau}{1+n\delta}) > n$. For n large enough and $\varepsilon = \frac{\tau}{1+n\delta}$, we shall have $Gz(B_X, \varepsilon) > n > \varepsilon^{-1} + 1$. Finally apply the above proved implication *v)* $\Rightarrow$ *i)* of Theorem 1.2. ∎

The properties of the function $\overline{\rho}_X(\cdot)$ imply that the hypothesis in Lemma 3.9 is equivalent to $\overline{\rho}'_X(0) < 1$ (right-hand derivative). Analogously, if $\xi_X(\varepsilon) < \varepsilon$ for some $\varepsilon \in (0, 1)$, then X is asymptotically uniformly smoothable.

Remark 3.10 *An analogue of Theorem 1.2 is true for dual Banach spaces endowed with the weak* topology and the modulus of asymptotic uniform smoothness changed to $\zeta^*_X(\varepsilon)$. Recall that $\zeta^*_X(\varepsilon)$ and $\overline{\rho}^*_X(\varepsilon)$ cannot be equivalent in that case.*

Proposition 3.11 *Let X be a Banach space such that $\lim_{\varepsilon\to 0}\varepsilon^p Gz(B_X, \varepsilon) = +\infty$ for every $p > 0$ (in particular, if $Gz(B_X, \varepsilon_0) \geq \omega$ for some $\varepsilon_0 > 0$). Then there exist an equivalent norm $\|\!|.|\!\|$ on X, such that $\lim_{\varepsilon\to 0}\varepsilon^{-p}\zeta_{\|\!|.|\!\|}(\varepsilon) = 0$ for any $p > 0$.*

Proof (Sketch). Follow the proof of Lemma 3.8 with appropriate changes. ∎

The Tsirelson space T (see [8] for the construction, but notice that the space defined there is in fact T^*, the dual of the Banach space built originally by Tsirelson) can be renormed for every $p > 1$ in such a way that $\overline{\delta}_T(\varepsilon) \geq c_p\,\varepsilon^p$, [5, Remarks 7.2]. By duality, we have that T^* can be renormed for every $q > 1$ in such a way that $\overline{\rho}_{T^*}(\varepsilon) = O(\varepsilon^q)$. In the next section, we shall show that actually $Gz(B_{T^*}, \varepsilon) > \omega$ for $0 < \varepsilon < 1/2$.

4 Embedding c_0 and distances to ℓ^n_∞

Let X a Banach space such that $\overline{\rho}_X(\varepsilon) = 0$ for some $\varepsilon > 0$. Milman [9] proved that X contains a copy of c_0. It is proved in [4, Theorem 2.9] that if X is in addition separable, then X is isomorphic to a subspace of c_0. The existence of a norm with such a property is characterized as follows.

Theorem 4.1 *If there exists a nonempty bounded set $A \subset X$ such that $[A]'_\varepsilon = A$ for some $\varepsilon > 0$, then X admits an equivalent renorming such that $\overline{\rho}_X$ vanishes in a neighborhood of* 0.

Proof. Without loss of generality we may assume that A is closed and convex. Then, after Lemma 2.5, we obtain that $B = A + (-A) + B_X$ is the ball of an equivalent norm such that $[B]'_\varepsilon = B$, and therefore $[B]'_\eta = B$ for every $\eta < \varepsilon$. ■

The aforementioned result [4, Theorem 2.9] implies the following.

Corollary 4.2 *A Banach space X is isomorphic to a subspace of c_0 if and only if it is separable and there exists a nonempty bounded set $A \subset X$ such that $[A]'_\varepsilon = A$ for some $\varepsilon > 0$.*

Corollary 4.3 *If c_0 does not imbed into X, then $[A]'_\varepsilon \subsetneq A$ for every nonempty bounded $A \subset X$ and every $\varepsilon > 0$. In particular, we have $Gz(B_X) < \infty$.*

The inequality $Gz(B_X, \varepsilon) \leq Sz(B_X, \varepsilon)$, which is true for ordinal values, implies that $Gz(B_X, \varepsilon) < \omega$ whenever X has an equivalent asymptotically uniformly convex norm. The next two results are concerned with the values that $Gz(B_X, \varepsilon)$ can take.

Theorem 4.4 *Let T^* be the dual of the Tsirelson space, then $Gz(B_{T^*}) = \omega^\omega$.*

Proof. Firstly we shall prove that $Gz(B_{T^*}) \geq \omega^\omega$. By Corollary 2.10 it is enough to show that $Gz(B_{T^*}, \varepsilon) > \omega$ for $\varepsilon \in (0, 1/2)$. In the following all the vectors considered are finitely supported. As usual in sequence spaces $u < v$ means that $\max \operatorname{supp}(u) < \min \operatorname{supp}(v)$. The standard basis of T^* is (e_i). Given $N \in \mathbb{N}$, our goal is to prove that $(1 - 2\varepsilon)B_{T^*} \subset [B_{T^*}]^N_\varepsilon$, which implies that $[B_{T^*}]^\omega_\varepsilon \neq \emptyset$. For every $n \in \mathbb{N}$ consider the set

$$A_n = \{\sum_{i=0}^n v_i : |v_0| + e_N < v_1 < \ldots < v_n,\ \|v_0\| \leq 1 - 2\varepsilon,\ \|v_i\| \leq \varepsilon \text{ if } i \geq 1\}.$$

By a remarkable property of T^*, we have $\|v_1 + \ldots + v_n\| \leq 2\varepsilon$ if v_i are as above and $n \leq N$, and therefore $A_N \subset B_{T^*}$. We claim that $\overline{A_{n-1}} \subset [\overline{A_n}]'_\varepsilon$. Indeed, if $v_0 + \ldots + v_{n-1} \in A_{n-1}$ and $Y \subset T^*$ is the finite codimensional subspace of the vectors which are zero on the first $\max \operatorname{supp}(v_{n-1})$ coordinates, then for every $v \in \varepsilon B_Y$ we have $v_0 + \ldots + v_{n-1} + v \in A_n$. That is, $v_0 + \ldots + v_{n-1}$ is the center of a finite codimensional ball of radius ε contained in $\overline{A_n}$ and so $v_0 + \ldots + v_{n-1} \in [\overline{A_n}]'_\varepsilon$. Once the claim is proven, apply it N times to obtain $\overline{A_0} \subset [\overline{A_N}]^N_\varepsilon$. The last inclusion implies that $(1 - 2\varepsilon)B_{T^*} \subset [B_{T^*}]^N_\varepsilon$, since A_0 dense in $(1 - 2\varepsilon)B_{T^*}$.
It was proved in [11, Proposition 16] that $Sz(B_{T^*}) = \omega^\omega$, the Szlenk index of T in our notation. Since we have $Gz(B_{T^*}) \leq Sz(B_{T^*})$, we deduce $Gz(B_{T^*}) = \omega^\omega$.■

Recall that the Tsirelson dual space T^* is a reflexive Banach space. On the other hand, the following result shows that for large reflexive Banach spaces X the goal index is always finite.

Theorem 4.5 *Let X be a Banach space such that c_0 does not imbed into X and X^* does not contain a weak*-total sequence (for instance, if X is a non-separable reflexive space). Then $Gz(B_X, \varepsilon) < \omega$ for every $\varepsilon > 0$.*

Proof. Notice that the hypothesis implies that countable codimensional subspaces of X are nontrivial. Fix $\varepsilon \in (0,1)$ and suppose $Gz(B_X, \varepsilon) \geq \omega$. We will find a copy of c_0 into X. For every $n \in \mathbb{N}$ we may consider the convex set given in Lemma 3.7 that was defined by

$$B_n = B_X + \frac{1}{n}\left(B_X + \ldots + [B_X]_\varepsilon^{n-1}\right)$$

in the proof. Recall that it satisfies the set inclusion $(1 - n^{-1})B_n \subset [B_n]'_\varepsilon$ and notice that the sequence (B_n) is decreasing. Let f_n be the Minkowski functional of B_n. We have for every $x \in X$ that $2^{-1}\|x\| \leq f_n(x) \leq \|x\|$ and $(f_n(x))$ is an increasing sequence. If we put $f(x) = \lim_n f_n(x)$, then f defines an equivalent norm on X whose unit ball will be denoted by B. Fix $x \in B$ and $n \in \mathbb{N}$. Since $x \in B_n$, using Proposition 2.11, we may fix $Y_{x,n} \subset X$ of finite codimension such that

$$B_{Y_{x,n}}(x, \varepsilon) \subset (1 + \frac{2}{n})B_n.$$

Now, for every $x \in B$ we can fix a countable codimensional space $Y_x = \bigcap_{n=1}^{\infty} Y_{x,n}$. If $z \in B_{Y_x}(x, \varepsilon)$ then $f_n(z) \leq 1+2/n$. Taking limits we have $f(z) \leq 1$ and therefore

$$B_{Y_x}(x, \varepsilon) \subset B.$$

We shall construct a sequence $(x_n) \subset X$ with $\|x_n\| = \varepsilon$ and such that $\sum_{k=1}^{n} \pm x_k \in B$ for any choice of signs. By a classical result of Bessaga and Pełczyński, this implies $c_0 \subset X$. There is no problem to find x_1. Suppose that we already have found x_k for $k \leq n$. Consider the following subspace of X which is of countable codimension

$$Y = \{y : \forall\, x = \sum_{k=1}^{n} \pm x_k, y \in Y_x\}$$

and take any $x_{n+1} \in Y$ with $\|x_{n+1}\| = \varepsilon$. The property of B ensures that $\sum_{k=1}^{n+1} \pm x_k \in B$ for any choice of signs. ∎

With some extra work it is easy to obtain an explicit copy of c_0 in the previous theorem instead of using Bessaga-Pełczyński. In fact, we shall develop this idea to prove the following proposition.

Proposition 4.6 *Let $A \subset B_X$ be a bounded symmetric closed convex set with nonempty interior, let $n \in \mathbb{N}$ be such that $[A]_\varepsilon^n \neq \emptyset$ for some $\varepsilon \in (0,1)$. Then for every $\delta > 0$ there exist a operator $T : \ell_\infty^n \to X$ such that $T(B_{\ell_\infty^n}) \subset A$ and $\|T^{-1}\| \leq \varepsilon^{-1}(1+\delta)$.*

Proof. Fix $\lambda = (1+\delta)^{1/2}$, take $\eta = \lambda^{-1}\varepsilon$ and $\xi = \lambda^{1/n}$. Define sets $A_k = \xi^k[\lambda^{-1}A]_\eta^{n-k}$ in order to use Proposition 2.11 between each pair of contiguous sets. We claim that there exist sets $\{e_1,\dots,e_n\} \subset S_X$ and $\{e_1^*,\dots,e_n^*\} \subset S_{X^*}$ such that $e_i^*(e_j) = 0$ for $i \neq j$, $e_i^*(e_i) \geq \lambda^{-1}$ and $\sum_{i=1}^k \pm\eta e_i \in A_k$ for any $1 \leq k \leq n$ and any choice of the signs. We have $0 \in A_0$ and take $e_1 \in X$ such that $\|e_1\| = 1$ and $\eta e_1 \in A_1$. Take e_1^* norming e_1. Suppose now that e_i and e_i^* are defined for $i \leq k < n$. Take $Y \subset X$ of finite codimension such that $\{e_1^*,\dots,e_k^*\} \subset Y^\perp$ and $B_Y(x,\varepsilon) \subset A_{k+1}$ for every $x \in \sum_{i=1}^k \pm e_i$, which is possible by using Proposition 2.11 2^k times. Take $e_{k+1}^* \in S_{X^*} \cap \{e_1,\dots,e_k\}^\perp$ such that $\sup\{e_{k+1}^*, B_Y\} = 1$. And finally, take $e_{k+1} \in S_Y$ such that $e_{k+1}^*(e_{k+1}) \geq \lambda^{-1}$. This finishes the proof of the claim. Now define the operator $T : \ell_\infty^n \to X$ by the formula $T((a_k)_{k=1}^n) = \sum_{k=1}^n \eta a_k e_k$. The choice of vectors implies that $T(B_{\ell_\infty^n}) \subset A$. On the other hand, T^{-1} is given explicitly by the formula

$$T^{-1}(x) = \eta^{-1}\left(\frac{e_1^*(x)}{e_1^*(e_1)},\dots,\frac{e_n^*(x)}{e_n^*(e_n)}\right)$$

for $x \in T(\ell_\infty^n)$. Therefore $\|T^{-1}\| \leq \varepsilon^{-1}\lambda^2 = \varepsilon^{-1}(1+\delta)$, finishing the proof. ■

Corollary 4.7 *Let X a Banach space such that $Gz(B_X,\varepsilon) \geq \omega$ for some $\varepsilon > 0$, then c_0 is finitely representable in X.*

Proof of Theorem 1.4. By Theorem 1.2, there is $c > 0$ such that $Gz(B_X,\varepsilon) > n$ for $\varepsilon = cn^{-1/p}$. Applying Proposition 4.6 with $A = B_X$, the operator T verifies that $\|T\| \leq 1$ and $\|T^{-1}\| < \varepsilon^{-1}(1+\delta)$ where $\delta > 0$ can be arbitrarily small. Therefore we have

$$\inf\{d_{BM}(Y,\ell_\infty^n) : Y \subset X, \mathcal{D}(Y) = n\} \leq \varepsilon^{-1}$$

which proves the statement. ■

Notice that Theorem 1.4 can be established also for Banach-Mazur distances from ℓ_1^n to n-dimensional subspaces of the dual space X^*.

Acknowledgements. I want to thank Richard Haydon and José Orihuela for fruitful discussions. I am also grateful to the anonymous referee for the careful reading of the manuscript, critical comments and helpful suggestions.

References

[1] T. Domínguez-Benavides, 'Modulus of nearly uniform smoothness and Lindenstrauss formulae', *Glasgow Math. J.* 353 (2001), no. 10, 3895–3918.

[2] G. Godefroy, N.J. Kalton, G. Lancien, 'Szlenk indices and uniform homeomorphisms', *Trans. Amer. Math. Soc.* 353 (2001), no. 10, 3895–3918.

[3] R. Huff, 'Banach spaces which are nearly uniformly convex', *Rocky Mountain J. Math.* 10 (1980), no. 4, 743–749.

[4] W.B. Johnson, J. Lindenstrauss, D. Preiss, G. Schechtman, Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces, *Proc. London Math. Soc.* (3) **84** (2002), no. 3, 711–746.

[5] H. Knaust, E. Odell, Th. Schlumprecht, 'On asymptotic structure, the Szlenk index and UKK properties in Banach spaces', *Positivity* 3 (1999), no. 2, 173–199.

[6] G. Lancien, 'A survey on the Szlenk index and some of its applications', *RACSAM Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat.* 100 (2006), no. 1-2, 209–235.

[7] G. Lancien, 'On uniformly convex and uniformly Kadec-Klee renormings', *Serdica Math. J.* 21 (1995), no. 1, 1–18.

[8] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I.* Vol. 92. Springer-Verlag, Berlin-New York, 1977.

[9] V. D. Milman 'Geometric theory of Banach spaces. II. Geometry of the unit ball', *Uspehi Mat. Nauk* 26 (1971), no. 6(162), 73-149.

[10] E. Odell, Th. Schlumprecht, 'Embedding into Banach spaces with finite dimensional decompositions', *RACSAM Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat.* 100 (2006), no. 1-2, 295-323.

[11] E. Odell, Th. Schlumprecht, A. Zsák, 'Banach spaces of bounded Szlenk index', *Studia Math.* 183 (2007), no. 1, 63-97.

[12] S. Prus, 'Nearly uniformly smooth Banach spaces', *Boll. Un. Mat. Ital.* B (7) 3 (1989), no. 3, 507–521.

[13] M. Raja, 'On weak* uniformly Kadec-Klee renorming', *Bull. London Math. Soc.* 42 (2010), 221–228.

[14] M. Raja, 'Compact spaces of Szlenk index ω', *J. Math. Anal. Appl.* 391 (2012), no. 2, 496509.

Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, SPAIN
E-mail: matias@um.es