Finite slicing in superreflexive Banach spaces*†

M. Raja‡

November, 2014

Abstract

For every superreflexive Banach space X there exists a supermultiplicative function which is the supremum, in a very natural ordering, of the set of all the moduli of convexity of equivalent norms. If this supremum is actually a maximum achieved under some equivalent renorming of X, then its modulus of convexity is the best possible in asymptotic sense. Otherwise, we can give an almost optimal uniformly convex renorming of X beyond the classical power type bound obtained by Pisier [23].

1 Introduction

A Banach space X is said to be *superreflexive* if every Banach space Y that is *finitely representable* in X is reflexive. Recall that being Y finitely representable in X roughly means that every finite subspace $F \subset Y$ can be embedded into X almost isometrically. Finite representability and superreflexivity, were introduced by R. C. James [15, 16] in the early 70's and nowadays they have become standard notions in Banach Space Theory. Among the different conditions which follow almost directly from the definition, we shall use the following: if X is superreflexive, then any ultraproduct $X^{\mathbb{N}}/\mathcal{U}$ is reflexive. We refer the reader to [2, 5] for more information on superreflexive Banach spaces.

The modulus of convexity of a Banach space $(X, \|\cdot\|)$ is defined for $t \in [0, 2]$ by the formula

$$\delta_{\|\cdot\|}(t) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in B_X, \|x - y\| \ \geq t\}.$$

The standard notation for the modulus of convexity is $\delta_X(t)$, but this is not the case in this paper since X is going to stand for an abstract Banach space endowed with many equivalent norms. A good source of information on properties of the modulus of convexity is [20]. The norm $\|\cdot\|$ of a Banach space X is said to be *uniformly convex* if $\delta_{\|\cdot\|}(t) > 0$ for every $t > 0$. This notion was introduced

*2010 Mathematics Subject Classification. Primary 46B20, 46B03.

†Keywords: superreflexive Banach space; uniform convexity; renorming.

‡This research was partially supported by the MEC/FEDER project MTM2011-25377.

by J. A. Clarkson [4] in the 30's. A Banach space having a uniformly convex equivalent norm is superreflexive. The converse is also true by a striking result of Enflo [6] (the original proof can be also found in [7]).

Theorem 1.1 (Enflo) *A superreflexive Banach space has an equivalent uniformly convex norm.*

The well known improvement of Enflo's Theorem made by Pisier [23] states that for a superreflexive Banach space there is an equivalent norm $\|\cdot\|$, $p \in [2, +\infty)$ and $c > 0$ such that $\delta_{\|\cdot\|}(t) \geq c\,t^p$ for every $t \in [0, 2]$. This is the so-called *power type modulus*. From an asymptotic point of view, "the most uniformly convex renorming" is the one with p as low as possible. In general, the best value of p is not attained, but at least there is a totally ordered hierarchy for the power type modulus of equivalent renormings. It is known that very general functions can be optimal modulus of convexity of Orlicz and Lorentz spaces, see [1, 21, 9, 20], but simply looking at the power type implies a loss of information. Our first result establishes the existence of a suitable upper bound for all the moduli of possible equivalent renormings. Consider the following partial order for functions defined on $(0, 1]$. We write $\phi \preceq \psi$ if there is a constant $c > 0$ such that $\phi(t) \leq c\,\psi(t)$ for all $t \in (0, 1]$. If $\phi \preceq \psi$ and $\psi \preceq \phi$, then we say that ϕ and ψ are *equivalent*.

Theorem 1.2 *Let X be a superreflexive Banach space. There exists a positive decreasing submultiplicative function $\mathfrak{N}_X(t)$ defined on $(0, 1]$ satisfying that $\mathfrak{N}_X(t)^{-1}$ is the supremum, up to equivalence, with respect to the order $\preceq$ of the set*

$$\{\delta_{|||\cdot|||}(t) : |||\cdot||| \text{ is an equivalent norm on } X\}.$$

Recall that the function ϕ is *submultiplicative* if $\phi(st) \leq \phi(s)\,\phi(t)$. Note that the function $\mathfrak{N}_X(t)^{-1}$ is *supermultiplicative*, which gives the alternative formulation of Theorem 1.2 claimed in the abstract. Such a property for the optimal modulus of convexity was first observed by Altshuler among Lorentz sequence spaces [1], see also [9, 20]. It is not difficult to see that a positive decreasing submultiplicative function ϕ defined on $(0, 1]$ is bounded by a power type function of the form $c\,t^{-p}$ with $c, p > 0$. Our next corollary shows the relation between the function $\mathfrak{N}_X(t)$ and the infimum of the possible exponents of power type modulus.

Corollary 1.3 *Given a superreflexive Banach space X, consider its type exponent defined as*

$$\mathfrak{p}_X = \inf_{0<t<1} \frac{\log(\mathfrak{N}_X(t))}{\log(1/t)}.$$

Then, for every $p > \mathfrak{p}_X$ there is $c > 0$ such that $\mathfrak{N}_X(t) \leq c\,t^{-p}$. Moreover, when the above infimum is attained, then $\mathfrak{N}_X(t)$ is equivalent to $t^{-\mathfrak{p}_X}$.

We say that the modulus of convexity is *optimal* if it is maximum with respect to the order $\preceq$ among all the moduli of convexity of equivalent norms.

Corollary 1.4 *A uniformly convex Banach space X with its norm $\|\cdot\|$ has an optimal modulus of convexity if, and only if, $\delta_{\|\cdot\|}(t)$ is equivalent to $\mathfrak{N}_X(t)^{-1}$.*

At this point, we have not given yet any positive result concerning the existence of an optimal renorming with modulus equivalent to $\mathfrak{N}_X(t)^{-1}$. We believe that this problem has a negative answer in general. We say that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *η-equivalent* for $\eta > 1$ if $\eta^{-1}\|x\|_1 \leq \|x\|_2 \leq \eta\|x\|_1$. Note that η-equivalence is a symmetric relation but it is not transitive. The best approximation of the optimal modulus of convexity that we can prove is the following.

Theorem 1.5 *Given a superreflexive Banach space X and $\eta > 1$, there exists a constant $c > 0$ such that for every $t \in (0,1]$, there is a η-equivalent norm $|\!|\!|\cdot|\!|\!|_t$ on X which satisfies*

$$\delta_{|\!|\!|\cdot|\!|\!|_t}(t) \geq c\,\mathfrak{N}_X(t)^{-1}.$$

Compare this result to [13, Theorem 4.7]. Blending the norms given by the previous theorem in a series we will get a norm with a nearly optimal modulus of uniform convexity.

Corollary 1.6 *Let X be a superreflexive Banach space. For every $\alpha > 1$ there exists an equivalent uniformly convex norm $|\!|\!|\cdot|\!|\!|$ such that for some $k > 0$ it satisfies*

$$\delta_{|\!|\!|\cdot|\!|\!|}(t) \geq k\,|\log t|^{-\alpha}\,\mathfrak{N}_X(t)^{-1}$$

for every $t \in (0,1]$. In particular, the modulus of convexity of $|\!|\!|\cdot|\!|\!|$ has power type p for every $p > \mathfrak{p}_X$.

We may now deduce the result that Pisier pointed out in the abstract of [23]: suppose that the norm of X satisfies $\lim_{t\to 0} t^{-p}\,\delta_{\|\cdot\|}(t) = \infty$, then there is $q < p$ and an equivalent norm such that $\lim_{t\to 0} t^{-q}\,\delta_{|\!|\!|\cdot|\!|\!|}(t) = \infty$. Indeed, this hypothesis implies that $\lim_{t\to 0}\mathfrak{N}_X(t)\,t^p = 0$ and thus $\mathfrak{p}_X \leq p$. On the other hand $\mathfrak{p}_X \neq p$, since the equality implies that $t^{-p} \preceq \mathfrak{N}_X(t)$ which is not possible. Therefore, $\lim_{t\to 0} t^{-q}\,\delta_{|\!|\!|\cdot|\!|\!|}(t) = \infty$ for any $q \in (\mathfrak{p}_X, p)$ with the norm given in Corollary 1.6.

The techniques used by Pisier [23] involve the use of basic sequences, martingales in $L^2(X)$ and several inequalities obtained analytically. In this part of the introduction we shall describe the tools that we have used to obtain the aforementioned results. First we need some geometrical definitions. The family of open halfspaces, $\{x \in X : f(x) > a\}$ with $f \in X^*$, is denoted by $\mathcal{H}$. A *slice* of $A \subset X$ is a nonempty subset of the form $A \cap H$ where $H \in \mathcal{H}$. When dealing with a bounded set A and $f \in S_{X^*}$, the following notation will be useful: $\sup\{f, A\} = \sup\{f(x) : x \in A\}$ and

$$S(A, f, \xi) = \{x \in A : f(x) > \sup\{f, A\} - \xi\},$$

where the infimum of all numbers $\xi > 0$ producing the same set is called the *width* of the slice S. For $\varepsilon > 0$, define the *slice derivation*

$$[A]'_\varepsilon = \{x \in A : \forall H \in \mathcal{H}, x \in H \Rightarrow \operatorname{diam}(A \cap H) \geq \varepsilon\}.$$

In other words, $[A]'_\varepsilon$ is obtained by removing from A its slices of diameter less than ε. It is clear that $[A]'_\varepsilon$ is closed and convex if so is A. A set A is called *dentable* if it has arbitrarily small slices. If X is reflexive, its bounded subsets are dentable and so $[A]'_\varepsilon \subsetneq A$. The useful *Lancien's midpoint argument* says that any slice S of a convex set A which does not meet $[A]'_\varepsilon$ has diameter at most 2ε. Indeed, if $x, y \in S = A \cap H$ and $[A]'_\varepsilon \cap H = \emptyset$ then $\frac{x+y}{2} \in A \setminus [A]'_\varepsilon$, so there is a $G \in \mathcal{H}$ such that $\operatorname{diam}(A \cap G) < \varepsilon$ and $\frac{x+y}{2} \in G$. Since x or y belongs to G, we get that a segment of length $\frac{\|x-y\|}{2}$ is contained in $A \cap G$; hence $\|x - y\| < 2\varepsilon$ and thus $\operatorname{diam}(A \cap H) \leq 2\varepsilon$. Lancien's midpoint argument may be used as well in the following way: if $x, y \in A$ and $\|x-y\| \geq 2\varepsilon$, then $\frac{x+y}{2} \in [A]'_\varepsilon$.

The operation $[\,\cdot\,]'_\varepsilon$ is a *set derivation*. The iterated derived sets are defined in a natural way as $[A]^n_\varepsilon = [[A]^{n-1}_\varepsilon]'_\varepsilon$. It is possible to define derived sets of order any ordinal number, but this is unnecessary for superreflexive spaces since Theorem 1.1 implies that given a bounded set $A \subset X$ and $\varepsilon > 0$, there is an $n \in \mathbb{N}$ such that $[A]^n_\varepsilon = \emptyset$. The least $n \in \mathbb{N}$ with this property is called the *dentability index* and it is denoted $Dz(A, \varepsilon)$. This index was used by Lancien [19] in order to give an alternative proof of Pisier's improvements of Theorem 1.1 and it is related to the so-called Szlenk index $Sz(A, \varepsilon)$ of the set. We refer the reader to [18] for the main properties and applications of these indices.

The next result describes a process that removes small slices from the "outer side" of a big slice without touching its complement. This process cannot exhaust the slice in finitely many steps, but the width of slices of bounded diameter can be uniformly reduced to one half.

Proposition 1.7 *Let X be a superreflexive Banach space. For every $\varepsilon \in (0, 1]$ there is an $N_\varepsilon \in \mathbb{N}$, such that for every slice $S = A \cap H$, with A convex and $H \in \mathcal{H}$, of diameter at most 1 and width $h > 0$ there are closed convex sets $(C_n)_{n=0}^{N_\varepsilon}$ having these properties:*

a) $A = C_0 \supset C_1 \supset \ldots \supset C_{N_\varepsilon} \supset (A \setminus H)$,

b) $[C_{n-1}]'_\varepsilon \subset C_n$, *and*

c) $C_{N_\varepsilon} \cap H$ *has width less than* $h/2$.

The least N_ε with all these properties will be denoted $\mathfrak{M}_X(\varepsilon)$.

The reader can check (with the help of a picture) that this process for slices of diameter 1 of big balls in Hilbert spaces takes $N_\varepsilon \simeq \varepsilon^{-2}$ steps. The construction of the function $\mathfrak{N}_X(t)$ is also geometrical, but just a bit more tricky. All the computations are done in X and the use of hyperspaces like $L^2(X)$ is not needed. Later we shall see that $\mathfrak{M}_X(t)$ and $\mathfrak{N}_X(t)$ are equivalent functions. As a consequence of the submultiplicativity of $\mathfrak{N}_X(t)$, we get that $\mathfrak{M}_X(t)^{-1}$ satisfies the so-called Δ_2 condition at zero, that is, there exists $c > 0$ such that $\mathfrak{M}_X(2t) \leq c\,\mathfrak{M}_X(t)$ for any $t \in (0, 1/2]$. Moreover, it follows that $\mathfrak{N}_X(t)$ and $\mathfrak{M}_X(t)$ are isomorphic invariants of X determined up to equivalence.

The structure of the rest of the paper is as follows. Section 2 contains an alternative proof of Enflo's Theorem (Theorem 1.1) using simple properties of superreflexive Banach spaces. In Section 3 we introduce the *slow slicing derivation*, showing its feasibility in uniformly convex spaces, which will provide the construction of the function $\mathfrak{N}_X$. Section 4 is devoted to renorming using functions $\mathfrak{N}_X(t)$ and $\mathfrak{M}_X(t)$. Finally, Section 5 contains several applications of these techniques. Throughout the paper X will be a superreflexive Banach space, but we will emphasize this fact in most of the statements.

2 A proof of Enflo's Theorem

The proof given by P. Enflo [6] of the existence of a uniformly convex renorming of a superreflexive Banach space is rather mysterious from a geometrical point of view. Alternative proofs in [23, 2, 19, 12] are not more elementary. The purpose of this section is to give a simple alternative proof of Enflo's Theorem, offering a first glance of the techniques developed in section 4. As well, the inclusion of this proof makes the paper self-contained because the construction of the function $\mathfrak{N}_X(t)$ requires at least the existence of one uniformly convex norm on X.

Let s denote a finite sequence with elements in $\{0,1\}$ and let $|s|$ be its length. We consider $\emptyset$ a sequence of length 0 and $s \frown i$ denotes the sequence of length $|s|+1$ obtained from s by adding $i \in \{0,1\}$. Given $\gamma \in [0, \frac{1}{2}]$ we say that a set of the form $\{x_s : |s| \leq n\}$ is a γ-weighted dyadic tree (of height n) if there exist $\lambda_s \in [\gamma, 1-\gamma]$ such that $x_s = \lambda_s x_{s\frown 0} + (1-\lambda_s) x_{s\frown 1}$ for every $|s| < n$. The point $x_\emptyset$ is called the root of the tree. The usual dyadic trees are $\frac{1}{2}$-weighted trees. We say that a dyadic tree $\{x_s : |s| \leq n\}$ is ε-separated if $\|x_{s\frown 0} - x_{s\frown 1}\| > \varepsilon$ for every $|s| < n$.

Proposition 2.1 *Let X be a superreflexive Banach space, $\gamma \in (0, \frac{1}{2}]$ and $\varepsilon > 0$. Then there exists an N such that $n \leq N$ for every $\{x_s : |s| \leq n\}$ γ-weighted and ε-separated dyadic tree contained in B_X.*

Proof. If there is not such a bound to the height of the trees, then we may take an ε-separated tree $\{x_s^n\}_{|s|\leq n} \subset B_X$ of height n for every $n \in \mathbb{N}$. These trees are γ-weighted with some coefficients $(\lambda_s^n)_{|s|\leq n} \subset [\gamma, 1-\gamma]$. For every finite sequence s of 0's and 1's the sequence $(x_s^n)_n$ defines an element x_s^u of the ultraproduct $X^{\mathbb{N}}/\mathcal{U}$ despite the fact that the first $|s|-1$ elements are missing. Taking $\lambda_s = \mathcal{U} - \lim_n \lambda_s^n$, it is elementary to check that $T = (x_s^u) \subset B_{X^{\mathbb{N}}/\mathcal{U}}$ is an infinite ε-separated dyadic tree with weights $(\lambda_s)_s \subset [\gamma, 1-\gamma]$. Clearly, any slice of T has diameter at least $\gamma\varepsilon$, which implies that $X^{\mathbb{N}}/\mathcal{U}$ cannot be reflexive which contradicts our assumption that X is superreflexive. ∎

The next simple lemma will be very useful throughout the paper.

Lemma 2.2 *If $\|x\| = \|y\| = 1$ and $r \in (0,1)$, then*

$$\min\{\|\alpha x - \beta y\| : \alpha, \beta \in [r,1]\} \geq \frac{r}{2}\,\|x-y\|.$$

Proof. Clearly, the minimum is attained for α or β equal to r. Without loss of generality we may assume that the quantity to be estimated from below is $\|(r+\lambda)x - ry\|$ where $\lambda \in [0, 1-r]$. Now, if $\lambda \geq \frac{r}{2}\|x-y\|$ we are done since

$$\|(r+\lambda)x - ry\| \geq |\,\|(r+\lambda)x\| - \|ry\|\,| = \lambda.$$

Otherwise, we have

$$\|(r+\lambda)x - ry\| \geq \|ry - rx\| - \|\lambda x\| \geq r\|y-x\| - \frac{r}{2}\|x-y\|,$$

which also leads to the thesis of the lemma. ■

Like in Enflo's original proof, the first step of our construction will provide us with a non-convex "uniformly non-square" homogeneous function.

Lemma 2.3 *Let X be a superreflexive Banach space and $\varepsilon \in (0,1]$. Then there exists a symmetric homogeneous function $F : X \to [0, +\infty)$ and an $\eta > 0$ such that $\frac{2}{3}\|x\| \leq F(x) \leq \|x\|$ and*

$$F(\frac{x+y}{2}) \leq 1 - \eta$$

whenever $x, y \in X$ satisfy $F(x) = F(y) = 1$ and $\|x - y\| > \varepsilon$.

Proof. Let T_n be the closure of the set of all roots of $\frac{1}{3}$-weighted $\frac{\varepsilon}{4}$-separated dyadic trees of height n contained in B_X, and let $N-1$ be an upper bound for this height. Consider the radial set

$$A_n = \{rx : r \in [0,1], x \in T_n\} \cup (1 - 3^{-n})B_X$$

and let F_n be its Minkowski functional. Note that $A_0 = B_X$ since any $x \in B_X$ is the root of a tree of height 0, $A_N = (1-3^{-N})B_X$, and $\|x\| \leq F_n(x) \leq \frac{3}{2}\|x\|$ for any n. Define the function

$$F(x) = (1 - 3^{-N-1})\|x\| + 2 \cdot 3^{-N-2}(N+1)^{-1} \sum_{n=0}^{N} F_n(x).$$

Clearly $(1 - 3^{-N-1})\|x\| \leq F(x) \leq \|x\|$. Moreover, it is easy to see that $N > \varepsilon^{-1}$ which allows us to give the more handy estimate $\|x\| \leq (1 + \varepsilon/8)F(x)$.
Let $x, y \in X$ be such that $F(x) = F(y) = 1$ and $\|x - y\| > \varepsilon$ and fix $0 \leq n < N$; then we have $1 \leq F_n(x), F_n(y) \leq 2$. Set $\overline{x} = F_n(x)^{-1}x$ and $\overline{y} = F_n(y)^{-1}y$; then $\overline{x}, \overline{y} \in A_n$.
We claim that $\|\overline{x} - \overline{y}\| > \varepsilon/4$. By previous estimates we already know that

$$\max\{\|x - \widehat{x}\|, \|y - \widehat{y}\|\} \leq \varepsilon/8,$$

where $\widehat{x} = \|x\|^{-1}x$ and $\widehat{y} = \|y\|^{-1}y$, and thus $\|\widehat{x} - \widehat{y}\| > 6\varepsilon/8$. Since we have $\|\overline{x}\|, \|\overline{y}\| \in [\frac{2}{3}, 1]$, we can apply Lemma 2.2 with $r = \frac{2}{3}$, to obtain that $\|\overline{x} - \overline{y}\| > \frac{2}{6}\frac{6}{8}\varepsilon = \varepsilon/4$.

Once the claim is proved, suppose that one of the points $\overline{x}$ or $\overline{y}$ belongs to $(1-3^{-n})B_X$, then $\|\lambda\overline{x}+(1-\lambda)\overline{y}\| \leq 1-3^{-n-1}$ for any $\lambda \in [\frac{1}{3}, \frac{2}{3}]$. If this is not the case, then we must have $\overline{x}, \overline{y} \in T_n$. Since $\|\overline{x}-\overline{y}\| > \varepsilon/4$ by Lemma 2.2, any point of the form $\lambda\overline{x}+(1-\lambda)\overline{y}$ with $\lambda \in [\frac{1}{3}, \frac{2}{3}]$ is arbitrarily close to the root of $\frac{1}{3}$-weighted $\frac{\varepsilon}{4}$-separated dyadic tree of height $n+1$ obtained by gluing two trees of height n. In any case, we have

$$\frac{x+y}{F_n(x)+F_n(y)} = \lambda\overline{x}+(1-\lambda)\overline{y} \in (1-3^{-n-1})B_X \cup T_{n+1} \subset A_{n+1}$$

and by the homogeneity of the functionals we arrive to the inequality

$$F_{n+1}(x+y) \leq F_n(x)+F_n(y).$$

This is the key point for the following shift argument:

$$\sum_{n=0}^{N} F_n(x+y) = \|x+y\| + \sum_{n=1}^{N} F_n(x+y)$$

$$\leq \|x+y\| + \sum_{n=0}^{N-1}(F_n(x)+F_n(y)) \leq \sum_{n=0}^{N}(F_n(x)+F_n(y)) - 3^{-N}$$

since

$$F_N(x)+F_N(y)-\|x+y\| \geq 2(1-3^{-N})^{-1} - 2(1-3^{-N-1})^{-1} > 3^{-N}.$$

Hence, we obtain that

$$F(x)+F(y)-F(x+y) \geq 3^{-2N-2}(N+1)^{-1}$$

which proves the lemma. ■

Proof of Theorem 1.1. Fix $\varepsilon \in (0,1]$ and let B be the closed convex hull of the set $A = \{x \in X : F(x) \leq 1\}$, where F is given by Lemma 2.3 and we may also assume that $\eta < \frac{2}{3}\varepsilon$. We claim that $\operatorname{diam}(S) < 8\varepsilon$ for any slice $S = B \cap H$ such that $H \in \mathcal{H}$ does not meet $(1-\eta\varepsilon)B$.
Indeed, without loss of generality we may suppose that $H = \{x \in X : f(x) > 1-\eta\varepsilon\}$ with $f \in X^*$ and $\sup\{f, B\} = 1$. The assumption on η gives us that $\operatorname{diam}(A\cap G) \leq 2\varepsilon$ for the halfspace $G = \{x \in X : f(x) > 1-\eta\}$. Consider the set

$$D = \{(1-\lambda)y+\lambda z : y \in \overline{\operatorname{conv}}(A\cap G), z \in B\setminus G, \lambda \in [\varepsilon, 1]\}.$$

Any point $x \in B$ is of the form $x = (1-\lambda)y+\lambda z$ with $y \in \overline{\operatorname{conv}}(A\cap G)$ and $z \in B\setminus G$. Take $x \in B\setminus D$; then necessarily $\lambda \in [0,\varepsilon]$ and, since $x-y = \lambda(z-y) \subset 2\varepsilon B$, we get $B\setminus D \subset \overline{\operatorname{conv}}(A\cap G)+3\varepsilon B_X$. Hence $\operatorname{diam}(B\setminus D) \leq 8\varepsilon$. Note that

$$\sup\{f, D\} \leq (1-\lambda)+\lambda(1-\eta) = 1-\eta\lambda$$

so we have $S \subset B\setminus D$, and therefore $\operatorname{diam}(S) < 8\varepsilon$ as claimed.
Now let $\|\cdot\|_n$ be the Minkowski functional of the symmetric convex set B given

by the above construction with $\varepsilon = \frac{1}{n}$ for every $n \in \mathbb{N}$ and set $\xi_n = \eta\varepsilon$. We claim that the equivalent norm

$$|||x||| = \frac{3}{4}\|x\| + \frac{1}{4}\sum_{n=1}^{\infty} 2^{-n}\|x\|_n$$

is uniformly convex. First notice that

$$|||x||| \le \|x\| \le \frac{3}{2}\|x\|_n \le \frac{3}{2}\|x\| \le 2|||x|||.$$

Given $\varepsilon > 0$, fix $n \in \mathbb{N}$ such that $n > 128\,\varepsilon^{-1}$. Suppose that $x, y \in X$ are such that $\|x - y\| > \varepsilon$ and $|||x||| = |||y||| = 1$. Since we have $\|x\|_n, \|y\|_n \le 2$, Lemma 2.2 gives that $\|\overline{x} - \overline{y}\| > \varepsilon/4 > 32/n$ for $\overline{x} = \frac{x}{\|x\|_n}$ and $\overline{y} = \frac{y}{\|y\|_n}$. The segment with endpoints $\overline{x}$ and $\frac{1}{2}(\overline{x} + \overline{y})$ has length greater than $16/n$, therefore $\|\frac{1}{4}(3\overline{x}+\overline{y})\| \le 1-\xi_n$ by Lancien's midpoint argument. A similar reasoning with the segment of endpoints $\overline{y}$ and $\frac{1}{2}(\overline{x}+\overline{y})$ allows us to deduce that $\|\frac{1}{4}(\overline{x}+3\overline{y})\| \le 1-\xi_n$. Therefore, we get that $\|\lambda\overline{x} + (1-\lambda)\overline{y}\|_n \le 1 - \xi_n$ for any $\lambda \in [\frac{1}{4}, \frac{3}{4}]$. In particular, we have

$$\frac{\|x + y\|_n}{\|x\|_n + \|y\|_n} \le 1 - \xi_n,$$

which implies $\|x\|_n + \|y\|_n - \|x + y\|_n \ge 4\xi_n$ and thus

$$|||x||| + |||y||| - |||x + y||| \ge 2^{-n}\xi_n.$$

This proves the uniform convexity of $|||\cdot|||$, as desired. ■

Remark 2.4 *Note that the uniformly convex norm $|||\cdot|||$ can be taken arbitrarily close to $\|\cdot\|$ (η-equivalent for any $\eta > 1$) by a suitable choice of coefficients. That implies the well known fact that uniformly convex norms are dense among the equivalent norms of a superreflexive Banach space X.*

Remark 2.5 *The best modulus of convexity obtained with this technique is of exponential type, like in Enflo's original proof. The rest of the paper is devoted to improve the quality of the possible modulus under renorming.*

Corollary 2.6 *If X is superreflexive, then for every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $[B_X]_\varepsilon^n = \emptyset$ and so $Dz(B_X, \varepsilon)$ is finite.*

3 Slicing in uniformly convex spaces

Along this section X will be a superreflexive Banach space with a fixed leading norm $\|\cdot\|$ which is not necessarily uniformly convex. We shall show that the functions $\mathfrak{N}_X$ and $\mathfrak{M}_X$ are controlled by the moduli of convexity of equivalent norms on X. To simplify computations, instead of using the usual modulus of convexity for an equivalent norm $|||\cdot|||$, we will consider the following modulus of convexity of a norm $|||\cdot|||$ (relative to $\|\cdot\|$)

$$\vartheta_{|||\cdot|||}(\varepsilon) = 1 - \frac{1}{2}\sup\{|||x + y||| : |||x||| = |||y||| = 1, \|x - y\| > \varepsilon\}.$$

Clearly we have $\delta_{|||\cdot|||}(\gamma^{-1}\varepsilon) \leq \vartheta_{|||\cdot|||}(\varepsilon) \leq \delta_{|||\cdot|||}(\gamma\varepsilon)$ if $|||\cdot|||$ is γ-equivalent to $\|\cdot\|$.

Lemma 3.1 *Suppose that we are given norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X such that $\|\cdot\| \leq 2\|\cdot\|_1 \leq 4\|\cdot\|$ and $\|\cdot\| \leq \|\cdot\|_2 \leq 2\|\cdot\|$. Then the norm $\|\cdot\|_\eta$ defined by*

$$\|x\|_\eta = (1-\eta)\|x\|_1 + \frac{\eta}{4}\,\|x\|_2$$

for any $\eta \in (0, 1/2)$ satisfies $\vartheta_{\|\cdot\|_\eta}(16\varepsilon) \geq \frac{\eta}{70}\,\vartheta_{\|\cdot\|_2}(\varepsilon)$ and $\|\cdot\|_\eta \leq \|\cdot\|_1$.

Proof. Clearly $\|\cdot\|_\eta \leq \|\cdot\|_1$. If $\|x\|_\eta = 1$, then $1 \leq \|x\|_1 \leq (1-\eta)^{-1}$ and thus $1/2 \leq \|x\|_2 \leq 8$. Suppose $\|x\|_\eta = \|y\|_\eta = 1$ and $\|x-y\| > 16\varepsilon$. By Lemma 2.2 applied to $\|\cdot\|_2$ we obtain

$$\|\frac{x}{\|x\|_2} - \frac{y}{\|y\|_2}\|_2 \geq \frac{1}{16}\|x-y\|_2 \geq \frac{1}{16}\|x-y\| \geq \varepsilon.$$

Take $\overline{x} = \frac{x}{\|x\|_2}$, $\overline{y} = \frac{y}{\|y\|_2}$ and $\lambda \in [\gamma, 1-\gamma]$ where $\gamma \in (0, 1/2]$. A simple convexity argument shows that

$$1 - \|\lambda\overline{x} + (1-\lambda)\overline{y}\|_2 \geq 2\gamma\,\vartheta_{\|\cdot\|_2}(\varepsilon).$$

It is easy to verify that

$$\lambda = \frac{\|x\|_2}{\|x\|_2 + \|y\|_2} \in [\frac{1}{17}, \frac{16}{17}]$$

and so we have

$$\frac{\|x\|_2 + \|y\|_2}{2} - \|\frac{x+y}{2}\|_2 = \frac{\|x\|_2 + \|y\|_2}{2}\left(1 - \frac{\|x+y\|_2}{\|x\|_2 + \|y\|_2}\right) \geq \frac{1}{17}\,\vartheta_{\|\cdot\|_2}(\varepsilon).$$

Therefore,

$$1 - \|\frac{x+y}{2}\|_\eta \geq \frac{\eta}{68}\,\vartheta_{\|\cdot\|_2}(\varepsilon)$$

which completes the proof of the lemma. ■

For the proof of the main result in this section we shall need another definition. We call *cylinder* of center x, radius $r > 0$ and height $h > 0$ to a convex body of the form

$$C = \{x + y + tz : y \in \ker f, \|y\| \leq r, t \in [-h/2, h/2]\}$$

where $f \in S_{X^*}$ and $z \in S_X$ satisfies $f(z) = 1$.

Lemma 3.2 *A cylinder centered at 0 of radius $r > 0$ and height $h > 0$ contains a ball of radius $\min\{r/2, h/2\}$ and it is contained in a ball of radius $r + h/2$.*

Proof. Without loss of generality, we may assume that the cylinder has its center at 0. If $\|x\| \leq \min\{r/2, h/2\}$, then $x = (x - f(x)z) + f(x)z$ is a decomposition like in the definition of cylinder. Thus, we have $|f(x)| \leq h/2$ and $\|x - f(x)z\| \leq r/2 + r/2 = r$. The other set inclusion is obvious. ■

Lemma 3.3 *Let $S = A \cap H$ be a slice of diameter at most $1/2$ and width h of a closed convex set $A \subset X$ with $H \in \mathcal{H}$. Then there is a cylinder C of radius 1 and height 2 such that $C \cap H$ has width h and $S \subset C \cap H$.*

Proof. Indeed, suppose that H is defined by $f \in S_{X^*}$ and $a = \sup\{f, A\}$. Pick a vector $z \in S_X$ with $f(z) = 1$. By the definition of width it is possible to pick a point $x_0 \in A \cap f^{-1}(a-h)$. Then set the center of the cylinder at $x_0 + (h-1)z$. The other properties are elementary to check. ■

Let us fix a closed convex set $B \subset X$ (we even admit $B = \emptyset$), the *slow slicing set derivation (with respect to B)* is defined for any bounded subset $A \subset X$ as

$$[A\,|_B]'_\varepsilon = \{x \in A : \forall\, S(A,f,\xi) \ni x \;\&\; S(A,f,2\xi) \cap B = \emptyset \Rightarrow \operatorname{diam}(S(A,f,2\xi)) \geq \varepsilon\}.$$

Removing only one half, with respect to the width, of the small slice will be the key to get the submultiplicative property of $\mathfrak{N}_X$. The sets $[A\,|_B]^n_\varepsilon$ are defined by iteration for any number $n \in \mathbb{N}$.

That derivation will allow us to slim slices without touching its complement.

Lemma 3.4 *Let X be a superreflexive Banach space, $A \subset X$ a closed convex subset and $S = S(A,f,2\xi)$ a slice of width $2\xi > 0$ and diam$(S) = \Delta > 0$. Given $\varepsilon \in (0, \Delta]$ there is an $N \in \mathbb{N}$ with the property*

$$S(A,f,\xi) \cap [A\,|_{A\setminus S}]^N_\varepsilon = \emptyset.$$

Moreover, we can take $N \leq 2240\, \vartheta_{|||\cdot|||}(\frac{\varepsilon}{64\Delta})^{-1}$ for any norm $|||\cdot|||$ on X satisfying $\|\cdot\| \leq |||\cdot||| \leq 2\|\cdot\|$.

Proof. By scaling we may suppose that $\Delta > 0$ is fixed, in particular we choose $\Delta = 1/2$ in order to keep nicer numbers in our construction. Set

$$H = \{x \in X : f(x) > \sup\{f, A\} - 2\xi\},$$

so we have $S = A \cap H$. Recall that $\|f\| = 1$ and $2\xi \leq \Delta$, so $\xi \leq 1/4$.
By Lemma 3.3 there is a cylinder C of radius 1 and height 2 such that $S \subset C \cap H$ and $C \cap H$ has width 2ξ. In order to estimate the number of steps of the slow slicing process to slim S to half width it is enough to work with such a kind of cylinders.
Without loss of generality, we may assume that the cylinder C is centered at the origin and so it is the unit ball of a 2-equivalent norm, so for some $z \in S_X$ norming f we have

$$C = \{y + tz : y \in B_X \cap \ker f, t \in [-1,1]\}.$$

Obviously, the halfspace H can be written now as $H = \{x \in X : f(x) > 1 - 2\xi\}$. Later we shall need also the halfspace $G = \{x \in X : f(x) > 1 - \xi\}$. Consider the cylinder

$$C_1 = \{y + tz : y \in B_X \cap \ker f, |t| \leq 1 - \frac{3\xi}{2}\}$$

also centered at the origin with equal parameters except its height which is $2(1-\frac{3}{2}\xi)$. Let $\|\cdot\|_1$ be the Minkowski functional of C_1 which clearly is a 2-equivalent norm as well. Since $\|\cdot\|_1$ and $\|\cdot\|_2 = |||\cdot|||$ satisfy the hypothesis of Lemma 3.1, we may consider the equivalent norm $\|\cdot\|_\eta$ given there for $\eta = \xi/4$. If B is the unit ball of $\|\cdot\|_\eta$, we claim that the choice of η implies that $B \cap G = \emptyset$. Indeed, $\|x\|_\eta \leq 1$ implies that $\|(1-\frac{\xi}{4})x\|_1 \leq 1$ and so

$$f(x) \leq \frac{1-3\xi/2}{1-\xi/4} < 1-\xi-\frac{\xi}{4}.$$

On the other hand, the inclusion $C_1 \subset B$ and the inequality $(1-\frac{3\xi}{2})(1+4\xi) > 1$ imply that $C \subset (1+4\xi)B$.
Lemma 3.1 says that $\vartheta_{\|\cdot\|_\eta}(16\varepsilon) \geq \frac{\xi}{280}\,\vartheta_{\|\cdot\|_2}(\varepsilon)$. Since there are norms $|||\cdot|||$ satisfying the inequality of the statement which are uniformly convex, we may assume at first that $\vartheta_{\|\cdot\|_2}(t) > 0$ for a given $t \in (0, 1/64)$. Let $N \in \mathbb{N}$ be the integer part of $2240\,\vartheta_{\|\cdot\|_2}(t)^{-1}$ and consider the convex sets

$$D_k = \left(1 + \frac{4\xi(N-k)}{N}\right) B$$

for $0 \leq k \leq N$. Note that $C \subset D_0 \subset 2B$ and $D_N = B$. As $\vartheta_{\|\cdot\|_\eta}(16t) > 2\frac{4\xi}{N}$, the definition of modulus of convexity implies that

$$[D_{n-2}]'_{32t} \subset D_n$$

and it is easy to check that $D_{N-1} \cap G = \emptyset$. Note that the above derivation shifts D_{n-2} into D_n, which in terms of the slow slice derivation means that $[D_{n-2}|_B]'_{32t} \subset D_{n-1}$. Bearing in mind that we started with an arbitrary slice of diameter $\Delta = 1/2$, we deduce by scaling for an arbitrary $\Delta > 0$ that

$$S(A, f, \xi) \cap [A\,|_{A \setminus S}]^N_{64t\Delta} = \emptyset.$$

If $\varepsilon \in [0, \Delta]$, taking $t = \frac{\varepsilon}{64\Delta}$, the previous computations provide the existence of N as claimed. Moreover, if $\vartheta_{\|\cdot\|_2}(t) > 0$ we have the upper estimation of $2240\,\vartheta_{\|\cdot\|_2}(t)^{-1}$, but if $\vartheta_{\|\cdot\|_2}(t) = 0$ then the estimation is trivially true. ■

We define here the function $\mathfrak{N}_X(\varepsilon)$. Let us recall that the existence of $\mathfrak{N}_X(\varepsilon)$ was claimed in Theorem 1.2.

Theorem 3.5 *Let X be a superreflexive Banach space. Then for every $\varepsilon \in (0,1]$ there is a least $\mathfrak{N}_X(\varepsilon) \in \mathbb{N}$ with the property*

$$S(C, f, \xi) \cap [C\,|_B]^{\mathfrak{N}_X(\varepsilon)}_\varepsilon = \emptyset$$

whenever $B \subset C$ are closed convex subsets of X, $f \in X^*$, $B \cap S(C, f, 2\xi) = \emptyset$ and diam$(S(C, f, 2\xi)) \leq 1$. Moreover, $\mathfrak{N}_X(\varepsilon)$ is a submultiplicative function, that is, $\mathfrak{N}_X(\varepsilon_1\varepsilon_2) \leq \mathfrak{N}_X(\varepsilon_1)\mathfrak{N}_X(\varepsilon_2)$ for every $\varepsilon_1, \varepsilon_2 \in (0,1]$.

Proof. Given $B \subset C$ convex closed sets, if $S \subset C$ is a slice disjoint from B, then $[C|_B]^n_\varepsilon \subset [C|_{C\setminus S}]^n_\varepsilon$ for every $n \in \mathbb{N}$. Thus the existence of $\mathfrak{N}_X(\varepsilon)$ is a straightforward consequence of Lemma 3.4 . Take $\varepsilon_1, \varepsilon_2 \in (0,1)$. In order to show that $\mathfrak{N}_X(\varepsilon_1\varepsilon_2) \leq \mathfrak{N}_X(\varepsilon_1)\mathfrak{N}_X(\varepsilon_2)$ it is enough to prove that $[C\,|_B]^{\mathfrak{N}_X(\varepsilon_2)}_{\varepsilon_1\varepsilon_2} \subset [C\,|_B]'_{\varepsilon_1}$. If $x \in C \setminus [C\,|_B]'_{\varepsilon_1}$, then for some slice $x \in S(C,f,\xi)$ and $\operatorname{diam}(S(C,f,2\xi)) \leq \varepsilon_1$. By scaling, it is clear that $[C\,|_B]^{\mathfrak{N}_X(\varepsilon_2)}_{\varepsilon_1\varepsilon_2} \cap S(C,f,\xi) = \emptyset$. Therefore, $x \notin [C\,|_B]^{\mathfrak{N}_X(\varepsilon_2)}_{\varepsilon_1\varepsilon_2}$, which completes proof. ■

Corollary 3.6 *$\mathfrak{N}_X(t)$ is an isomorphic invariant of X up to equivalence.*

Proof. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be a pair of equivalent norms on X. If they are γ-equivalent, a simple geometric argument gives us

$$\mathfrak{N}_{(X,\|\cdot\|_1)}(t) \leq \mathfrak{N}_{(X,\|\cdot\|_2)}(\gamma^{-2}t) \leq c\,\mathfrak{N}_{(X,\|\cdot\|_2)}(t)$$

where $c = \mathfrak{N}_{(X,\|\cdot\|_2)}(\gamma^{-2})$. By the symmetry of the argument, we have the equivalence between $\mathfrak{N}_{(X,\|\cdot\|_1)}(t)$ and $\mathfrak{N}_{(X,\|\cdot\|_2)}(t)$. ■

Proof of Proposition 1.7. It is enough to consider the sets $C_k = [A\,|_{A\setminus H}]^k_\varepsilon$ and to take $N_\varepsilon = \mathfrak{N}_X(\varepsilon)$. ■

Corollary 3.7 *Let X be a superreflexive space. Then $\mathfrak{M}_X(t) \leq \mathfrak{N}_X(t)$.*

Once we know the submultiplicativity of $\mathfrak{N}_X$ we can rewrite the estimation given in Lemma 3.4 in a more handy way.

Theorem 3.8 *Let X be a superreflexive space and $\gamma > 1$. There exist a constant $c > 0$ such that $\mathfrak{N}_X(\varepsilon) \leq c\,\vartheta_{|\!|\!|\cdot|\!|\!|}(\varepsilon)^{-1}$ for any γ-equivalent norm $|\!|\!|\cdot|\!|\!|$ on X.*

Proof. Suppose that the norm $\|\cdot\|_2$ satisfies the inequality $\|\cdot\| \leq \|\cdot\|_2 \leq 2\|\cdot\|$. For $\varepsilon \in (0, 1/64)$ Lemma 3.4 gives us that

$$\mathfrak{N}_X(64\varepsilon) \leq 2240\,\vartheta_{\|\cdot\|_2}(\varepsilon)^{-1}.$$

Since $\mathfrak{N}_X(\varepsilon) \leq \mathfrak{N}_X(1/64)\mathfrak{N}_X(64\varepsilon)$, it follows that

$$\mathfrak{N}_X(\varepsilon) \leq 2240\,\mathfrak{N}_X(1/64)\,\vartheta_{\|\cdot\|_2}(\varepsilon)^{-1}.$$

This proves the statement under the special restrictions of the norm $\|\cdot\|_2$. In case of starting with an arbitrary γ-equivalent norm $|\!|\!|\cdot|\!|\!|$ take $\|\cdot\|_2 = \|\cdot\| + \gamma^{-1}|\!|\!|\cdot|\!|\!|$ which satisfies the above assumptions and notice that

$$\vartheta_{\|\cdot\|_2}(a\varepsilon) \geq b\,\vartheta_{|\!|\!|\cdot|\!|\!|}(\varepsilon)$$

where the constants $a, b > 0$ depends only on γ, by the same arguments used in the proof of Lemma 3.1. ■

4 Improving the modulus of uniform convexity

We will transfer the asymptotic behavior of functions $\mathfrak{N}_X$ and $\mathfrak{M}_X$ to the modulus of convexity of a new norm.

Lemma 4.1 *Let X be a superreflexive Banach space and fix $0 < \gamma < 1/12$. Then for every $\varepsilon \in (0, 1]$ there is an equivalent norm $|||\cdot|||$ such that*

$$(1-\gamma)\|x\| \leq |||x||| \leq \|x\|$$

and

$$1 - |||\frac{x+y}{2}||| \geq \frac{\gamma}{36\,\mathfrak{M}_X(\varepsilon)},$$

whenever $|||x||| = |||y||| = 1$ and $\|x-y\| \geq 16\varepsilon$.

Proof. Fix $\gamma < \eta \leq 1/12$ and take $\alpha = 1 - 4\eta$, $\beta = 1 - 2\eta$ and $\xi = 1 - \gamma$. Given ε, fix $N = \mathfrak{M}_X(\varepsilon)$. There are sets (C_n) such that $C_0 = B_X$, $[C_{n-1}]'_\varepsilon \subset C_n$ and $\alpha B_X \subset C_N \subset \beta B_X$. Indeed, C_n is obtained from C_{n-1} by removing all the slices of diameter less than ε which are disjoint from βB_X. Any slice $S = B_X \cap H$ disjoint from βB_X has width less than 4η. The definition of $\mathfrak{M}_X(\varepsilon)$, implies that $C_N \cap H$ has width less than 2η. As S was arbitrary, we get that $C_N \subset \alpha B_X$.
Denote by F_n the Minkowski functional of C_n. We have $\|x\| \leq F_n(x) \leq \alpha^{-1}\|x\|$. Consider the equivalent norm on X defined by

$$|||x||| = (1-\gamma)\,\|x\| + \frac{\gamma\alpha}{N+1}\sum_{n=0}^{N} F_n(x).$$

The definition clearly implies that $\xi\|x\| \leq |||x||| \leq \|x\|$. In particular, if $|||x||| = 1$, then $1 \leq \|x\| \leq \xi^{-1}$ and $1 \leq F_n(x) \leq \alpha^{-1}\xi^{-1}$. Note that $\alpha\xi^2 > 1/2$.
Assume $|||x||| = |||y||| = 1$ and $\|x-y\| \geq 16\varepsilon$, then

$$\frac{1}{2} < \alpha\xi \leq \frac{F_n(x)}{F_n(y)} \leq (\alpha\xi)^{-1} < 2.$$

Take $\overline{x} = F_n(x)^{-1}x$ and $\overline{y} = F_n(y)^{-1}y$ with $n < N$. Clearly $\overline{x}, \overline{y} \in C_n$ and applying Lemma 2.2 to norm $|||\cdot|||$ we get

$$\|\overline{x} - \overline{y}\| \geq |||\overline{x} - \overline{y}||| \geq \frac{\alpha\xi}{2}|||x-y||| \geq \frac{\alpha\xi^2}{2}\|x-y\| \geq 4\varepsilon.$$

The segment with endpoints $\overline{x}$ and $\frac{1}{2}(\overline{x}+\overline{y})$ has length greater than 2ε. Lancien's midpoint argument we have $\frac{1}{4}(3\overline{x}+\overline{y}) \in C_{n+1}$. After a similar reasoning with the segment of endpoints $\overline{y}$ and $\frac{1}{2}(\overline{x}+\overline{y})$, we get that $\frac{1}{4}(\overline{x}+3\overline{y}) \in C_{n+1}$. Therefore, we have $\lambda\overline{x} + (1-\lambda)\overline{y} \in C_{n+1}$ for any $\lambda \in [\frac{1}{4}, \frac{3}{4}]$. In particular, we may choose

$$\lambda = \frac{F_n(x)}{F_n(x) + F_n(y)}$$

and thus

$$\frac{x+y}{F_n(x) + F_n(y)} \in C_{n+1}$$

which implies

$$F_{n+1}(\frac{x+y}{2}) \leq \frac{F_n(x)+F_n(y)}{2}.$$

We now apply this inequality to the norm $\|\cdot\|$ with the shift argument

$$\sum_{n=0}^{N} F_n(\frac{x+y}{2}) = \|\frac{x+y}{2}\| + \sum_{n=1}^{N} F_n(\frac{x+y}{2})$$

$$\leq \|\frac{x+y}{2}\| + \sum_{n=0}^{N-1} \frac{F_n(x)+F_n(y)}{2} \leq \sum_{n=0}^{N} \frac{F_n(x)+F_n(y)}{2} - (\beta^{-1} - \xi^{-1})$$

since $\|\frac{x+y}{2}\| \leq \xi^{-1}$ and $F_N(x), F_N(y) \geq \beta^{-1}$. Observe that

$$\beta^{-1} - \xi^{-1} = \frac{1}{1-2\eta} - \frac{1}{1-\gamma} = \frac{2\eta - \gamma}{(1-2\eta)(1-\gamma)} \geq \eta.$$

Therefore,

$$\sum_{n=0}^{N} F_n(\frac{x+y}{2}) \leq \sum_{n=0}^{N} \frac{F_n(x)+F_n(y)}{2} - \eta.$$

Thus,

$$1 - \|\frac{x+y}{2}\| \geq \frac{\gamma\alpha\eta}{2(N+1)}$$

and the statement of the lemma follows by taking $\eta = 1/12$. ■

Proof of Theorem 1.5. Without loss of generality we may assume $\eta < 12/11$. Fix $\gamma = 1 - \eta^{-1}$ and take $t > 0$. For $\varepsilon = \frac{t}{16}$ consider the norm $\|\cdot\|_t$ given by Lemma 4.1. If $\|x\|_t = \|y\|_t = 1$ and $t \leq \|x-y\|_t \leq \|x-y\|$. Then we have

$$1 - \|\frac{x+y}{2}\|_t \geq \frac{1-\eta^{-1}}{36\,\mathfrak{N}_X(t/16)} \geq \frac{1-\eta^{-1}}{36\,\mathfrak{N}_X(1/16)}\,\mathfrak{N}_X(t)^{-1}$$

which shows that $\delta_{\|\cdot\|_t}(t) > c\,\mathfrak{N}_X(t)^{-1}$, where the constant c only depends on X and a fixed $\eta > 1$. ■

Lemma 4.2 *Let (ε_n) and (θ_n) be non increasing sequences of positive numbers with limit zero. Suppose in addition that*

$$\sum_{n=1}^{\infty} \mathfrak{M}_X(\varepsilon_n)\,\theta_n < +\infty.$$

Then there is an equivalent norm $\|\cdot\|$ on X and a constant $k > 0$ such that for every $n \in \mathbb{N}$ we have

$$1 - \|\frac{x+y}{2}\| \geq k\,\theta_n$$

whenever $\|x\| = \|y\| = 1$ and $\|x-y\| \geq 8\varepsilon_n$.

Proof. Put $\alpha_n = \mathfrak{M}_X(\varepsilon_n)\,\theta_n$ and $\sigma = \sum_{n=1}^\infty \alpha_n$ for simplicity. Let $\|\cdot\|_n$ be the equivalent norm built in Lemma 4.1 for $\varepsilon = \varepsilon_n$ and $\gamma = 1/12$. Recall that $\frac{9}{10}\|x\| \leq \|x\|_n \leq \|x\|$ for every $x \in X$. Consider the equivalent norm

$$|||x||| = \|x\| + \frac{1}{5\sigma}\sum_{n=1}^\infty \alpha_n \|x\|_n.$$

It is easy to see that if $|||x||| = 1$ then $\frac{3}{4} \leq \|x\|_n \leq 1$ for every $n \in \mathbb{N}$. Suppose that $|||x||| = |||y||| = 1$ and $\|x-y\| > 8\varepsilon_n$. For $n \in \mathbb{N}$ we have

$$\lambda = \frac{\|x\|_n}{\|x\|_n + \|y\|_n} \in (\frac{1}{4}, \frac{3}{4})$$

and using convexity we obtain that

$$\|\lambda \frac{x}{\|x\|_n} + (1-\lambda)\frac{y}{\|y\|_n}\|_n \leq 1 - \frac{1}{432\,\mathfrak{M}_X(\varepsilon_n)}.$$

After some elementary computations we have

$$\frac{\|x\|_n + \|y\|_n}{2} - \|\frac{x+y}{2}\|_n \geq \frac{3}{1728\,\mathfrak{M}_X(\varepsilon_n)}.$$

This information applied to the norm $|||\cdot|||$ gives

$$1 - |||\frac{x+y}{2}||| \geq \frac{\theta_n}{2880\,\sigma}$$

and the proof is complete. ■

Proof of Corollary 1.6. Take $\varepsilon_n = 2^{-n}$ and $\theta_n = \mathfrak{M}_X(2^{-n})^{-1} n^{-\alpha}$ in Lemma 4.2. Given $\varepsilon \in (0,1]$, take $n \in \mathbb{N}$ such that $2^{3-n} \leq \varepsilon < 2^{4-n}$. We have $\|x-y\| > \varepsilon \geq 8\varepsilon_n$ and thus

$$1 - |||\frac{x+y}{2}||| \geq k\,\mathfrak{M}_X(2^{-n})^{-1} n^{-\alpha} \geq k\,\mathfrak{M}_X(\frac{\varepsilon}{16})^{-1}(4 - \frac{\log\varepsilon}{\log 2})^{-\alpha}.$$

Since $\mathfrak{M}_X(\frac{\varepsilon}{16})^{-1} \geq \mathfrak{N}_X(1/16)^{-1}\,\mathfrak{N}_X(\varepsilon)^{-1}$, for some $K > 0$ we have

$$1 - |||\frac{x+y}{2}||| \geq K\,(-\log\varepsilon)^{-\alpha}\,\mathfrak{N}_X(\varepsilon)^{-1}$$

if $\|x-y\| > \varepsilon$. It is not difficult to enforce the above inequality for points that satisfies $|||x-y||| > \varepsilon$ just changing the value of $K > 0$. ■

Let us remark that for $p \geq 2$, there exist constants $c_p, C_p > 0$ such that $c_p t^{-p} \leq \mathfrak{M}_{\ell_p}(t) \leq C_p t^{-p}$. However the norm given in Corollary 1.6 will not be of power type p. With the same idea, we can produce norms with modulus of convexity arbitrarily close to $\mathfrak{M}_X(t)^{-1}$ (equivalently $\mathfrak{N}_X(t)^{-1}$) in asymptotic sense but unfortunately they are not equivalent.

Proof of Theorem 1.2. The construction of the function $\mathfrak{N}_X(t)$ and its submultiplicativity was established in Theorem 3.5 and the property of being $\mathfrak{N}_X(t)^{-1}$ an upper bound in the order $\preceq$ for the set of all the moduli of convexity follows from Theorem 3.8 as well as the relation between the functions $\vartheta_{|||\cdot|||}$ and $\delta_{|||\cdot|||}$. It only remains to show that it is a minimum. Suppose that $\phi(t) \preceq \mathfrak{N}_X(t)^{-1}$ and $\phi(t)$ is also an upper bound for all the moduli of convexity.
The set $\mathcal{E}$ of equivalent norms on X is a complete metric space endowed with the metric

$$d(\|\cdot\|_1, \|\cdot\|_2) = \sup\{|\log(\frac{\|x\|_1}{\|x\|_2})| : x \neq 0\}.$$

Indeed, the map $\Lambda : \mathcal{E} \to \ell_\infty(S_X)$ given by $\Lambda(|||\cdot|||) = (\log(|||x|||))_{x\in S_X}$ is an isometry onto its image. Now, if $(\|\cdot\|_n)$ is a Cauchy sequence in $\mathcal{E}$, the pointwise convergence of $(\Lambda(\|\cdot\|_n))$ in $\ell_\infty(S_X)$ implies the existence of $\lim_n \|x\|_n = |||x|||$ for every $x \in X$. It is elementary to check that $|||\cdot|||$ is an equivalent norm on X. Note that the set of norms which are η-equivalent to $|||\cdot|||$ is exactly the ball of center $|||\cdot|||$ and radius $\log(\eta)$ in $\mathcal{E}$.
By the Baire category theorem applied to the cover of $\mathcal{E}$ given by the sets $C_n = \{|||\cdot||| : \delta_{|||\cdot|||}(t) \leq n\,\phi(t)\}$ there exists an equivalent norm $\|\cdot\|_0$ and a $\gamma > 1$ such that $\delta_{\|\cdot\|}(t) \leq a\,\phi(t)$ for all $t \in (0,1]$ and every γ-equivalent (to $\|\cdot\|_0$) norm $\|\cdot\|$ on X. Without loss of generality we may assume X is already endowed with the norm $\|\cdot\|_0$ since the function $\mathfrak{N}_X$ remains equivalent. By Theorem 1.5 there exists a constant $b > 0$ such that for a given $t \in (0,1]$ there is a γ-equivalent norm $\|\cdot\|_t$ such that

$$\delta_{\|\cdot\|_t}(t) \geq b\,\mathfrak{N}_X^{-1}(t).$$

Multiplying by $\phi(t)^{-1}$ we get

$$b\,\mathfrak{N}_X^{-1}(t)\phi(t)^{-1} \leq \delta_{\|\cdot\|_t}(t)\,\phi(t)^{-1} \leq a$$

and hence $\mathfrak{N}_X(t)^{-1} \leq (a/b)\,\phi(t)$ independently of the renorming and so for every $t \in (0,1]$. We deduce that $\phi(t)$ is equivalent to $\mathfrak{N}_X(t)^{-1}$ as desired. ■

Proof of Corollary 1.3. For any $s \in (0,1)$ and $0 < t < s$, take $n \in \mathbb{N}$ the integer part of $\frac{\log t}{\log s}$. We have $s^{n+1} \leq t$ and so

$$\mathfrak{N}_X(t) \leq \mathfrak{N}_X(s^{n+1}) \leq \mathfrak{N}_X(s)^{n+1} \leq \mathfrak{N}_X(s)\,\mathfrak{N}_X(s)^{\frac{\log t}{\log s}} = c\,t^{\frac{\log(\mathfrak{N}_X(s))}{\log s}}.$$

If $p > \mathfrak{p}_X$, a suitable choice of s provides that $\mathfrak{N}_X(t) \leq c\,t^{-p}$ for $t \in (0,s)$, and thus $\mathfrak{N}_X(t) \leq c\,t^{-p}$ for every $t \in (0,1)$ just taking a larger $c > 0$. If the infimum is attained at some $s \in (0,1)$, the previous estimation gives that $\mathfrak{N}_X(t) \leq c\,t^{-\mathfrak{p}_X}$ for $t \in (0,1]$. On the other hand, if we have

$$\mathfrak{p}_X \leq \frac{\log(\mathfrak{N}_X(t))}{\log(1/t)}$$

for every $t \in (0,1)$, a simple computation gives us that $\mathfrak{N}_X(t) \geq t^{-\mathfrak{p}_X}$. Consequently, $t^{-\mathfrak{p}_X}$ is a decreasing multiplicative function equivalent to $\mathfrak{N}_X(t)$. ■

Proposition 4.3 *The function $\mathfrak{M}_X(t)$ is an isomorphic invariant of X equivalent to $\mathfrak{N}_X(t)$ and $\mathfrak{M}_X(t)^{-1}$ satisfies the Δ_2 condition at zero.*

Proof. The combination of Theorem 3.8 and Lemma 4.1 give for some 2-equivalent norm $|||\cdot|||$ and constants

$$\mathfrak{N}_X(16t) \leq a\,\delta_{|||\cdot|||}(16t) \leq a\,b\,\mathfrak{M}_X(t)$$

and so $\mathfrak{N}_X(t) \leq a\,b\,\mathfrak{N}_X(1/16)\,\mathfrak{M}_X(t)$. Since we always have $\mathfrak{M}_X(t) \leq \mathfrak{N}_X(t)$, we obtain the equivalence. Observe now

$$\mathfrak{M}_X(t) \leq \mathfrak{N}_X(t) \leq \mathfrak{N}_X(1/2)\,\mathfrak{N}_X(2t) \leq c\,\mathfrak{M}_X(2t)$$

which is equivalent to the Δ_2 condition of $\mathfrak{M}_X(t)^{-1}$. ∎

5 Further applications

This final section gathers together several scattered results related to superreflexive Banach spaces or their uniformly convex renormings.

Lower bounds for $\mathfrak{N}_X$. A submultiplicative function is bounded above by power functions. A bound from below also has some interesting consequences. In our next lemma we consider the plane $\mathbb{R}^2$ with the Euclidean norm.

Lemma 5.1 *If $\varepsilon > 0$, then $\mathfrak{M}_{\mathbb{R}^2}(\varepsilon) \geq \varepsilon^{-2}/8$.*

Proof. Given $r \in [1/4, 1/2]$ observe that $\sqrt{(r+2\varepsilon^2)^2 - r^2} > \varepsilon$, and so

$$rB_{\mathbb{R}^2} \subset \left[(r+2\varepsilon^2)B_{\mathbb{R}^2}\right]'_{\varepsilon}.$$

It follows that $(1/4)B_{\mathbb{R}^2} \subset [(1/2)B_{\mathbb{R}^2}]^n_\varepsilon$ if $n \in \mathbb{N}$ satisfies that $n2\varepsilon^2 \leq 1/4$. On the other hand, it is obvious that $[(1/2)B_{\mathbb{R}^2}]^{\mathfrak{M}_{\mathbb{R}^2}(\varepsilon)}_\varepsilon \subset (1/4)B_{\mathbb{R}^2}$. ∎

This last result is related to the well known approximate upper bound $\varepsilon^2/8$ for any modulus of convexity, which is often presented in literature as a consequence of the celebrated Dvoretzky's Theorem or Nördlander's estimation, see [3].

Proposition 5.2 *If X has dimension bigger than one, then $\mathfrak{N}_X(\varepsilon) \geq c\,\varepsilon^{-2}$ for some $c > 0$. In particular, we have $\mathfrak{p}_X \geq 2$.*

Proof. It is clear that $\mathfrak{N}_X(\varepsilon) \geq \mathfrak{N}_Y(\varepsilon)$ for any subspace $Y \subset X$. If Y has dimension 2, then $\mathfrak{N}_Y(\varepsilon)$ is equivalent to ε^{-2} by the previous lemma. ∎

Relation with the dentability index. A simple geometric argument gives the following inequality

$$Dz(B_X, \varepsilon) \leq \mathfrak{M}_X(\varepsilon/2)$$

for a superreflexive Banach space X and $\varepsilon > 0$. However, it seems more complicated to control $\mathfrak{M}_X(\varepsilon)$ or $\mathfrak{N}_X(\varepsilon)$ by means of $Dz(B_X, \varepsilon)$ using a direct argument. Lancien [19] proved the next result.

Theorem 5.3 (Lancien) *Let X be a superreflexive Banach space, then there exists an equivalent norm $\|\!|\cdot|\!\|$ and a $c>0$ such that for every $\varepsilon>0$ we have*

$$\delta_{\|\!|\cdot|\!\|}(\varepsilon)\geq \frac{c\,\varepsilon^2}{Dz(B_X,\varepsilon/8)^2}.$$

This result together Theorem 3.8 implies that

$$\mathfrak{N}_X(\varepsilon)\leq C\,\varepsilon^{-2}\,Dz(B_X,\varepsilon)^2$$

for some constant $C>0$ and every $\varepsilon\in(0,1]$. We believe that this bound can be improved. Unfortunately we do not know how to adapt the "shift argument" to the hypothesis $Dz(B_X,\varepsilon)<+\infty$ to provide an alternative approach to Lancien's Theorem.

Extension of UC norms. The next result solves Problem IV.4 of [5] for the extension of uniformly convex norms.

Theorem 5.4 *Let X be a superreflexive space and $Y\subset X$ a subspace. Suppose that Y has a uniformly convex norm $\|\cdot\|_Y$ (resp. with modulus of power type). Then there exists an equivalent uniformly convex norm on X (resp. with modulus of power type) such that its restriction to X is $\|\cdot\|_Y$.*

Proof. Fix $\varepsilon>0$ and $\eta=\delta_{\|\cdot\|_Y}(\varepsilon)$. We may assume that the norm $\|\cdot\|$ of X is the convex hull of $B\cup B_Y$, where B is an equivalent ball on X such that $B\cap Y\subset(1-\eta)B_Y$. Clearly the restriction of $\|\cdot\|$ to Y is $\|\cdot\|_Y$. Let $J\subset X^*$ be the set of norm one functionals which attains its maximum on B_Y. Take $f\in J$ and $G=\{x\in X: f(x)>1-\eta\}$. Observe that $\operatorname{diam}(A)\leq\varepsilon$ where $A=B_Y\cap G$. Consider the convex set

$$D=\{(1-\lambda)y+\lambda z: y\in A, z\in B_X\setminus G, \lambda\in[\varepsilon,1]\}.$$

Reasoning as in our proof of Theorem 1.1, we get that $\operatorname{diam}(B_X\cap H)\leq 5\varepsilon$, where $H=\{x\in X: f(x)>1-\eta\varepsilon\}$. If we take

$$C=B_X\setminus\bigcup_{f\in J}\{x\in X: f(x)>1-\eta\varepsilon\}$$

then we have $C\cap Y=(1-\eta\varepsilon)B_Y$ and $[B_X]'_{5\varepsilon}\subset C$. Take $C_0=B_Y$, $C_1=C$ and $C_n=[C_1\,|_{(1-\eta\varepsilon)B_X}]^{n-1}_{5\varepsilon}$ for $n\geq 2$. We will have $C_n\subset(1-\frac{\eta\varepsilon}{2})B_X$ for $n=\mathfrak{N}_X(5\varepsilon)$ The norm $\|\!|\cdot|\!\|$ provided by the proof of Lemma 4.1 with this sequence of sets satisfies that $\|\!|y|\!\|=c\|y\|_Y$ for some $c>0$ and every $y\in Y$. The combination of these norms for different values of ε (see Lemma 4.2) produces a uniformly convex norm on X such that its restriction to Y is a multiple of $\|\cdot\|_Y$. If $\|\cdot\|_Y$ has modulus of power type, then we can take $\eta=a\,\varepsilon^{p_1}$ for some $a,p_1>0$. We also know that $\mathfrak{N}_X(\varepsilon)\leq b\,\varepsilon^{-p_2}$ for some $b,p_2>0$. Repeating the computations of Lemma 4.1 we obtain a lower bound for the modulus of the form $c\,\varepsilon^{p_1+p_2}$. The norm $\|\!|\cdot|\!\|$ given by Lemma 4.1 can be done with modulus of convexity of power type any $p>p_1+p_2$.■

Generalized cotype. Following [10], a function $\phi(t)$ is a *generalized cotype* of the space X if there exist constants $a, b > 0$ such that $\sum_{k=1}^{n} \phi(\|x_k\|) \leq b$ whenever $x_1, \dots, x_n \in X$ satisfy

$$\int_0^1 \| \sum_{k=1}^{n} r_k(t)\, x_k \| \, dt \leq a$$

where $(r_k(t))$ are the Rademacher functions, see [7]. Note that this last inequality implies $\|x_k\| \leq a$ for $1 \leq k \leq n$. A remarkable result of Figiel and Pisier [11] establishes that any modulus of convexity is a generalized cotype.

Proposition 5.5 [1] *The function $\mathfrak{N}_X(t)^{-1}$ is a generalized cotype of X.*

Proof. We say that a finite sequence of vectors $(x_k)_{k=1}^{n}$ is γ-*delimited* if it satisfies

$$\int_0^1 \| \sum_{k=1}^{n} r_k(t)\, x_k \| \, dt \leq \gamma.$$

This notion can be easily extended to infinite sequences. Denote by $\mathcal{S}(\gamma)$ the set of all the γ-delimited (finite or infinite) sequences in X. Note that all the elements of a γ-delimited sequence are norm bounded by γ. Consider the numbers (c_n) defined by

$$c_n = \sup\{\sum_{k=1}^{n} \mathfrak{N}_X(\|x_k\|)^{-1} : \{x_1, \dots, x_n\} \in \mathcal{S}(1)\}.$$

If the sequence (c_n) is bounded then the statement is true. We shall proceed by contradiction, so assume that $\lim_n c_n = +\infty$. For $j \in \mathbb{N}$, find $n_j \in \mathbb{N}$ such that $c_{n_j} > \mathfrak{N}_X(2^{-j})$ and $\{x_{j,1}, \dots, x_{j,n_j}\} \in \mathcal{S}(1)$ such that $\sum_{k=1}^{n_j} \mathfrak{N}_X(\|x_{j,k}\|)^{-1} > \mathfrak{N}_X(2^{-j})$ and so we have $\sum_{k=1}^{n_j} \mathfrak{N}_X(\|2^{-j} x_{j,k}\|)^{-1} > 1$ by the submultiplicativity of $\mathfrak{N}_X$. Arrange all the vectors $\{2^{-j} x_{j,k} : j \in \mathbb{N}, 1 \leq k \leq n_j\}$ into a sequence (x_n). Note that $(x_n) \in \mathcal{S}(1)$ and $\sum_{n=1}^{\infty} \mathfrak{N}_X(\|x_n\|)^{-1} = +\infty$. Moreover, without loss of generality, assume that all $x_n \neq 0$ and $(\|x_n\|)$ is not increasing. Take $\xi \in (0, 1]$. The submultiplicativity of $\mathfrak{N}_X$ implies that $\sum_{n=1}^{\infty} \mathfrak{N}_X(\xi\|x_n\|)^{-1} = +\infty$. Now, find $\theta_n > 0$ such that $\sum_{n=1}^{\infty} \theta_n = +\infty$ and $\sum_{n=1}^{\infty} \mathfrak{N}_X(\xi\|x_n\|)\, \theta_n < +\infty$. Apply Lemma 4.2 to find a 2-equivalent norm $|\!|\!| \cdot |\!|\!|$ such that $\delta_{|\!|\!| \cdot |\!|\!|}(8\xi\|x_n\|) \geq k\, \theta_n$ for some $k > 0$ and every $n \in \mathbb{N}$. By construction we have

$$\sum_{n=1}^{\infty} \delta_{|\!|\!| \cdot |\!|\!|}(16\xi\, |\!|\!| x_n |\!|\!|) \geq \sum_{n=1}^{\infty} \delta_{|\!|\!| \cdot |\!|\!|}(\lambda_n \xi\, |\!|\!| x_n |\!|\!|) = +\infty$$

where $\lambda_n |\!|\!| x_n |\!|\!| = 8\|x_n\|$. Since $\xi > 0$ can be taken as small as desired, we arrive to a contradiction with the afore mentioned result of Figiel and Pisier [11] stating that a modulus of convexity is a generalized cotype. ■

When this paper was almost finished, Prof. S. Troyanski brought our attention to the work of T. Figiel on superreflexive latices and the use of generalized cotype functions in uniformly convex renorming. We believe that a full understanding

[1] The statement is false and the proof is wrong.

of the connections between our "coordinate-free approach" and Figiel's results on spaces with bases would be an interesting research line in future developments.

A topological setting. All the results in this paper are established for a superreflexive Banach space, but it is possible to adapt them to special sets in general Banach spaces. In our paper [24] we have studied the class of *finitely dentable* convex sets. A bounded convex closed set $A \subset X$ of a Banach space is finitely dentable if $Dz(A, \varepsilon) < +\infty$ for every $\varepsilon > 0$. We proved that these sets are weakly compact subsets of X and uniformly Eberlein compacta when considered in its weak topology. The techniques of Section 3 and ideas from [24, 26] can be combined to prove the following.

Proposition 5.6 *Let $A \subset X$ be a finitely dentable bounded convex closed set. Then there exists a $B \subset X$ bounded convex symmetric which is also finitely dentable, satisfying that $A \subset B$ and for every $\varepsilon > 0$ there is $\theta > 0$ such that*

$$[B]'_\varepsilon \subset (1 - \theta)\, B.$$

We are not giving the proof of this result, but the reader can compare it with [26, Theorem 1.1]. Note that from a topological point of view, a compact with *Szlenk index at most* ω is just a *descriptive compactum*, that is, a compact space having a σ*-isolated network*, see [26]. This notion is a kind of covering property, related to paracompactness, that has been used in the characterization of *locally uniformly convex* renorming of Banach spaces (see the book [22] for missing definitions account of results and bibliography). The application of these techniques to uniform convexity has been studied in [8].

Acknowledgements. I am indebted with Professors Ondrej Kalenda, José Orihuela and Stanimir Troyanski for fruitful discussions on the subject of this paper, and Antonio Pérez for his careful reading of the manuscript.

References

[1] Z. Altshuler, 'Uniform convexity in Lorentz sequence spaces', *Israel J. Math.* 20 (1975), no. 3-4, 260-274.

[2] B. Beauzamy *Introduction to Banach spaces and their geometry*, North-Holland Mathematics Studies, 68. Notas de Matemática [Mathematical Notes], 86. North-Holland Publishing Co., Amsterdam-New York, 1982.

[3] Y. Benyamini, J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, American Mathematical Society Colloquium Publications 48, 2000.

[4] J. A. Clarkson, 'Uniformly convex spaces', *Trans. Amer. Math. Soc.* 40 (1936), no. 3, 396-414.

[5] R. Deville, G. Godefroy, V. Zizler *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64. Longman Scientific & Technical, Harlow, 1993.

[6] P. Enflo, 'Banach spaces which can be given an equivalent uniformly convex norm' *Israel J. Math.* 13 (1972), 281-288.

[7] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, *Functional analysis and infinite-dimensional geometry*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 8. Springer-Verlag, New York, 2001.

[8] S. Ferrari, *Localization techniques for renorming*, Tesi di Dottorato, Universita degli Studi di Milano, 2013.

[9] T. Figiel, 'On the moduli of convexity and smoothness', *Studia Math.* 56 (1976), no. 2, 121-155.

[10] T. Figiel, 'Uniformly convex norms on Banach lattices', *Studia Math.* 68 (1980), no. 3, 215-247.

[11] T. Figiel, G. Pisier, 'Series aleatoires dans les espaces uniformement convexes ou uniformement lises', *C.R. Acad. Sci. Paris*, Serie A, 279 (1974), 611-614.

[12] G. Godefroy, 'Renormings of Banach spaces' *Handbook of the Geometry of Banach spaces* Vol. 1, W.B. Johnson and J. Lindenstrauss editors, Elsevier, Amsterdam (2001), 781–835.

[13] G. Godefroy, N.J. Kalton, G. Lancien, 'Szlenk indices and uniform homeomorphisms', *Trans. Amer. Math. Soc.* 353 (2001), no. 10, 3895–3918.

[14] P. Hájek, V. Montesinos, J. Vanderwerff, V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 26. Springer, New York, 2008.

[15] R. C. James, 'Super-reflexive Banach spaces', *Canad. J. Math.* 24 (1972), 896-904.

[16] R. C. James, 'Super-reflexive spaces with bases', *Pacific J. Math.* 41 (1972), 409-419.

[17] W.B. Johnson, J. Lindenstrauss, *Handbook of Banach space*

[18] G. Lancien, 'A survey on the Szlenk index and some of its applications', *RACSAM Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* 100 (2006), no. 1-2, 209–235.

[19] G. Lancien, 'On uniformly convex and uniformly Kadec-Klee renormings', *Serdica Math. J.* 21 (1995), no. 1, 1–18.

[20] J. Lindenstrauss, L. Tzafriri *Classical Banach spaces II. Function spaces.*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], 97. Springer-Verlag, Berlin-New York, 1979.

[21] R. P. Maleev, S. L. Troyanski 'On the moduli of convexity and smoothness in Orlicz spaces', *Studia Math.* 54 (1975), no. 2, 131-141.

[22] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia *A Nonlinear Transfer Technique for Renorming*, Lect. Notes in Math. 1951, Springer Verlag, 2009.

[23] G. Pisier, 'Martingales with values in uniformly convex spaces', *Israel J. Math.* 20 (1975), no. 3-4, 326-350.

[24] M. Raja, 'Finitely dentable functions, operators and sets', *J. Convex Anal.* 15 (2008), 219–233.

[25] M. Raja, 'On weak* uniformly Kadec-Klee renorming', *Bull. London Math. Soc.* 42 (2010), 221–228.

[26] M. Raja, 'Compact spaces of Szlenk index ω', *J. Math. Anal. App.* 391 (2012), 496–509.

Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, SPAIN
E-mail: matias@um.es