WEAKLY METRIZABILITY OF SPHERES AND RENORMINGS OF BANACH SPACES

S. FERRARI[1], J. ORIHUELA[2], M. RAJA[2*]

Abstract. We study the metrizability of weak topologies when restricted to the unit sphere of some equivalent norm on a Banach space, and its relationships with other geometrical properties of norms. In case of dual Banach space X^* we prove that there exists a dual norm such that its unit sphere is weak* metrizable if and only if $(\mathscr{B}_{X^*}, w^*)$ is a descriptive compact, which provides a complete characterization.

1. Introduction

The norm topology and the weak topology of a normed space $(X, \|\cdot\|)$ are very different when X is infinite dimensional. Nevertheless, these topologies may agree on very representative subsets of X, as for instance its unit sphere $\mathcal{S}_X = \{x \in X : \|x\| = 1\}$. This is the so called *Kadec property*, which is easily checkable in Hilbert spaces and some classical spaces as ℓ_p for $p \in [1, +\infty)$ (see [2] and [3]). We will deal with a more general property. Given $F \subset X^*$ a subspace of the topological dual of X, the topology on X of pointwise convergence on F is denoted $\sigma(X, F)$. We are mainly interested in norming subspaces F, that is, if

$$\|x\|_F = \sup\{|x^*(x)| : x^* \in F, \|x^*\| \leq 1\}$$

is an equivalent norm on X. As expected, the norm of X is said to $\sigma(X, F)$-*Kadec* if the norm topology and the $\sigma(X, F)$-topology agree on $\mathcal{S}_X$. Kadec properties are related to geometric properties of the norm. The norm is said to be locally uniformly rotund (LUR) if given $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subset X$, then $\lim_n \|x_n - x\| = 0$ whenever

$$\lim_n \|x_n\| = \lim_n \left\| \frac{x_n + x}{2} \right\| = \|x\|$$

or equivalently, $\lim_n \left(2\|x_n\|^2 + 2\|x\|^2 - \|x_n + x\|^2\right) = 0$ (see Lemma 2.2). It is not difficult to check that LUR norms are both strictly convex (or rotund) and Kadec. Moreover, if a LUR norm is $\sigma(X, F)$-lower semicontinuous (lsc) for some norming subspace $F \subset X^*$, then it is $\sigma(X, F)$-Kadec and every point $x \in \mathcal{S}_X$ is $\sigma(X, F)$-denting point of the unit ball $\mathscr{B}_X$, that is, x is contained in $\sigma(X, F)$-open slices on $\mathscr{B}_X$ of arbitrarily small diameter. Recall that an open slice of some set $A \subset X$ is a nonempty set of the form $A \cap H$ where H is an open halfspace. Note that if F is closed, then the $\sigma(X, F)$-open halfspaces are exactly the

Date: October 2015

* Corresponding author.

2010 *Mathematics Subject Classification.* 46B03, 54E35, 46B20, 46B26.

Key words and phrases. metrization, symmetrizable space, renorming, descriptive compact, locally uniformly rotund norm, sphere.

halfspaces defined by elements from F.

All these notions are tightly related through renoming. The following theorem contains early results of S. Troyanski [15], the linear topological characterization of LUR renormability from [6] and their generalizations for weaker topologies [12].

Theorem 1.1. *Let X a Banach space and $F \subset X^*$ a norming subspace. The following statements are equivalent:*

(i) *X admits an equivalent $\sigma(X,F)$-lsc LUR norm;*
(ii) *X admits an equivalent rotund norm and an equivalent $\sigma(X,F)$-Kadec norm;*
(iii) *X admits an equivalent norm such that every point of $\mathcal{S}_X$ is $\sigma(X,F)$-denting in $\mathscr{B}_X$;*
(iv) *there is a sequence $(A_n)_{n\in\mathbb{N}} \subset X$ of sets such that for every $x \in X$ and every $\varepsilon > 0$ there is a $\sigma(X,F)$-open halfspace H containing x and some $n \in \mathbb{N}$ such that $\|\cdot\|\text{-}\operatorname{diam}(A_n \cap H) < \varepsilon$.*

Let us say that in the case of a dual Banach space with its weak* topology the corresponding result is stronger: in statement *(ii)* we may remove the existence of a rotund norm (see [12]). Both in [7, Problem 3] and in [8, Question 6.4], it was asked if, in the case of $F = X^*$ (i.e. $\sigma(X,F)$ is the weak topology), it is possible to add to the equivalences of Theorem 1.1 the following condition:

($\star$) *X admits an equivalent rotund norm and an equivalent norm with a metrizable unit sphere for the weak topology.*

We were not able to prove the above conjecture, but we have managed to prove some results in that direction. Particularly, we have considered the study of the metrizability of $(\mathcal{S}_X, \sigma(X,F))$ as our main goal. The first of our results is a characterization in the spirit of the Kadec renorming characterization [11].

Theorem 1.2. *Let X a Banach space and $F \subset X^*$ a norming subspace. The following statements are equivalent:*

(i) *X admits a $\sigma(X,F)$-lsc equivalent norm such that $(\mathcal{S}_X, \sigma(X,F))$ is metrizable;*
(ii) *there is a metric d on X whose topology is finer than $\sigma(X,F)$ and a sequence $(A_n)_{n\in\mathbb{N}} \subset X$ of convex $\sigma(X,F)$-closed sets such that for every $x \in X$ and every $\varepsilon > 0$ there is a $\sigma(X,F)$-neighborhood U of x such that $d\text{-}\operatorname{diam}(A_n \cap U) < \varepsilon$.*

Compare with Theorem 1.1 and note that we require the convexity of the sets of the sequence $(A_n)_{n\in\mathbb{N}}$. In the characterization of Kadec renorming [11] it is not known if convexity can be removed. If the sets $(A_n)_{n\in\mathbb{N}}$ are not convex in statement (ii) we can produce a $\sigma(X,F)$-lsc "gauge function" $F : X \to [0,+\infty)$ such that the $\sigma(X,F)$-topology and the d-topology agree on the sets $\{x \in X : F(x) = r\}$ with $r \in [0,+\infty)$.

We found that the metrizabilty of spheres is tightly related to this generalization of the LUR property. Let τ be a topology on the normed space X, then a norm on X is said to be τ-locally uniformly rotund (τ-LUR) if given $x \in X$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ we have $\tau\text{-}\lim_n x_n = x$, whenever

$$\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\right) = 0.$$

Clearly, when τ is the norm topology we retrieve the notion of LUR norm. The second named author proved the following characterization of $\sigma(X,F)$-LUR renormability in [10] that compares to 1.2.

Theorem 1.3. *Let X a Banach space and $F \subset X^*$ a norming subspace. The following statements are equivalent:*

(i) *X admits an equivalent $\sigma(X,F)$-lsc and $\sigma(X,F)$-LUR;*
(ii) *there is a metric d on X whose topology is finer than $\sigma(X,F)$ and a sequence of sets $(A_n)_{n\in\mathbb{N}} \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$ there is a $\sigma(X,F)$-open halfspace H containing x such that $d\text{-}\operatorname{diam}(A_n \cap H) < \varepsilon$.*

Following the proof of this result in [10], we may check that the metric d plays a technical role in the construction of a particular kind of *network* and its relation with the $\sigma(X,F)$-topology does not go on beyond the "convexification" of the sets $(A_n)_{n\in\mathbb{N}}$ made implicitly with slice localization principles. In the case of the weak topology, it was established in [7] that a w-LUR Banach space is LUR renormable. Dual w^*-LUR norms in dual Banach spaces has been studied, but we will discuss this later in relation with our results in the dual case.

Let us pass to our second main result. We want to discuss the conditions that give the metrizability of the unit sphere together with a basis made of slices. Note that the condition in statement (i) is stronger than the $\sigma(X,F)$-LUR property.

Theorem 1.4. *Let X a normed space and F a norming subspace in X^*. The following facts are equivalent:*

(i) *X admits an equivalent $\sigma(X,F)$-lsc norm with the following property: let $x \in X$ and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}} \subseteq X$, we have $\sigma(X,F)\text{-}\lim_n y_n = x$, whenever*
$$\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\right) = \lim_n \left(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\right) = 0;$$
(ii) *X admits an equivalent $\sigma(X,F)$-lsc such that $(\mathcal{S}_X,\sigma(X,F))$ is metrizable by a metric d such that for every $x \in \mathcal{S}_X$ and every $\varepsilon > 0$ there exists a $\sigma(X,F)$-open halfspace H such that $x \in H$ and $d\text{-}\operatorname{diam}(H \cap \mathcal{S}_X) < \varepsilon$;*
(iii) *X admits both an equivalent $\sigma(X,F)$-lsc and $\sigma(X,F)$-LUR norm and an equivalent $\sigma(X,F)$-lsc norm $\|\cdot\|$ such that $(\mathcal{S}_{\|\cdot\|}, \sigma(X,F))$ is metrizable.*

We will see in the proof that statement (i) implies that the sphere $(\mathcal{S}_X, \sigma(X,F))$ of a norm satisfying the property in (i) is actually metrizable. Statement (ii) above can be modified to an equivalent one assuming that the metric d is defined on X, generates a topology finer than $\sigma(X,F)$ and $d\text{-}\operatorname{diam}(H \cap \mathscr{B}_X) < \varepsilon$. In other words, saying that any point of $\mathcal{S}_X$ is a sort of denting point of $\mathscr{B}_X$ with respect to d.

The description of the results for dual Banach spaces requires some special topological definitions. Let $\mathfrak{N}$ be a family of subsets of a topological space (X,τ). The family is said to be isolated if it is discrete in its union endowed with the relative topology, or in other words, if for every $N \in \mathfrak{N}$ we have
$$N \cap \overline{\bigcup_{M\in\mathfrak{N}\setminus\{N\}} M}^{\tau} = \emptyset.$$

If there is a decomposition $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ such that every family $\mathfrak{N}_n$ is isolated, then the family $\mathfrak{N}$ is said to be σ-isolated. A family $\mathfrak{N}$ of subsets of X is said to be a network if every open set is a union of members of $\mathfrak{N}$.

Definition 1.5. A compact Hausdorff space K is said to be a descriptive compact space if its topology has a σ-isolated network.

Descriptive topological spaces and spaces having a σ-isolated network (the definition differs of the compact case) were introduced and first studied by Hansell in [5]. Descriptive compact spaces has been studied in relation with renorming theory in [9] and [13]. Examples of descriptive compact spaces are the metrizable ones (trivially), Eberlein and Gul'ko compacta and scattered compacta K with $K^{(\omega_1)} = \emptyset$.

In the case of dual Banach space and the weak* topology we can prove a stronger result which completes the characterization [13, Theorem 1.3] and provides a complete answer to the weakly metrization of spheres in that case.

Theorem 1.6. *Let X^* a dual Banach space. The following condition are equivalent:*

(i) *$(\mathscr{B}_{X^*}, w^*)$ is a descriptive compact;*
(ii) *there is a metric d on X^* whose topology is finer than the weak* topology and a sequence $(A_n)_{n\in\mathbb{N}} \subset X$ of w^*-closed sets such that for every $x^* \in X^*$ and every $\varepsilon > 0$ there is a w^*-neighborhood U of x^* and $n \in \mathbb{N}$ such that $d\text{-}\operatorname{diam}(A_n \cap U) < \varepsilon$;*
(iii) *X^* admits an equivalent dual w^*-LUR norm;*
(iv) *X^* admits an equivalent dual norm such that $(\mathcal{S}_{X^*}, w^*)$ is metrizable;*
(v) *X^* admits an equivalent dual norm with the following property: let $x^* \in X^*$ and $(x_n^*)_{n\in\mathbb{N}}$, $(y_n^*)_{n\in\mathbb{N}} \subseteq X^*$, we have $w^*\text{-}\lim_n y_n^* = x^*$, whenever*

$$\lim_n \left(2\|x^*\|^2 + 2\|x_n^*\|^2 - \|x^* + x_n^*\|^2\right) = \lim_n \left(2\|x_n^*\|^2 + 2\|y_n^*\|^2 - \|x_n^* + y_n^*\|^2\right) = 0.$$

In particular, by [13, Theorem 1.3] we may deduce that K is a descriptive compact if and only if $\mathscr{C}(K)^*$ admits a dual norm such that $(\mathcal{S}_{\mathscr{C}(K)^*}, w^*)$ is metrizable.

A systematic study of properties close to weakly metrizability of unit spheres under renormings was carried out in the Ph.D. Thesis of the first named author, done under the advising of the second and third named authors. This paper contains an excerpt of those results with some improvements on the statements and simplifications on their proofs achieved after the thesis defense.

We believe that our notation is mostly standard and we have already introduced almost all the symbols that we will need. Just one remark. When we speak about some equivalent norm of the space X, as after a renorming, the symbols $\mathscr{B}_X$ and $\mathcal{S}_X$ stand for the unit ball and unit sphere in that norm. Only if there is some confusion, as dealing with a second equivalent norm $|||\cdot|||$, then we will use $\mathscr{B}_{|||\cdot|||}$ or $\mathcal{S}_{|||\cdot|||}$.

2. Proofs and further consequences

The proof of our first main result follows ideas from [11] for Kadec renorming.

Proof of Theorem 1.2. (i)⇒(ii) Let ρ be a metric defined on $\mathcal{S}_X$ that metrizes the sphere $(\mathcal{S}_X, \sigma(X,F))$. Define a metric d on X by

$$d(x,y) = \max\left\{\rho\left(\frac{x}{\|x\|}, \frac{y}{\|y\|}\right), |\|x\| - \|y\||\right\}$$

if $x, y \neq 0$ and $d(x,y) = \|x\| + \|y\|$ otherwise. It is elementary to check that the topology of d is finer than $\sigma(X,F)$, and coarser than the $\|\cdot\|$-topology. We define the sequence of sets (A_n) as a renumbering of the countable family $\{r\,\mathscr{B}_X : r \in \mathbb{Q}^+\}$. Note that statement (ii) is clearly satisfied for $x = 0$. If $x \neq 0$, take $r = \|x\|$ and observe that on the sphere $r\,\mathcal{S}_X$ the topologies of d and $\sigma(X,F)$ agree. Given $\varepsilon > 0$ we may take a $\sigma(X,F)$-open neighborhood U of x such that d-$\operatorname{diam}(r\,\mathscr{B}_X \cap U) < \varepsilon$. Indeed, in order to see that we may suppose that $r = 1$. Find V $\sigma(X,F)$-open neighborhood of x such that ρ-$\operatorname{diam}(r\,\mathcal{S}_X \cap V) < \varepsilon$ and find now $\delta > 0$ and W another $\sigma(X,F)$-open neighborhood of x such that $W + \delta\,\mathscr{B}_X \subset V$ and define $U = W \setminus (1-\delta)\,\mathscr{B}_X$ which will have the required property. Now, a standard perturbation argument (see [11, Lemma 1] for instance) allows us to find $r < s \in \mathbb{Q}$ and U' such that d-$\operatorname{diam}(s\,\mathscr{B}_X \cap U') < \varepsilon$.

(ii)⇒(i) Without loss of generality we may assume that F is 1-norming, i.e. $\|\cdot\| = \|\cdot\|_F$. In particular, we may consider X as a subspace of F^*. Recall that in this case the $\sigma(X,F)$-topology is actually the restriction to X of the weak* topology on F^*. Firstly we claim that the sets (A_n) can be taken in such a way that the point x is in the norm interior after a suitable change of the sequence of sets and the metric d by another one. To see that, recall that the topology of d has a σ-discrete basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. For $B \in \mathcal{B}$ consider the w^*-open sets

$$U_B = \bigcup\{U : U \text{ is } w^*\text{-open and } A_n \cap U \subset B\}$$

Clearly $H_n = \{U_B \cap \overline{A_n}^{w^*} : B \in \mathcal{B}_n\}$ is a disjoint family of relatively w^*-open subsets of $\overline{A_n}^{w^*}$ since two open sets that meet in $\overline{A_n}^{w^*}$, they must meet in A_n. Now we may write $U_B = \bigcup_{m=1}^{\infty} U_{B,m}$ with $U_{B.m}$ a w^*-open and $\|x - y\| > 2/m$ if $x \in U_{B,m}$ and $y \notin U_B$. Observe now that the family

$$\mathcal{B}_{n,m} = \{U_{B,m} \cap (\overline{A_n}^{w^*} + m^{-1}\,\mathscr{B}_{F^*}) : B \in \mathcal{B}_n\}$$

is disjoint for every $n, m \in \mathbb{N}$. There exists a pseudometric ρ making clopen all the sets of the family $\mathcal{B}_{n,m}$ and of diameter less or equal than 2^{-n-m}. Indeed, define $\rho_{n,m}(x,y) = 1$ if there is some $B \in \mathcal{B}_{n,m}$ such that $\#(B \cap \{x,y\}) = 1$ and $\rho_{n,m}(x,y) = 0$ otherwise. Then take $\rho(x,y) = \sum_{n,m\in\mathbb{N}} 2^{-n-m}\rho_{n,m}(x,y)$. The restriction of ρ to X is a metric because it separates points, and its topology is finer than the weak*-topology. Note that the sets

$$A_{n,m} = X \cap (\overline{A_n}^{w^*} + m^{-1}\,\mathscr{B}_{F^*})$$

are $\sigma(X,F)$-closed, any $x \in A_n$ satisfying the property of the statement is an interior point of some $A_{n,m}$ and

$$\{U_{B,m} \cap A_{n,m} : B \in \mathcal{B}_n, n, m \in \mathbb{N}\}$$

is a network for the topology of ρ, which completes the proof of the claim.
Now we return to the notation of the begining, with sets (A_n) having nonempty interior and a metric on X called d. Take x_n an interior point of A_n and let f_n be the Minkowski functional

of $A_n - x_n$, which is convex and $\sigma(X,F)$-lsc. Consider the symmetric convex $\sigma(X,F)$-lsc function

$$\Gamma(x) = \|x\| + \sum_{n=1}^{\infty} a_n(f_n(x-x_n) + f_n(x_n - x))$$

where the coefficients $a_n > 0$ ensure the uniform convergence of the series on bounded sets and such that $\Gamma(0) < 1$. The set $\{x \in X : \Gamma(x) \leq 1\}$ is the unit ball of an equivalent $\sigma(X,F)$-lsc norm $|||\cdot|||$. Suppose that a net $(x_\varpi)_{\varpi\in\Omega} \subset \mathcal{S}_{|||\cdot|||}$ is $\sigma(X,F)$-converging to some $x \in \mathcal{S}_{|||\cdot|||}$. Note that $\Gamma(x_\varpi) = \Gamma(x) = 1$. A standard argument of lower semicontinuity shows that $\lim_\varpi f_n(x_\varpi - x_n) = f_n(x - x_n)$ for every $n \in \mathbb{N}$. Fix $\varepsilon > 0$ and take $n \in \mathbb{N}$ and a $\sigma(X,F)$-open U such that $x \in A_n^\circ \cap U$ and d- $\mathrm{diam}(A_n \cap U) < \varepsilon$, where A_n° is the norm interior of A_n. We have $f_n(x - x_n) < 1$ since $x \in A_n^\circ$, so for ϖ large enough we also have $f_n(x_\varpi - x_n) < 1$. In particular, $x_\varpi \in A_n$. For ϖ large we have $x_\varpi \in U$, therefore $d(x_\varpi, x) < \varepsilon$. This shows that the net $(x_\varpi)_{\varpi\in\Omega}$ converges to x in the d-topology. Since the d-topology is finer than $\sigma(X,F)$, this concludes the proof. □

Remark 2.1. If we know that the metric d is $\sigma(X,F)$-lsc then we can remove the hypothesis of the sets A_n being $\sigma(X,F)$-closed because the $\sigma(X,F)$-closure does not increase the d-diameter. In fact, that is the idea used in the characterization of $\sigma(X,F)$-Kadec renormability, when d is the norm (see [11]).

The next lemma contains some facts about convexity which are useful in renorming theory. The easy proofs are left to the reader.

Lemma 2.2. *Given $f : X \to [0,+\infty)$ convex, the following inequalities hold:*

$$0 \leq \frac{(f(x)-f(y))^2}{4} \leq \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2; \tag{2.1}$$

$$\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right)^2 \leq \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2; \tag{2.2}$$

$$\min\{f(z) : z \in [x,y]\} \geq \min\{f(x), f(y)\} - 2\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right). \tag{2.3}$$

The following definition is used by topologists in metrization theory and generalized metric spaces, see [4, Definition 9.1].

Definition 2.3. Let S a nonempty set. A function $\rho : S \times S \to [0,+\infty)$ is called a *symmetric* if $\rho(x,y) = \rho(y,x)$ for every $x, y \in S$ and $\rho(x,y) = 0$ if, and only if, $x = y$.

If a set S has a symmetric ρ, then we can define a topology τ_ρ on S in the following way: $U \in \tau_\rho$ if, and only if, for every $x \in U$ there exists $\varepsilon > 0$ such that

$$B_\rho(x,\varepsilon) := \{y \in S \mid \rho(x,y) < \varepsilon\} \subseteq U.$$

Observe that without an additional assumption (such as the triangle inequality for ρ), we cannot assume $B_\rho(x,\varepsilon)$ to be a neighborhood of x. When such a condition happens, then ρ is called a *semimetric*. Observe that thanks to (2.1) of Lemma 2.2

$$\rho(x,y) := 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2$$

defines a symmetric on X if and only if $\|\cdot\|$ is strictly convex. We will see that this symmetric plays a major role along the proofs of the results.

Proposition 2.4. *Let $A \subset X$ be a convex cone such that $(\mathcal{S}_X \cap A, \sigma(X,F))$ is metrizable and any $x \in \mathcal{S}_X \cap A$ has a local base for the $\sigma(X,F)$-topology made of slices. Then there exists an equivalent $\sigma(X,F)$-lsc norm $\|\cdot\|_A$ on X such that $\sigma(X,F)$-$\lim_n y_n = x$ whenever the points x, x_n, y_n are in A and satisfy*

$$\lim_n (2\|x\|_A^2 + 2\|x_n\|_A^2 - \|x + x_n\|_A^2) = \lim_n (2\|x_n\|_A^2 + 2\|y_n\|_A^2 - \|x_n + y_n\|_A^2) = 0.$$

Proof. Let d the metric defined on $\mathcal{S}_X \cap A$. We may assume that the metric extends to a pseudometric on $A \setminus \{0\}$ by $d(x,y) := d\Big(\frac{x}{\|x\|}, \frac{y}{\|y\|}\Big)$. Let $\mathcal{H}$ be the collection of $\sigma(X,F)$-open halfspaces of X and

$$\mathcal{H}_n = \big\{H \in \mathcal{H} : \; H \cap \mathscr{B}_X \cap A \neq \emptyset \text{ and } d\text{-}\operatorname{diam}(H \cap \mathscr{B}_X \cap A) < 2^{-n}\big\}.$$

Consider for $n \in \mathbb{N}$ the sets

$$B_n = \overline{\operatorname{co}}^{\sigma(X,F)}\Big((\mathscr{B}_X \setminus \bigcup \mathcal{H}_n) \cup 2^{-1}\mathscr{B}_X\Big)$$

where we are taking $\bigcup \mathcal{H}_n = \bigcup_{H \in \mathcal{H}_n} H$. We claim that the following holds: if the segment $[x,y] \subset (\mathscr{B}_X \cap A) \setminus B_n$, then $d(x,y) < 2^{1-n}$. Indeed, define sets

$$L_n = \{z \in [x,y] : \exists H \in \mathcal{H}_n \text{ such that } [x,z] \subset H\},$$

$$R_n = \{z \in [x,y] : \exists H \in \mathcal{H}_n \text{ such that } [z,y] \subset H\}.$$

By definition of B_n, we have $L_n \cup R_n = [x,y]$ as any $z \in [x,y]$ is contained in some $H \in \mathcal{H}_n$ and either $[x,z] \subset H$ or $[z,y] \subset H$. Note that L_n and R_n are nonempty and relatively open. Therefore, there is some $z \in L_n \cap R_n$ and so $d(x,y) \leq d(x,z) + d(z,y) < 2^{1-n}$. Consider now the equivalent $\sigma(X,F)$-lsc norm on X defined by

$$\|x\|_A^2 = \|x\|^2 + \sum_{n=1}^{\infty} 2^{-n}(f_n(x)^2 + f_n(-x)^2)$$

where f_n are the Minkowski functionals of the sets B_n defined previously.
Consider $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq A$ and $x \in A$ such that

$$\lim_n \,(2\|x_n\|_A^2 + 2\|y_n\|_A^2 - \|x_n + y_n\|_A^2) = \lim_n \,(2\|x_n\|_A^2 + 2\|x\|_A^2 - \|x_n + x\|_A^2) = 0.$$

If $x = 0$, the proof is trivial. Otherwise, without loss of generality we may assume that $\|x\| = 1$ since that multiplies all the sequences by the same factor. Moreover, we may assume that $\|x_n\| = \|y_n\| = 1$ because this modifies the former sequences by adding norm null sequences. By (2.1) of Lemma 2.2 we obtain, for every $k \in \mathbb{N}$,

$$\lim_n \frac{f_k(x_n)^2 + f_k(y_n)^2}{2} - f_k\left(\frac{x_n + y_n}{2}\right)^2 = \lim_n \frac{f_k(x_n)^2 + f_k(x)^2}{2} - f_k\left(\frac{x_n + x}{2}\right)^2 = 0.$$

Since $x \in \mathcal{S}_X \cap \bigcup \mathcal{H}_k$, for every $k \in \mathbb{N}$, then $f_k(x) > 1$ and assumptions above implies, together with Lemma 2.2 that there exists $N_k \in \mathbb{N}$ such that $\min\{f_k(x), f_k(x_n), f_k(y_n)\} >$

$1+\delta$ for some $\delta > 0$ and every $n \geq N_k$. By (2.2) of Lemma 2.2, we may suppose that N_k is large enough to guarantee that

$$\max\left\{\frac{f_k(x_n)+f_k(y_n)}{2} - f_k\left(\frac{x_n+y_n}{2}\right), \frac{f_k(x_n)+f_k(x)}{2} - f_k\left(\frac{x_n+x}{2}\right)\right\} < \frac{\delta}{2}$$

for every $n \geq N_k$. By (2.3) of Lemma 2.2, we have

$$[x, x_n] \cup [x_n, y_n] \subset (\mathscr{B}_X \cap A) \setminus B_k$$

and so we deduce $d(x, y_n) \leq d(x, x_n) + d(x_n, y_n) < 2^{2-k}$ for $n \geq N_k$. Together with $\|y_n\| = \|x\| = 1$, that implies $\sigma(X, F)$-$\lim_n y_n = x$. □

We are now ready to prove the main theorem.

Proof of Theorem 1.4. (ii)⇒(i) Apply Proposition 2.4 with $A = X$.

(i)⇒(iii) Clearly the norm is $\sigma(X, F)$-LUR. We claim that the $\sigma(X, F)$-topology on the unit sphere is symmetrized by the symmetric

$$\rho(x, y) = 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2.$$

Indeed, if $x \in \mathcal{S}_X$ the definition of $\sigma(X, F)$-LUR norm implies that every $\sigma(X, F)$-neighborhood of x contains a "ball" of the form $B_\rho(x, r)$ for some $r > 0$. On the other hand, $B_\rho(x, r)$ is a $\sigma(X, F)$-neighborhood of x. Just take a $\sigma(X, F)$-open halfspace H containing x and disjoint with $(1 - r)\mathscr{B}_X$. Thus, if $y \in \mathcal{S}_X \cap H$ then $\frac{x+y}{2} \notin (1 - r)\mathscr{B}_X$, that is, $\|x + y\| > 1 - r$. Therefore, $\mathcal{S}_X \cap H \subset B_\rho(x, r)$. The metrization of the unit sphere is now consequence of the following metrization result of Arhangel'skiĭ (1966), see [1] and [4, Theorem 9.14]:

Let (X, τ) be a topological space symmetrizable with respect to a symmetric ρ such that τ-$\lim_n y_n = x$ whenever $x \in X$, $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ are such that $\lim_n \rho(x, x_n) = 0$ and $\lim_n \rho(x_n, y_n) = 0$. Then X is metrizable.

(iii)⇒(ii) Let $\|\cdot\|_R$ a $\sigma(X, F)$-lsc and $\sigma(X, F)$-LUR norm on X and let $\|\cdot\|_M$ another $\sigma(X, F)$-lsc norm such that its unit sphere $(\mathcal{S}_{\|\cdot\|_M}, \sigma(X, F))$ is metrized by a metric ρ. Consider the norm $\|\cdot\|^2 = \|\cdot\|_R^2 + \|\cdot\|_M^2$ and the metric

$$d(x, y) = \rho\left(\frac{x}{\|x\|_M}, \frac{y}{\|y\|_M}\right).$$

It is easy to check that $\|\cdot\|$ shares the properties of $\|\cdot\|_R$ and $\|\cdot\|_M$, with respect to d, which easily gives statement (ii). □

We will consider the Radon positive measures $\mathfrak{M}^+(K)$ on a Hausdorff compact K. Note that $\mathfrak{M}^+(K)$ is identified with a positive cone in $\mathscr{C}(K)^*$.

Proposition 2.5. *Let K be a descriptive compact space. There exists a strictly convex dual norm $|||\cdot|||_+$ on $\mathscr{C}(K)$ such that the weak* topology is metrizable on $\mathcal{S}_{|||\cdot|||_+} \cap \mathfrak{M}^+(K)$.*

Proof. The arguments for the proof of this results relies completely on the techniques developed in [13] which are scattered along that paper. In [13, Lemma 3.1] there is the description of a network $\mathfrak{N} = \bigcup_{n=1}^\infty \mathfrak{N}_n$ where each $\mathfrak{N}_n$ is a family of disjoint relatively open subsets of some closed set $A_n \subset K$. This network plays an important role in the construction of norms and metrics on $\mathscr{C}(K)^*$ with special properties. Define the metric on $\mathfrak{M}^+(K)$ by

$$d(\mu,\nu) = |\mu(K) - \nu(K)| + \sum_{n=1}^{\infty} 2^{-n} \sum_{N\in\mathfrak{N}_n} |\mu(N) - \nu(N)|.$$

The convergence of the series is ensured since each family $\mathfrak{N}_n$ is disjoint and so the series is dominated by the total variation of the measures. Now [13, Lemma 3.2] implies that the d-topology is finer than the weak* topology on $\mathfrak{M}^+(K)$. The converse (restricted to the sphere) requires the norm constructed in the proof of [13, Theorem 3.3]. We will sketch the steps of the construction skipping some details. For every $n \in \mathbb{N}$ define a function F_n on $\mathscr{C}(K)^*$ by

$$F_n(\mu)^2 = \sum_{N\in\mathfrak{N}_n} |\mu|(N)^2.$$

For $m, n \in \mathbb{N}$ define a seminorm $\|\cdot\|_{m,n}$ on $\mathscr{C}(K)^*$ by the formula

$$\|\mu\|_{m,n}^2 = \inf\left\{m^{-1}F_n(\nu)^2 + \|\mu-\nu\|^2 : \operatorname{supp}(\nu) \subset A_n\right\}.$$

Finally, define an equivalent norm $|||\cdot|||_+$ on $\mathscr{C}(K)^*$ by

$$|||\mu|||_+^2 = \|\mu\|^2 + \sum_{m,n\in\mathbb{N}} 2^{-m-n}\|\mu\|_{m,n}^2 + \sum_{n\in\mathbb{N}} 2^{-n}|\mu|(K\setminus A_n)^2 + \sum_{n\in\mathbb{N}} 2^{-n}|\mu|(K\setminus A_n')^2$$

where $A_n' = A_n \setminus \bigcup_{N\in\mathfrak{N}_n} N$. In order to see that the weak* topology and d coincides on $\mathcal{S}_{|||\cdot|||_+} \cap \mathfrak{M}^+(K)$ take a weak* converging sequence $w^*\text{-}\lim_n \mu_n = \mu$ with $|||\mu_n|||_+ = |||\mu|||_+ = 1$. Note that in such a case $\lim_n |||\mu_n + \mu|||_+ = 2$, so the hypothesis of a claim inside the proof of [13, Theorem 3.3] is fulfilled. The claim says that (μ_n) converges to μ in the norm of the space $\ell_1(\mathfrak{N}_k)$ for any $k \in \mathbb{N}$, that is,

$$\lim_n \sum_{H\in\mathfrak{N}_k} |\mu_n(H) - \mu(H)| = 0.$$

Since we also have $\lim_n \mu_n(K) = \mu(K)$, we deduce $\lim_n d(\mu_n,\mu) = 0$ as we wanted. □

Proof of Theorem 1.6. (i) ⇔ (ii) is just an equivalent definition of descriptiveness (put together [13, Lemma 2.1] and [13, Lemma 2.2]).

(i) ⇔ (iii) is proved in [13, Theorem 1.3].

(iv) ⇒ (i) is proved in [13, Lemma 2.5].

(v) ⇒ (iv) (and (iii) as well) follows from Theorem 1.4.

To close the cycle of implications we will prove (i) ⇒ (v). Assume that $\mathscr{B}_{X^*}$ is a descriptive compact. Proposition 2.5 with Proposition 2.4 and Choquet's Lemma [3] provide us with a dual renorming $\|\cdot\|_*$ such that given $\{\mu\}, (\mu_n), (\nu_n) \subset \mathfrak{M}^+(\mathscr{B}_{X^*})$, then $w^*\text{-}\lim_n \nu_n = \mu$ whenever

$$\lim_n(2\|\mu_n\|_*^2 + 2\|\nu_n\|_*^2 - \|\mu_n+\nu_n\|_*^2) = \lim_n(2\|\mu_n\|_*^2 + 2\|\mu\|_*^2 - \|\mu_n+\mu\|_*^2) = 0. \tag{2.4}$$

Let $T : \mathscr{C}(\mathscr{B}_{X^*})^* \to X^*$ be the barycentric map, which is linear and w^*-w^*-continuous. Define an equivalent dual norm $|||\cdot|||$ on X^* by

$$|||x^*|||^2 = \inf\{\left\|\mu^1\right\|_*^2 + \left\|\mu^2\right\|_*^2 : \mu^1, \mu^2 \in \mathfrak{M}^+(\mathscr{B}_{X^*}), T(\mu^1 - \mu^2) = x^*\}.$$

We claim that $|||\cdot|||$ satisfies statement (iv). Suppose that we are given $x^* \in X^*$ and $(x_n^*)_{n\in\mathbb{N}}$, $(y_n^*)_{n\in\mathbb{N}} \subseteq X^*$, such that

$$\lim_n (2\,|||\, x^*\,|||^2 + 2\,|||\, x_n^*\,|||^2 - |||\, x^* + x_n^*|||^2) = \lim_n (2\,|||\, x_n^*\,|||^2 + 2\,|||\, y_n^*\,|||^2 - |||\, x_n^* + y_n^*|||^2) = 0;$$

The infimum in the definition of $|||\cdot|||$ is attained, so we may find measures

$$\{\mu^1, \mu^2\}, (\mu_n^1), (\mu_n^2), (\nu_n^1), (\nu_n^2) \subset \mathfrak{M}^+(\mathscr{B}_{X^*})$$

such that

$$|||x^*|||^2 = \left\|\mu^1\right\|_*^2 + \left\|\mu^2\right\|_*^2 \text{ and } x^* = T(\mu^1 - \mu^2);$$
$$|||x_n^*|||^2 = \left\|\mu_n^1\right\|_*^2 + \left\|\mu_n^2\right\|_*^2 \text{ and } x_n^* = T(\mu_n^1 - \mu_n^2);$$
$$|||y_n^*|||^2 = \left\|\nu_n^1\right\|_*^2 + \left\|\nu_n^2\right\|_*^2 \text{ and } y_n^* = T(\nu_n^1 - \nu_n^2).$$

An application of Lemma 2.2 gives easily that conditions (2.4) are fulfilled for $\mu^1, (\mu_n^1), (\nu_n^1)$ and $\mu^2, (\mu_n^2), (\nu_n^2)$. Now observe that

$$\lim_n y_n^* = \lim_n T(\nu_n^1 - \nu_n^2) = \lim_n T(\nu_n^1) - \lim_n T(\nu_n^2) = T(\mu^1) - T(\mu^2) = x^*$$

which concludes the proof of the theorem. □

Remark 2.6. Once we know that the sphere $(\mathcal{S}_{X^*}, w^*)$ is metrizable, we can deduce that it is metrizable by a complete metric since $\mathcal{S}_{X^*} = \mathscr{B}_{X^*} \setminus \bigcup_{n=1}^{\infty}(1 - n^{-1})\,\mathscr{B}_{X^*}$ is a $\mathcal{G}_\delta$-subset of a compact space $(\mathscr{B}_{X^*}, w^*)$.

We will describe an alternative argument that is useful in the context of dual Banach spaces. Let X be a locally convex space. Denote by $\mathcal{V}(x)$ a local base of the topology at $x \in X$ and by $\mathcal{H}$ the family of open halfspaces of X. Let $\delta : \mathcal{P}(X) \to [0, +\infty]$ be a "diameter function", which here means a requirement of monotonicity, that is, $\delta(A) \leq \delta(B)$ if $A \subset B$. Given $A \subset X$ and $\varepsilon > 0$ denote

$$\langle A\rangle_\varepsilon' = \{x \in A : \forall U \in \mathcal{V}(x), \delta(A \cap U) \geq \varepsilon\};$$
$$[A]_\varepsilon' = \{x \in A : \forall H \in \mathcal{V}(x) \cap \mathcal{H}, \delta(A \cap H) \geq \varepsilon\}.$$

Note that $[A]_\varepsilon'$ is convex if A is so. These operations can be iterated, for instance $[A]_\varepsilon^{k+1} = [[A]_\varepsilon^k]_\varepsilon'$ and $[A]_\varepsilon^\omega = \bigcap_{n=1}^\infty [A]_\varepsilon^n$. Finally, let $\mathrm{Ext}(A)$ denote the set of extreme points of a convex set $A \subset X$. With this notation we have the following result, already proved in [14, Lemma 2.10].

Proposition 2.7. *If $A \subset X$ is compact and convex, then* $\mathrm{Ext}([A]_\varepsilon^\omega) \subset \langle A\rangle_\varepsilon'$.

If we assume (iii) of Theorem 1.6, we can prove (iv) using Proposition 2.7 with the following idea. Without loss of generality, the metric can be extended to a metric d' defined in $\mathscr{B}_{X^*}$. Then $([\mathscr{B}_{X^*}]_{2^{-m}}^n)_{n,m\in\mathbb{N}}$ is a countable family of symmetric convex sets that can be used in the same fashion that the sets B_n in the proof of Proposition 2.4. However, it is more interesting to apply this idea to the characterization of existence of dual strictly convex renorming through symmetrics.

Theorem 2.8. *Let X^* be a dual Banach space. The following statements are equivalent:*

(i) *X^* admits an equivalent dual strictly convex norm;*
(ii) *X^* admits an equivalent dual norm and a symmetric ρ on $\mathcal{S}_{X^*}$ such that every point $x^* \in \mathcal{S}_{X^*}$ admits a relative weak* neighbourhood of small ρ-diameter.*

Proof. (i) $\Rightarrow$ (ii) If X^* is endowed with a dual rotund norm, then we already know that the symmetric $\rho(x^*, y^*) = 4 - \|x^* + y^*\|^2$ (or $\rho(x^*, y^*) = 2 - \|x^* + y^*\|$) does the job even with weak*-open slices as neighborhoods.
(ii) $\Rightarrow$ (i) Assume that ρ is a symmetric on $\mathcal{S}_{X^*}$ and define $\delta(A) = \rho\text{-diam}(\mathcal{S}_{X^*} \cap A)$. Note that the hypothesis implies that $\mathcal{S}_{X^*} \cap \langle \mathscr{B}_{X^*} \rangle'_\varepsilon = \emptyset$. Consider the sets

$$B_{n,m} = \overline{\text{co}}^{w^*}([\mathscr{B}_{X^*}]^n_{2^{-m}} \cup 2^{-1}\mathscr{B}_{X^*})$$

and their Minkowski's functionals $f_{n,m}$. Define an equivalent dual norm $|||\cdot|||$ as

$$|||x^*|||^2 = \|x^*\|^2 + \sum_{n,m} 2^{-n-m} f_{n,m}(x^*)^2.$$

We claim that $|||\cdot|||$ is strictly convex. We will arrive to a contradiction, so assume that for some points $x^* \neq y^* \in X^*$ we have

$$2\,|||\, x^*\, |||^2 + 2\,|||\, y^*\, |||^2 - |||\, x^* + y^* |||^2 = 0.$$

Then by Lemma 2.2 we also have

$$2\|x^*\|^2 + 2\|y^*\|^2 - \|x^* + y^*\|^2 = 0;$$

$$2f_{n,m}(x^*)^2 + 2f_{n,m}(y^*)^2 - f_{n,m}(x^* + y^*)^2 = 0.$$

for every $n, m \in \mathbb{N}$. Without loss of generality, we may assume that $\|x^*\| = \|y^*\| = 1$. Fix $m \in \mathbb{N}$ such that

$$2^{-m} < \min\left\{\rho\left(x^*, \frac{x^* + y^*}{2}\right), \rho\left(y^*, \frac{x^* + y^*}{2}\right)\right\}.$$

Let $n \in \mathbb{N}$ the maximum of the set $\{k \in \mathbb{N} : f_{k,m}(x^*) = 1\}$, which by Proposition 2.7 exists and is finite. Then we have

$$f_{n,m}(x^*) = f_{n,m}(y^*) = f_{n,m}\left(\frac{x^* + y^*}{2}\right) = 1;$$

$$f_{n+1,m}(x^*) = f_{n+1,m}(y^*) = f_{n+1,m}\left(\frac{x^* + y^*}{2}\right) = \alpha > 1.$$

In particular, $\frac{x^*+y^*}{2} \in B_{n,m} \setminus B_{n+1,m}$, so there is $H \in \mathcal{H}$ such that $\frac{x^*+y^*}{2} \in H$ and $\delta(B_{n,m} \cap H) < 2^{-m}$. Since either $x^* \in B_{n,m} \cap H$ or $y^* \in B_{n,m} \cap H$, this leads to a contradiction. $\square$

Funding. This work was supported by the grants of Ministerio de Economía y Competividad MTM2014-57838-C2-1-P; and Fundación Séneca Región de Murcia 19368/PI/14.

References

[1] A.V. Arhangel'skiĭ, Mappings and spaces, *Russian Math. Surveys* 21 (1966), 115–162.

[2] R. Deville, G. Godefroy, V. Zizler, *Smoothness and renormings in Banach spaces*, volume 64 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, 1993.

[3] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[4] G. Gruenhage, Generalized metric spaces, In *Handbook of set-theoretic topology*, pages 423–501. North-Holland, Amsterdam, 1984.

[5] R. W. Hansell, Descriptive sets and the topology of nonseparable Banach spaces, *Serdica Math. J.* 27 (2001) 1–66.

[6] A. Moltó, J. Orihuela, S. Troyanski, Locally uniformly rotund renorming and fragmentability, *Proc. London Math. Soc.* (3) 75 (1997), no. 3, 619–640.

[7] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, On weakly locally uniformly rotund Banach spaces, *J. Funct. Anal.*, 163 (1999), 252–271.

[8] A. Moltó, J. Orihuela, S. Troyanski, M. Valdivia, *A nonlinear transfer technique for renorming*, volume 1951 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 2009.

[9] L. Oncina, M. Raja, Descriptive compact spaces and renorming, *Studia Math.* 165 (1) (2004), 39–52.

[10] J. Orihuela, On $\mathcal{T}_p$-locally uniformly rotund norms, *Set-Valued Var. Anal.* 21 (2013), no. 4, 691–709.

[11] M. Raja, Kadec norms and Borel sets in a Banach space, *Studia Math.*, 136 (1999), 1–16.

[12] M. Raja, Locally uniformly rotund norms, *Mathematika*, 46 (1999), 343–358.

[13] M. Raja, Weak* locally uniformly rotund norms and descriptive compact spaces. *J. Funct. Anal.*, 197 (2003), 1–13.

[14] M. Raja, Dentability indices with respect to measures of non-compactness, *J. Funct. Anal.*, 253 (2007), 273–286.

[15] S. L. Troyanski, Locally uniformly convex norms, *C. R. Acad. Bulgare Sci.*, 32 (1979), 1167–1169.

[1] Dipartimento di Matematica e Informatica, Università degli Studi di Parma, Parco Area delle Scienze 53/A, 43124 Parma, Italy.

E-mail address: simone.ferrari1@unipr.it

[2] Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Murcia, Spain

E-mail address: joseori@um.es; matias@um.es