METRIZATION THEORY AND THE KADEC PROPERTY

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Abstract. The uniform structure of a descriptive normed space $(X, \|\cdot\|)$ always admits a description with an (F)-norm $\|\cdot\|_1$ such that weak and norm topologies coincide on

$$\{x \in X : \|x\|_1 = \rho\}$$

for every $\rho > 0$.

1. Introduction

Paracompactness is a generalization of the concept of compactness and it belongs to the class of concepts related with covering properties of topological spaces. On the other hand, the concept of full normality can be regarded as belonging to another genealogy of concepts, the separation axioms which include regularity, normality and many other properties. Stone's theorem says that those two concepts, belonging to different categories, coincide for Hausdorff topological spaces, see Chapter V in [22]. In particular, the fact that every metrizable space is paracompact is going to be a fundamental tool when looking for convex renorming properties of a Banach space. Indeed, the use of Stone's theorem has been extensively considered in order to build new techniques to construct equivalent locally uniformly rotund norms on a given normed space X, [7], [19], [21]. The σ-discreteness of the basis for the metric topologies gives the necessary rigidity condition that appears in all the known cases of existence of such a renorming property. It is our aim here to study the impact of Stone's theorem for Kadec renormings.

Throughout this paper $(X, \|\cdot\|)$ will denote a normed space and X^* its dual. With $\mathscr{B}_X$ and $\mathcal{S}_X$ (resp. $\mathscr{B}_{X^*}$ and $\mathcal{S}_{X^*}$) we will denote the unit ball and the unit sphere of X (resp. X^*). If Z is a subspace of X^*, we shall denote by $\|\cdot\|_Z$ the norm-continuous seminorm given by

$$\|x^{**}\|_Z = \sup_{f \in \mathscr{B}_{X^*} \cap Z} |x^{**}(f)|$$

for $x^{**} \in X^{**}$. We say that Z is *norming* if $\|\cdot\|_Z$ is an equivalent norm on X. If $\|\cdot\|_Z$ coincides with $\|\cdot\|$ on X we say that Z is 1-*norming*. We shall denote by $\sigma(X, Z)$ the topology on X of pointwise convergence on Z, but in the particular

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cases of the weak and the weak-star topologies we shall use w and w^* respectively. If τ is a topology on a normed space X, we say that a norm $\|\cdot\|$ on X is

- *rotund*, or *strictly convex*, if for every $x, y \in X$ the condition $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\|$ implies $x = y$. Geometrically this means that $\mathcal{S}_X$ has no non-trivial line segments, or equivalently, every point of $\mathcal{S}_X$ is an extreme point of $\mathscr{B}_X$.
- τ-*Kadec* if the norm and the τ topologies coincide on the unit sphere. If $\tau = w$ we shall say that $\|\cdot\|$ is Kadec.
- τ-*locally uniformly rotund* (τ-**LUR**, for short), if given a point x and a sequence $(x_n)_{n\in\mathbb{N}}$ in X we have $\lim_{n\to\infty} x_n = x$ in the τ topology whenever
$$\lim_{n\to\infty} \left(2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2\right) = 0.$$
If τ is the norm topology we will say that $\|\cdot\|$ is locally uniformly rotund (**LUR**, for short).

It is a known result that a space X admits an equivalent **LUR** norm if, and only if, it admits an equivalent Kadec norm and an equivalent rotund norm (see [32] and [28]).

If we have a Kadec norm $\|\cdot\|$ on the normed space X the identity map from $(\mathcal{S}_X, w)$ to $(X, \|\cdot\|)$ is continuous. If we have a subset C of the normed space X, a normed space Y and a map $\phi : (C, w) \to (Y, \|\cdot\|)$, then ϕ is said to be *piecewise continuous* if there is a countable cover $C = \bigcup_{n\in\mathbb{N}} C_n$ such that each one of the restrictions $\phi_{|C_n}$ is weak to norm continuous. A norm pointwise limit of a sequence of piecewise continuous maps is called a σ-*continuous map*, see [21] for an account of results around this notion and references. In a normed space $(X, \|\cdot\|)$ with a Kadec norm the identity map in X, from the w to the norm topology, is σ-continuous, and Stone's Theorem can be applied to the norm topology to get that the norm topology has a network $\mathcal{N}$ that can be written as a countable union of subamilies, $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, where each one of the subfamilies $\mathcal{N}_n$ is a discrete family in its union (isolated, for short)

$$\bigcup \mathcal{N}_n := \bigcup \{N \mid N \in \mathcal{N}_n\}$$

endowed with the w topology, i.e. for every $x \in \bigcup \mathcal{N}_n$ there exists a w neighborhood V of x such that

$$\operatorname{card}\{N \in \mathcal{N}_n \mid N \cap V \neq \emptyset\} = 1,$$

see Theorem 1.5 in [7]. No example is known of Banach space with this kind of network, which is called *descriptive* Banach space, and without an equivalent Kadec norm, see chapter 3 in [21].

In the classical theory of Banach spaces, not only normed spaces were considered, but also those spaces on which a metric is defined which is compatible with the vector space operations, see chapter 3 in Banach's book [1]. Indeed, the uniform structure of a metrizable topological vector space is described with the following notion, [15, p. 163]:

Definition 1.1. An (F)-norm in a vector space X is function $\|\cdot\| : X \to [0, +\infty)$ such that:

(1) $x = 0$ if, and only if, $\|x\| = 0$;
(2) $\|\lambda x\| \leq \|x\|$, if $|\lambda| \leq 1$ and $x \in X$;
(3) $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$;
(4) $\lim_n \|\lambda x_n\| = 0$, if $\lim_n \|x_n\| = 0$ for every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $\lambda \in \mathbb{R}$;
(5) $\lim_n \|\lambda_n x\| = 0$, if $\lim_n \lambda_n = 0$ for every $(\lambda_n)_{n\in\mathbb{N}}$ and $x \in X$.

The space X is said to be (F)-normed. The sets $\{x \in X : \|x\| < \varepsilon\}$, for $\varepsilon > 0$ form a basis of neighborhoods of the origin for the topology determined by the (F)-norm. A basis of the uniformity associated is formed with the sets $\{(x, y) \in X \times X : \|x - y\| < \varepsilon\}$, for $\varepsilon > 0$.

Banach called a complete (F)-normed space an (F)-space, after Fréchet, see Chapter III in [1].

In this paper we shall prove the following result:

Theorem 1.2 (Kadec F-renorming)**.** *Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. Then the following conditions are equivalent:*

(1) *There is a norm-equivalent $\sigma(X,Z)$-lower semicontinuous and $\sigma(X,Z)$-Kadec (F)-norm $\|\cdot\|_0$ on X, i.e. an (F)-norm $\|\cdot\|_0$ such that $\sigma(X,Z)$ and norm topologies coincide on the unit "sphere" $\{x \in X \mid \|x\|_0 = 1\}$, and the topology determined by the (F)-norm $\|\cdot\|_0$ on X coincides with the topology of the norm $\|\cdot\|$.*
(2) *The normed space X is $\sigma(X,Z)$-descriptive; i.e. there are isolated families $\mathcal{B}_n$ for the $\sigma(X,Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and a set $B \in \mathcal{B}_n$ with the property that $x \in B$ and that $\|\cdot\|$-$\operatorname{diam}(B) < \varepsilon$.*
(3) *The norm topology admits a basis $\mathcal{B} = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ such that each one of the families $\mathcal{B}_n$ is $\sigma(X,Z)$-isolated and norm discrete.*

In Section 3 the equivalence (1) $\Leftrightarrow$ (2) is established. Section 4 takes care of (2) $\Leftrightarrow$ (3).

Prior estimates were first obtained by the fourth named author in Theorem 1 of [27] when he constructed a positively homogeneous symmetric function $F : X \to [0, +\infty)$, with $\|\cdot\| \leq F(\cdot) \leq 3\|\cdot\|$, such that the weak and norm topologies coincide on $\{x \in X : F(x) = 1\}$. The norm continuity of Raja's function F does not follow immediately from his construction and it was asked by different people if it actually could be done. In an unpublished note of Raja appears a construction to make F continuous when dealing with topologies of the form $\sigma(X,Z)$, let us transcript it here:

Lemma 1.3. *Let X be a normed space with an 1-norming subspace $Z \subset X^*$. Assume that X is $\sigma(X,Z)$-descriptive. Then there exists a symmetric homogeneous $\sigma(X,Z)$-lower semicontinuous and norm continuous function Φ on X with $\|\cdot\| \leq \Phi(\cdot) \leq 3\|\cdot\|$ such that the topologies of the norm and $\sigma(X,Z)$ coincide on the set $S = \{x \in X \mid \Phi(x) = 1\}$.*

Proof. We may identify X isometrically as a subspace of Z^*. In this way the $\sigma(X,Z)$ topology is induced on X by the w^*-topology of Z^*. In this proof the closed balls will always be referred to Z^*, that is: $\mathscr{B}[x, \varepsilon] := \overline{\mathscr{B}(x, \varepsilon)}^{w^*}$.

We shall build a norm continuous function with the same properties of F, the function constructed in [27, Theorem 1]. Our task will be to add the norm continuity to the other properties of F, so we will use it in the proof. Let K be the w^*-closure of the star-shaped set $\{x \in X \mid F(x) \leq 1\}$. It is easy to verify that K is also star-shaped. Let q_n be the Minkowski functional of $K + \mathscr{B}[0, 1/n]$ for $n \geq 2$. As this set is w^*-closed, q_n is w^*-lower semicontinuous. It is easy to realize that q_n is also symmetric and verifies the inequality

$$\left(1 - \frac{1}{n}\right)\|\cdot\| \leq q_n(\cdot) \leq 3\|\cdot\|.$$

We claim that every q_n is norm continuous. Indeed, it is clear that q_n is norm lower semicontinuous. By homogeneity it is enough to show that the set

$$U_n := \{z^* \in Z^* \mid q_n(z^*) < 1\}$$

is norm open. Take $z^* \in U_n$, we know $q_n(z^*) < 1$, then take $\lambda \in (0, 1)$ such that $q_n(z^*) < \lambda^2$. This implies that $z^* \in \lambda^2 K + \mathscr{B}[0, \lambda^2/n]$. In particular $z^* \in K + \mathscr{B}(0, \lambda/n)$ which is norm open and contained in $K + \mathscr{B}[0, 1/n]$.

Let us consider the function

$$\Phi(z^*) = \|z^*\| + \sum_{n \geq 2} 2^{-n} q_n(z^*)$$

which is homogeneous, symmetric, w^*-lower semicontinuous, norm continuous and satisfies $\|\cdot\| \leq \Phi(\cdot) \leq 3\|\cdot\|$. We claim that Φ has the Kadec property at the points of X, that is, if (z^*_ω) is a net w^*-converging to $x \in X$ such that $\Phi(z^*_\omega)$ converges to $\Phi(x)$, then (z^*_ω) is norm convergent to x. Clearly, we may assume $x \neq 0$, and by homogeneity of F also assume that $F(x) = 1$. If (z^*_ω) is a net as above, using the lower semicontinuity in a standard way we obtain that $q_n(z^*_\omega)$ converges to $q_n(x)$ for each n. As $q_n(x) < F(x) = 1$, for ω large $q_n(z^*_\omega) < 1$ and thus $z^*_\omega \in K + \mathscr{B}[0, 1/n]$. Given any $\varepsilon > 0$ it is possible to take a $\sigma(X, Z)$-open neighborhood U of x such that $U \cap \{x \in X \mid F(x) \leq 1\}$ has diameter less than ε. We may assume that U is w^*-open in Z^* and passing to closure we obtain that $\operatorname{diam}(U \cap K) \leq \varepsilon$. By [27, Lemma 1], given $\varepsilon > 0$ there is $r > 0$ and another w^*-neighborhood V of x such that $\operatorname{diam}(V \cap (K + \mathscr{B}(0, r))) < \varepsilon$. If we take $n \geq 2$ such that $1/n < r$, then

$$\operatorname{diam}(V \cap \{z^* \in Z^* \mid q_n(z^*) \leq 1\}) < \varepsilon.$$

For ω large enough, $z^*_\omega \in V$ by the w^*-convergence and $q_n(z^*_\omega) < 1$, so

$$z^*_\omega \in V \cap \{z^* \in Z^* \mid q_n(z^*) \leq 1\}$$

and this implies $\|z^*_\omega - x\| < \varepsilon$. Now is clear that the restriction of Φ to X will satisfy all the properties required and this ends the proof of the lemma. □

Our new construction in this paper provides the triangle inequality for the F-norm which turns out to be a Lipschitz function with respect to the metric associated with it, thus uniformly continuous for the original norm. From the above proof and having in mind Theorem 4 in [10] it is not clear that Φ should have to be uniformly continuous.

Our results in this paper answer Question 6.2 in [21]. Main ideas are provided by a Decomposition Lemma 3.3, together with extended versions of the Connection Lemma 3.2 in [25] given by Theorem 3.5.

Nevertheless the following question remains open:

Problem 1.4. Is it possible to convexify the construction in Theorem 1.2 in order to get an equivalent $\sigma(X,Z)$-lower semicontinuous norm $|||\cdot|||$ on X such that the $\sigma(X,Z)$ and norm topologies coincide on the unit sphere $\{x \in X \mid |||x||| = 1\}$?

Note that a norm on X such that the $\sigma(X,Z)$ and norm topologies coincide on its unit sphere is necessarily $\sigma(X,Z)$-lower semicontinuous. As a matter of fact, all the statements of isometric nature involving the $\sigma(X,Z)$-topology in this paper will include the hypothesis of $\sigma(X,Z)$-lower semicontinuity. Moreover, when handling topological non-isometric statements we will always assume that the norm is $\sigma(X,Z)$-lower semicontinuous. Indeed, changing the norm of X by an equivalent one do not alter the validity of the statement. Let us remark that a consequence is that the norming subspace $Z \subset X^*$ can be supposed 1-norming without loss of generality together any non-isometric topological statement involving the $\sigma(X,Z)$-topology.

2. p-convex constructions

In this section we shall prove some results regarding generalized convexity that shall be used in what follows. First of all let us recall the following definition, [15, p. 160]:

Definition 2.1. Let A be a subset of a vector space X and $p \in (0,1]$. A is said to be *p-convex* if for every $x, y \in A$ and $\tau, \mu \in [0,1]$ such that $\tau^p + \mu^p = 1$ we have $\tau x + \mu y \in A$. We denote by $\mathrm{co}_p(A)$ the *p-convex hull* of a set A, i.e. the smallest p-convex set of X containing A.

Notice that the p-convex hull of a set A can be represented explicitly as

$$\mathrm{co}_p(A) = \{\sum_{i=1}^{n} \tau_i x_i : (x_i)_{i=1}^n \subset A, \tau_i \geq 0, \sum_{i=1}^{n} \tau_i^p = 1\}.$$

It is easy to check that $\tau x \in \mathrm{co}_p(A)$ whenever $\tau \in (0,1]$ and $x \in A$ if $p \in (0,1)$. If we have a p-convex and absorbent subset A in a vector space X, we define its p-Minkowski functional as

$$p_A(x) := \inf\{\lambda^p \mid \lambda > 0, x \in \lambda A\}.$$

The p-convexity of A implies that $p_A(x+y) \leq p_A(x) + p_A(y)$ and $p_A(\lambda x) = \lambda^p p_A(x)$ for $\lambda > 0$. Moreover, if A is balanced as well, then p_A is a *p-seminorm* in the terminology of [15, p. 160]. The usual Minkowski functional is defined as usual:

$$q_A(x) := \inf\{\lambda \mid \lambda > 0, x \in \lambda A\}$$

and we obviously have $q_A(x) = p_A(x)^{1/p}$ for every $x \in X$. The functional q_A is a quasinorm and we have that $q_A(x+y) \leq 2^{(1/p)-1}(q_A(x) + q_A(y))$.

We shall now study some fundamental properties of the functions whose epigraph is a p-convex set.

Definition 2.2. A real function ϕ from a vector space X is said to be *p-convex* (resp. to satisfy the *p-property*), for $p \in (0,1]$, if

$$\phi(\tau x + \mu y) \le \tau\phi(x) + \mu\phi(y) \qquad (\text{resp. } \phi(\tau x + \mu y) \le \tau^p\phi(x) + \mu^p\phi(y))$$

whenever $\tau \ge 0$, $\mu \ge 0$ and $\tau^p + \mu^p = 1$.

The following observations are easily checked :

- the epigraph of ϕ is p-convex if and only if ϕ is p-convex;
- the sum of non-negative p-convex functions is p-convex as well;
- if ϕ is convex and $\phi(0) = 0$, then ϕ is p-convex for every $p \in (0,1]$;
- if ϕ is p-convex and non-negative, then $\phi(0) = 0$ and ϕ satisfies the p-property;
- If ϕ is p-convex for $0 < p < 1$ and non-negative, then ϕ is q-convex for any $0 < q \le p$.

The following lemma will provide an idea of how the p-convex hull of some set looks like specially when p is close to 0.

Lemma 2.3. *Let $(X, \|\cdot\|)$ be a normed space, $A \subset \mathscr{B}_X$ and $p \in (0,1)$. Then*

$$\{\lambda x \,|\, 0 < \lambda \le 1,\ x \in A\} \subseteq \mathrm{co}_p(A)$$

and

$$\mathrm{co}_p(A) \subseteq \{\lambda x \,|\, 0 < \lambda \le 1,\ x \in A\} + \mathscr{B}[0, p(1-p)^{1/p-1}].$$

As a consequence, if $A \subset X$ is closed and bounded, then

$$\bigcap_{p\in(0,1)} \mathrm{co}_p(A) = \{\lambda x \,|\, 0 < \lambda \le 1,\ x \in A\}.$$

Proof. First note that if $0 < \lambda_1 < \lambda_2$ and $\lambda_1 x, \lambda_2 x \in \mathrm{co}_p(A)$, then $\lambda x \in \mathrm{co}_p(A)$ for every $\lambda \in [\lambda_1, \lambda_2]$. Now, if $x \in A$ then $n^{1-1/p}x \in \mathrm{co}_p(A)$ for every $n \in \mathbb{N}$ because $n^{-1/p}x + \cdots + n^{-1/p}x$ with n addends is a p-convex combination of elements from A. The fact that $\lim_{n\to\infty} n^{1-1/p} = 0$ finishes the proof of the first set inclusion.
Any point of $\mathrm{co}_p(A)$ is of the form $\tau_1 x_1 + \cdots + \tau_n x_n$ where $x_1, \ldots, x_n \in A$, $\tau_1, \ldots, \tau_n \in [0,1]$ and $\tau_1^p + \cdots + \tau_n^p = 1$. Suppose that $\tau_i = \max\{\tau_1, \ldots, \tau_n\}$. We want to estimate the distance $d = \|\tau_1 x_1 + \cdots + \tau_n x_n - \tau_i x_i\|$. Without loss of generality, we may suppose that $\tau_1 \ge \tau_2 \ge \cdots \ge \tau_n$, therefore

$$d = \|\tau_2 x_2 + \cdots + \tau_n x_n\| \le \tau_2 + \cdots + \tau_n$$

It is easy to see that the maximum value of the term on the right-hand side is attained when $\tau_1 = \tau_2 = \cdots = \tau_k > \tau_{k+1} \ge \tau_{k+2} = 0$ for some $k \in \{1, \ldots, n\}$. Observe that $k = \tau_1^{-p}(1 - \tau_{k+1}^p)$ and so

$$\tau_1 + \tau_2 + \cdots + \tau_n = k\tau_1 + \tau_{k+1} = \tau_1^{1-p} - \tau_1^{1-p}\tau_{k+1}^p + \tau_{k+1} \le \tau_1^{1-p}.$$

Therefore, we have $\tau_2 + \cdots + \tau_n \le \tau_1^{1-p} - \tau_1$. This last expression, as a function of $\tau_1 \in [0,1]$, attains its maximum at $\tau_1 = (1-p)^{1/p}$. An easy computation gives us that $d \le \tau_2 + \cdots + \tau_n \le p(1-p)^{1/p-1}$ as desired.
The consequence follows easily from these facts: $\{\lambda x \,|\, 0 < \lambda \le 1,\ x \in A\} \cup \{0\}$ is closed and $\lim_{p\to 0^+} p(1-p)^{1/p-1} = 0$. □

Finally, we shall show two inequalities and facts about functions which satisfy the p-property that shall be needed.

Proposition 2.4. *Suppose that ϕ satisfies the p-property for some $p \in (0,1]$, then for every $x, y \in X$,*

$$\tau^p \mu^p (\phi(x) - \phi(y))^2 \leq \tau^p \phi(x)^2 + \mu^p \phi(y)^2 - \phi(\tau x + \mu y)^2$$

whenever $\tau^p + \mu^p = 1$ and $\tau \geq 0$, $\mu \geq 0$.

Proof. We have

$$\begin{aligned} \tau^p \phi(x)^2 + \mu^p \phi(x)^2 - \phi(\tau x + \mu y)^2 &\geq \tau^p \phi(x)^2 + \mu^p \phi(x)^2 - (\tau^p \phi(x) + \mu^p \phi(y))^2 \\ &= (\tau^p - \tau^{2p})\phi(x)^2 + (\mu^p - \mu^{2p})\phi(y)^2 - 2\tau^p \mu^p \phi(x)\phi(y) \\ = \tau^p(1-\tau^p)\phi(x)^2 + \mu^p(1-\mu^p)\phi(y)^2 &- 2\tau^p\mu^p\phi(x)\phi(y) = \tau^p\mu^p(\phi(x) - \phi(y))^2. \end{aligned}$$

□

Corollary 2.5. *For a p-seminorm $\|\cdot\|_p$ on the vector space X we have*

$$(\|x\|_p - \|y\|_p)^2 \leq 2\|x\|_p^2 + 2\|y\|_p^2 - \|x+y\|_p^2.$$

Proof. A p-seminorm is a nonnegative function that satisfies the p-property to which we apply the former lemma for $\tau = \mu = (1/2)^{1/p}$. □

We follow with a p-version of fact II.2.3 of [3].

Proposition 2.6. (1) *If $\|\cdot\|_p$ is a p-seminorm on X and $x_j, x \in X$ then the following are equivalent:*
   (a) $\lim_j \|x_j\|_p = \|x\|_p$ *and* $\lim_j \left\|\frac{x+x_j}{2^{1/p}}\right\|_p = \|x\|_p$*;*
   (b) $\lim_j (2\|x\|_p^2 + 2\|x_j\|_p^2 - \|x+x_j\|_p^2) = 0$

(2) *If $\alpha_n > 0$, $\|\cdot\|_{p_n}$ is a p_n-seminorm on X for some sequence $(p_n) \subseteq (0,1)$ and*

$$\lim_j \left(2F^2(x) + 2F^2(x_j) - F^2(x+x_j)\right) = 0,$$

where $F^2(x) = \sum_{n\in\mathbb{N}} \alpha_n \|x\|_{p_n}^2$, then for every $n \in \mathbb{N}$

$$\lim_j (2\|x\|_{p_n}^2 + 2\|x_j\|_{p_n}^2 - \|x+x_j\|_{p_n}^2) = 0.$$

Proof. Both can be derived from Corollary 2.5. □

We are now going to state a version of Proposition 2.1 in [25] for the p-convex case. These *distance functions* will be an essential tool in our reasoning.

Proposition 2.7. *Let X be a normed space and Z a norming subspace in the dual space X^*. If C is a w^*-compact and p-convex subset of X^{**}, $0 < p \leq 1$, and we define for $x \in X$*

$$\varphi(x) := \inf_{c^{**}\in C} \|x - c^{**}\|_Z$$

*Then φ is a p-convex, $\sigma(X,Z)$-lower semicontinuous and 1-Lipschitz from X to $[0,+\infty)$. We call such a function the Z-*distance *to the set C.*

Proof. The fact that C is p-convex implies that φ is a p-convex function. Indeed, let us take $x, y \in X$ and fix $0 \leq \tau, \mu \leq 1$ with $\tau^p + \mu^p = 1$, and $\varepsilon > 0$. If we choose c_x^{**} and c_y^{**} such that

$$\|x - c_x^{**}\|_Z \leq \varphi(x) + \varepsilon \text{ and } \left\|y - c_y^{**}\right\|_Z \leq \varphi(y) + \varepsilon,$$

then

$$\left\|\tau x + \mu y - (\tau c_x^{**} + \mu c_y^{**})\right\|_Z \leq \|\tau x - \tau c_x^{**}\|_Z + \left\|\mu y - \mu c_y^{**}\right\|_Z$$

$$\leq \tau(\varphi(x) + \varepsilon) + \mu(\varphi(y) + \varepsilon) \leq \tau\varphi(x) + \mu\varphi(y) + (\tau + \mu)\varepsilon \leq \tau\varphi(x) + \mu\varphi(y) + \varepsilon$$

because $\tau + \mu \leq \tau^p + \mu^p = 1$. Since $\tau c_x^{**} + \mu c_y^{**} \in C$ we have

$$\varphi(\tau x + \mu y) \leq \tau\varphi(x) + \mu\varphi(y) + \varepsilon$$

for every $\varepsilon > 0$ and $0 \leq \tau, \mu \leq 1$ with $\tau^p + \mu^p = 1$.

Let us prove the lower semicontinuity of φ. Fix $r \geq 0$ and take a net $\{x_\alpha \,|\, \alpha \in A\}$ in X with $\varphi(x_\alpha) \leq r$ for every $\alpha \in A$ and let $x \in X$ be the $\sigma(X, Z)$-limit of the net $\{x_\alpha \,|\, \alpha \in A\}$. We will see that $\varphi(x) \leq r$ too. Firstly note that the net $\{x_\alpha \,|\, \alpha \in A\}$ is necessarily bounded. Indeed, the triangular inequality implies that $\|x_\alpha\|_Z \leq r + \sup_{c^{**} \in C} \|c^{**}\|_Z$ and of course $\sup_{c^{**} \in C} \|c^{**}\|_Z < +\infty$. Let us fix an $\varepsilon > 0$ and choose $c_\alpha^{**} \in C$ such that $\|x_\alpha - c_\alpha^{**}\| \leq r + \varepsilon$ for every $\alpha \in A$. Since C is w^*-compact we can find a cluster point (x^{**}, c^{**}) of the net $\{(x_\alpha, c_\alpha^{**}) \,|\, \alpha \in A\}$ in $X^{**} \times X^{**}$ for the topology $\sigma(X^{**}, X^*)$ on every factor, since $\{x_\alpha \,|\, \alpha \in A\}$ was bounded. Then we have that x^{**} does coincide with x when both linear functionals are restricted to Z and thus for every $f \in \mathscr{B}_{X^*} \cap Z$

$$f(x^{**} - c^{**}) = f(x - c^{**}) \leq r + \varepsilon$$

and so $\varphi(x) \leq r + \varepsilon$. Since the reasoning is valid for every $\varepsilon > 0$ we get $\varphi(x) \leq r$ as required.

The Lipschitz condition follows from the triangle inequality of the seminorm $\|\cdot\|_Z$ on X^{**}. Indeed, for every $x, y \in X$ and $c^{**} \in \overline{C}^{\sigma(X^{**}, X^*)}$ we have $\|x - c^{**}\|_Z \leq \|x - y\|_Z + \|y - c^{**}\|_Z$, thus $\varphi(x) \leq \|x - y\|_Z + \varphi(y)$. If we interchange x and y we see that

$$|\varphi(x) - \varphi(y)| \leq \|x - y\|_Z$$

which also implies 1-Lipschitz with respect to the norm of X as $\|\cdot\|_Z \leq \|\cdot\|$. $\square$

Remark 2.8. Note that if $B \subset X$ is $\sigma(X, Z)$-closed, then the Z-distance to the weak*-closure of B in X^{**} is positive on $X \setminus B$. Indeed, B is in particular weakly closed, so if $C \subset X^{**}$ is the weak*-closure of B, then $B = C \cap X$.

Looking for the "scalpel parameter" measuring a rigidity condition involved in our renormings we introduce the following:

Definition 2.9. Let $(X, \|\cdot\|)$ be a normed space, Z be a norming subspace in X^* and $0 < p \leq 1$. A family $\mathcal{B} := \{B_i \,|\, i \in I\}$ of subsets in the normed space X is said to be *p-isolated* for the $\sigma(X, Z)$-topology when for every $i \in I$

$$B_i \cap \overline{\mathrm{co}_p \bigcup \{B_j \,|\, j \neq i,\ j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

Let us observe that by the Hahn-Banach theorem, the definition of 1-isolated family corresponds to the notion of $\sigma(X,Z)$-slicely isolated, see [25]. We now state the following interplay result describing this new concept as a biorthogonal system of sets and p-convex functions:

Theorem 2.10. *Let $(X, \|\cdot\|)$ be a normed space and Z be a norming subspace in X^*. Let $\mathcal{B} := \{B_i \,|\, i \in I\}$ be an uniformly bounded family of subsets of X. The following are equivalent:*

(1) *The family $\mathcal{B}$ is p-isolated for the $\sigma(X,Z)$-topology;*

(2) *There exists a family*

$$\mathcal{L} := \{\varphi_i : X \to [0, +\infty) \,|\, i \in I\}$$

of p-convex and $\sigma(X,Z)$ lower semicontinuous functions such that for every $i \in I$

$$\{x \in X \,|\, \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.$$

(3) *There exists a family*

$$\mathcal{L} := \{\psi_i : X \to [0, +\infty) \,|\, i \in I\}$$

of p-convex and $\sigma(X,Z)$-lower semicontinuous functions and numbers $0 \leq \alpha \leq \beta$ such that for every $i, j \in I$

$$\psi_i(B_i) > \beta \geq \alpha \geq \psi_i(B_j).$$

Proof. Let us assume that the family $\mathcal{B}$ is $\sigma(X,Z)$ p-isolated. Applying Proposition 2.7 we may consider φ_i to be the Z-distance to

$$\overline{\mathrm{co}_p \bigcup \{B_j : j \neq i, j \in I\}}^{\sigma(X^{**},X^*)}$$

for every $i \in I$. Consider on X^{**} the topology $\sigma(X^{**}, Z)$. This topology is not Hausdorff in general, but it is coarser than the topology generated by the seminorm $\|\cdot\|_Z$. In particular, the $\|\cdot\|_Z$-distance to a $\sigma(X^{**},Z)$-closed subset of X^{**} from outer points is strictly positive. Our hypothesis on the p-isolated character of the family $\mathcal{B}$ tells us that when a point x belongs to the set B_i of the family $\mathcal{B}$, then there is a $\sigma(X,Z)$-open subset $W \ni x$ such that

$$W \cap \mathrm{co}_p \bigcup \{B_j \,|\, j \neq i, j \in I\} = \emptyset.$$

There is a $\sigma(X^{**},Z)$-open set $\widetilde{W}$ such that $W = X \cap \widetilde{W}$. We have

$$\mathrm{co}_p \bigcup \{B_j \,|\, j \neq i, j \in I\} \subset X^{**} \setminus \widetilde{W}$$

and so

$$\overline{\mathrm{co}_p \bigcup \{B_j \,|\, j \neq i, j \in I\}}^{\sigma(X^{**},X^*)} \subset \overline{\mathrm{co}_p \bigcup \{B_j \,|\, j \neq i, j \in I\}}^{\sigma(X^{**},Z)} \subset X^{**} \setminus \widetilde{W}.$$

After the previous considerations, that implies $\varphi_i(x) > 0$. Clearly we also have $\varphi_j(x) = 0$ for every $j \in I$ with $j \neq i$.

Condition (2) clearly implies (3) with $\alpha = \beta = 0$.

Finally, if we assume (3), given a family $\mathcal{L} := \{\psi_i : X \to [0, +\infty) \,|\, i \in I\}$ of p-convex and $\sigma(X,Z)$-lower semicontinuous functions such that the conditions in

3 are satisfied we will have, by the p-convexity of the function ψ_i, that $\psi_i(y) \le \alpha$ for every $y \in \operatorname{co}_p \bigcup\{B_j \,|\, j \neq i,\ j \in I\}$ and also, by the lower semicontinuity of ψ_i, for every $y \in \overline{\operatorname{co}_p \bigcup\{B_j \,|\, j \neq i,\ j \in I\}}^{\sigma(X,Z)}$. Therefore, for every $i \in I$ and $x \in B_i$ we have $x \notin \overline{\operatorname{co}_p \bigcup\{B_j \,|\, j \neq i,\ j \in I\}}^{\sigma(X,Z)}$ and this finishes the proof of the $\sigma(X,Z)$-p-isolated property of the family $\mathcal{B}$. □

3. Construction of a Kadec F-norm

The following concept is a crucial one in the study of locally bounded topological vector spaces, see [15, p. 159]:

Definition 3.1 (Quasinorm)**.** A *quasinorm* in a vector space X is a function $q : X \to [0, +\infty)$ such that:

(1) $x = 0$ if, and only if, $q(x) = 0$;
(2) $q(\alpha x) = |\alpha| q(x)$ for every $\alpha \in \mathbb{R}$ and $x \in X$;
(3) there exists $k \geq 1$ such that $q(x + y) \leq k(q(x) + q(y))$ for every $x, y \in X$.

We begin to deal first with the construction of a Kadec quasinorm from where the F-norm will follow. Our approach is based on the network property that characterizes descriptive Banach spaces, see theorems 1.2 and 1.5 in [7].

Let us summarize facts in the following:

Theorem 3.2 (Kadec quasi-renorming)**.** *Let $(X, \|\cdot\|)$ be a normed space with an 1-norming subspace Z in X^*. Then the following conditions are equivalent:*

(1) *There is a sequence (A_n) of subsets of X such that for every $\varepsilon > 0$ and $x \in X$ there is some integer p together with a $\sigma(X,Z)$-open set W such that $x \in A_p \cap W$ and $\|\cdot\|$-diam$(A_p \cap W) < \varepsilon$*
(2) *For every $\varepsilon > 0$ there is an equivalent $\sigma(X,Z)$-lower semicontinuous quasinorm $q_\varepsilon(\cdot)$ on X such that*
   (a) *$(1-\varepsilon)\|x\| \leq q_\varepsilon(x) \leq (1+\varepsilon)\|x\|$ for every $x \in X$.*
   (b) *$q_\varepsilon(x+y) \leq \frac{1+\varepsilon}{1-\varepsilon}(q_\varepsilon(x) + q_\varepsilon(y))$ for every $x, y \in X$*
   (c) *$\sigma(X,Z)$ coincides with the norm topology on the "unit sphere"*
$$\{x \in X : q_\varepsilon(x) = 1\}.$$
(3) *The normed space X is $\sigma(X,Z)-$descriptive; i.e. there are isolated families for the $\sigma(X,Z)$-topology*
$$\{\mathcal{B}_n \,|\, n = 1, 2, \ldots\}$$
*in X such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and a set $B \in \mathcal{B}_n$ with the property that $x \in B$ and that $\|\cdot\|$-*diam*$(B) < \varepsilon$.*
(4) *There is a metric d on X generating a topology finer than the weak topology on X and such that the identity map from $(X, \sigma(X,Z))$ into (X,d) is σ-continuous.*
(5) *There exists a network $\mathcal{N}$ for the $\sigma(X, X^*)$ topology that can be written as a countable union of subfamilies, $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$, where each one of the subfamilies $\mathcal{N}_n$ is $\sigma(X,F)$-isolated.*

The former theorem is a strong reformulation of the conditions used by Raja to construct a Kadec function F in every normed space X with countable cover by sets of small local diameter. Our different approach here will permit the construction of a Kadec F-norm as well as to show a precise connection between metrization theory and Kadec renormability in next section. Our proof is based on two fundamental lemmata. The first is a decomposition lemma and shows how to decompose an isolated family of sets into countable many p_n-isolated families; the second one is a connection lemma between the existence of a p-isolated family and the Kadec property.

Lemma 3.3 (Decomposition lemma)**.** *Let $(X, \|\cdot\|)$ be a normed space, Z be a norming subspace in X^*. Let $\mathcal{B}$ be a uniformly bounded and isolated family of sets for the $\sigma(X,Z)$ topology. Then for every $B \in \mathcal{B}$ we can write*

$$B = \bigcup_{n=1}^{\infty} B_n$$

in such a way that, for every $n \in \mathbb{N}$ fixed, the family $\{B_n \mid B \in \mathcal{B}\}$ is $\sigma(X,Z)$-q_n-isolated for some $q_n \in (0,1]$. The sequence (q_n) can be taken to be nonincreasing and with limit 0.

Proof. Without loss of generality we may assume that Z is 1-norming, so the closed balls $\mathscr{B}[0,r]$ are $\sigma(X,Z)$-closed for any radius $r > 0$. Given a neighborhood W of the origin in the $\sigma(X,Z)$-topology let us define the width of W as:

$$\mathrm{wd}(W) := \sup\{\delta > 0 \mid \mathscr{B}(0,\delta) \subseteq W\}.$$

Without lose of generality we may assume that $\mathcal{B}$ is contained in the unit ball of X. Set $A_{n,k} := \mathscr{B}[0, \frac{k}{4n}] \setminus \mathscr{B}[0, \frac{k-1}{4n}]$ and denote by $\mathcal{U}$ the family of all convex and $\sigma(X,Z)$-open neighborhoods of the origin in X. The isolated family $\mathcal{B}$ for the $\sigma(X,Z)$-topology can be decomposed as follows

$$B = \bigcup_{n\in\mathbb{N}} \bigcup_{k\leq 4n} B_{n,k}$$

where

$$B_{n,k} := \big\{x \in B \cap A_{n,k} \,\big|\, \exists W \in \mathcal{U}, \mathrm{wd}(W) > n^{-1}, (x+W) \cap B' = \emptyset \ \forall B' \in \mathcal{B} \setminus \{B\}\big\}.$$

We will see that the family $\{B_{n,k} \mid B \in \mathcal{B}\}$ is q-isolated whenever q satisfies the inequality

$$q(1-q)^{1/q-1} < \frac{1}{4n}$$

which clearly implies the statement of the lemma after reindexing the sets. The statement about the sequence (q_n) can be derived from the fact that the term on the left-hand side has limit 0 as q goes to 0. In order to show the q-isolatedness of $\{B_{n,k} \mid B \in \mathcal{B}\}$, fix a point $x \in B_{n,k}$. By the definition of the set there is a open neighborhood of the origin W in the $\sigma(X,Z)$-topology, with $\mathscr{B}(0,1/n) \subseteq W$, and $(x+W) \cap B' = \emptyset$ for every $B' \in \mathcal{B} \setminus \{B\}$. In particular we see that

$$(x + \frac{1}{4}W) \cap ((B' + \mathscr{B}(0, \frac{3}{4n})) = \emptyset$$

for every $B' \in \mathcal{B} \setminus \{B\}$. Now we claim that

$$((x + \frac{1}{4}W) \setminus \mathscr{B}[0, \frac{k-1}{4n}]) \cap \mathrm{co}_q \bigcup \{B' \cap A_{n,k} \mid B' \in \mathcal{B} \setminus \{B\}\} = \emptyset$$

which implies the desired $\sigma(X,Z)$-q-isolatedness as the first set is a $\sigma(X,Z)$-open neighborhood of x. Indeed, if suppose that $y \in \mathrm{co}_q \bigcup \{B' \cap A_{n,k} \mid B' \in \mathcal{B} \setminus \{B\}\}$. Then, for some $B' \in \mathcal{B} \setminus \{B\}$ there are $x' \in B' \cap A_{n,k}$ and $\lambda \in [0,1]$ such that $y \in \lambda x' + \mathscr{B}[0, \frac{1}{4n}]$ by Lemma 2.3. Since $y \notin \mathscr{B}(0, \frac{k-1}{4n})$ we have $\lambda x' \notin \mathscr{B}[0, \frac{k-2}{4n}]$. Since $x' \in A_{n,k}$, we get $\|x' - \lambda x'\| \leq \frac{2}{4n}$. Therefore $y \in x' + \mathscr{B}(0, \frac{3}{4n}) \subset B' + \mathscr{B}(0, \frac{3}{4n})$ which is incompatible with $y \in x + \frac{1}{4}W$. □

The following variant of Deville's master lemma was proved, for sequences, by Haydon (see Proposition 1.2 of [8]) to construct Kadec norms is spaces $C(\Upsilon)$ where Υ is a tree. The following net version has been used in [2]; we will use it here to describe the connection between Haydon's approach and Stone's discreteness:

Lemma 3.4 (see lemma 5.3 of [2]). *Let X be a topological space, S be a set and $\varphi_s, \psi_s : X \to [0, +\infty)$ lower semicontinuous functions such that*

$$\sup_{s \in S}(\varphi_s(x) + \psi_s(x)) < +\infty$$

for every $x \in X$. Define

$$\varphi(x) = \sup_{s \in S} \varphi_s(x), \quad \theta_m(x) = \sup_{s \in S} \left(\varphi_s(x) + 2^{-m}\psi_s(x)\right), \quad \theta(x) = \sum_{m \in \mathbb{N}} 2^{-m}\theta_m(x).$$

Assume further that $\{x_\sigma\}_{\sigma \in \Sigma}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma \in \Gamma}$ and a net $\{i_\gamma\}_{\gamma \in \Gamma} \subseteq S$ such that

$$\lim_{\gamma \in \Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma \in \Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma \in \Gamma} \varphi(x_\gamma) = \sup_{s \in S} \varphi_s(x)$$

and

$$\lim_{\gamma \in \Gamma}(\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0.$$

We can now state the connection lemma between Haydon's approach and Stone's discreteness in the following:

Theorem 3.5 (p-connection). *Let $(X, \|\cdot\|)$ be a normed space and Z be a norming subspace in X^*. Let $\mathcal{B} := \{B_i \mid i \in I\}$ be an uniformly bounded and p-isolated family of subsets of X for the $\sigma(X,Z)$-topology and some $p \in (0,1]$. Then there is an equivalent $\sigma(X,Z)$-lower semicontinuous quasinorm, with p-power a p-norm, $\|\cdot\|_{\mathcal{B}}$ on X such that: for every net $\{x_\alpha \mid \alpha \in A\}$ and x in X with $x \in B_{i_0}$ for $i_0 \in I$, the conditions $\sigma(X,Z)$-$\lim_\alpha x_\alpha = x$ and $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ imply that*

1. *there exists $\alpha_0 \in A$ such that x_α is not in $\overline{\mathrm{co}_p \bigcup \{B_i \mid i \neq i_0,\ i \in I\}}^{\sigma(X,Z)}$ for $\alpha \geq \alpha_0$;*
2. *for every positive δ there exists $\alpha_\delta \in A$ such that*

$$x, x_\alpha \in \overline{(\mathrm{co}\,(B_{i_0} \cup \{0\}) + \mathscr{B}(0,\delta))}^{\sigma(X,Z)}$$

whenever $\alpha \geq \alpha_\delta$.

Proof. Without loss of generality we may assume that Z is 1-norming. Let us fix the index $i \in I$ and define the function φ_i as the Z-distance to the set

$$\overline{\mathrm{co}_p \bigcup \{B_j \mid j \neq i,\ j \in I\}}^{\sigma(X^{**},X^*)}.$$

Recall that φ_i is 1-Lipschitz, p-convex and $\sigma(X,Z)$-lower semicontinuous thanks to Proposition 2.7. Let us set $D_i := \mathrm{co}\,(B_i \cup \{0\})$, $D_i^\delta := D_i + \mathscr{B}(0,\delta)$, where

$$\mathscr{B}(0,\delta) := \{x \in X \mid \|x\|_Z < \delta\},$$

for every $\delta > 0$ and $i \in I$. We shall denote by p_i^δ the Minkowski functional of the convex body $\overline{D_i^\delta}^{\sigma(X,Z)}$ which is obviously sublinear, Lipschitz and $\sigma(X,Z)$-lower semicontinuous. Then, for $x \in X$ we define the $\sigma(X,Z)$-lower semicontinuous norms ψ_i with the formula

$$\psi_i(x) = \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_i^{1/n}(x)$$

for every $x \in X$. It is well defined and $\sigma(X,Z)$-lower semicontinuous. Indeed, since $\mathscr{B}(0,\delta) \subset \overline{D_i^\delta}^{\sigma(X,Z)}$ we have for every $x \in X$, and $\delta > 0$, that $p_i^\delta(\delta x/\|x\|_Z) \leq 1$, thus $\delta p_i^\delta(x) \leq \|x\|_Z$ and hence the above series converge. Note that this also gives that ψ_i is 1-Lipschitz. We are now in position to apply Lemma 3.4 to get an equivalent quasinorm $\|\cdot\|_{\mathcal{B}}$ on X such that the condition $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ together with $\sigma(X,Z)$-$\lim_\alpha x_\alpha = x$ for a net $\{x_\alpha \mid \alpha \in A\}$ and x in X imply the existence of a finer net $\{x_\beta\}_{\beta\in B}$ and a net $(i_\beta)_{\beta\in B}$ in I satisfying these conditions

(1) $\lim_\beta \varphi(x_\beta) = \lim_\beta \varphi_{i_\beta}(x) = \lim_\beta \varphi_{i_\beta}(x_\beta) = \sup_{i\in I} \varphi_i(x)$;
(2) $\lim_\beta(\psi_{i_\beta}(x_\beta) - \psi_{i_\beta}(x)) = 0$.

Indeed, using the definitions in Haydon's Lemma 3.4 we introduce the functions:

$$\theta_m(x) := \sup\left\{\varphi_i(x) + 2^{-m}\psi_i(x) \,\middle|\, i \in I\right\};$$
$$\theta(x) := \|x\|_Z + \sum_{m\in\mathbb{N}} 2^{-m}(\theta_m(x) + \theta_m(-x)).$$

Note that θ_m is p-convex, $\sigma(X,Z)$-lower semicontinuous and 2-Lipschitz. That gives us that θ is a symmetric, p-convex, $\sigma(X,Z)$-lower semicontinuous and 5-Lipschitz function such that $\lim_\alpha \theta(x_\alpha) = \theta(x)$ together with $\sigma(X,Z)$-$\lim_\alpha x_\alpha = x$ imply the conditions 1 and 2 above by Haydon's lemma. Since $\mathcal{B}$ is a uniformly bounded family, the $\|\cdot\|_Z$-1-Lipschitz functions $\{\varphi_i, \psi_i : i \in I\}$ are uniformly bounded on bounded sets, thus there is $\rho > 0$ such that $\mathscr{B}_X \subset \{x \in X \mid \theta(x) \leq \rho\}$

The Minkowski functional of the p-convex set

$$D := \{x \in X \mid \theta(x) \leq \rho\}$$

provide us with the quasinorm $\|\cdot\|_{\mathcal{B}}$ we are looking for. Theorem 6.4.4, p.107 of[10] tells us that its p-power $\|\cdot\|_{\mathcal{B}}^p$ is uniformly continuous and so an equivalent p-norm on X with

$$\|\cdot\|_{\mathcal{B}} \leq \|\cdot\| \leq \rho\|\cdot\|_{\mathcal{B}}.$$

Let us take a net $\{x_\alpha \,|\, \alpha \in (A, \succ)\}$ and x in X with $\|x\|_{\mathcal{B}} = 1$ verifying that $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ and such that x is the $\sigma(X,Z)$-limit of the net (x_α). We claim that

$$\lim_\alpha \theta(x_\alpha) = \rho$$

Indeed, for every $\alpha \in A$ we can write $x_\alpha = (1+\eta_\alpha)y_\alpha$ where $1+\eta_\alpha > 0$ and $\theta(y_\alpha) = \rho$. Notice that $\lim_{\alpha\in A} \eta_\alpha = 0$ since $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}} = 1$. Thus $\lim_\alpha \|x_\alpha - y_\alpha\| = \lim_\alpha \eta_\alpha \|y_\alpha\| = 0$ by the boundness of D, and $\lim_\alpha \theta(x_\alpha) = \lim_\alpha \theta(y_\alpha) = \rho$ since θ is Lipschitz.

Our hypothesis on the p-isolated character of the family $\mathcal{B}$ gives us that

$$x \notin \overline{\mathrm{co}_p \bigcup \{B_i \,|\, i \neq i_0,\ i \in I\}}^{\sigma(X,Z)}$$

whenever $x \in B_{i_0}$, and so $\varphi_{i_0}(x) > 0$ but $\varphi_i(x) = 0$ for all $i \in I$ with $i \neq i_0$, see Theorem 2.10.

From the condition 1 above there exists β_0 such that $i_\beta = i_0$ and $\varphi_{i_0}(x_\beta) > 0$ for all $\beta \geq \beta_0$, from where the conclusion 1 of the theorem will follow. Moreover, the condition 2 above implies that $\lim_\beta(\psi_{i_\beta}(x_\beta) - \psi_{i_\beta}(x)) = 0$, thus $\lim_\beta \psi_{i_0}(x_\beta) = \psi_{i_0}(x)$. Then we have

$$\psi_{i_0}(x) = \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_{i_0}^{1/n}(x) \leq \sum_{n\in\mathbb{N}} \frac{1}{n2^n} \liminf_\beta p_{i_0}^{1/n}(x_\beta) \leq \liminf_\beta \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_{i_0}^{1/n}(x_\beta)$$

$$= \lim_\beta \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_{i_0}^{1/n}(x_\beta) = \lim_\beta \psi_{i_0}(x_\beta) = \psi_{i_0}(x) = \sum_{n\in\mathbb{N}} \frac{1}{n2^n} p_{i_0}^{1/n}(x)$$

where the first inequality comes from the lower semicontinuity and the second one is just Fatou's Lemma for positive series. It now follows for every positive integer n

$$\liminf_\beta p_{i_0}^{1/n}(x_\beta) = p_{i_0}^{1/n}(x).$$

If we fix a positive number δ and we set the integer q such that $1/q < \delta$, since $x \in D_{i_0}^{1/q}$ we have that $p_{i_0}^{1/q}(x) < 1$ because $D_{i_0}^{1/q}$ is norm open and therefore, for every $\beta \in B$ there is $\gamma_\beta \geq \beta$ such that $p_{i_0}^{1/q}(x_{\gamma_\beta}) < 1$, thus $x_{\gamma_\beta} \in \overline{D_{i_0}^{\delta}}^{\sigma(X,Z)}$, and indeed $x_{\gamma_\beta} \in \overline{(\mathrm{co}(B_{i_0} \cup \{0\}) + \mathscr{B}(0,\delta))}^{\sigma(X,Z)}$. The proof is over since our reasoning is valid for any subnet of the original one $\{x_\alpha \,|\, \alpha \in A\}$. □

Remark 3.6. The following observations will be useful.

(1) For every $\alpha > 1$ it is possible to construct the former quasinorm $\|\cdot\|_{\mathcal{B}}$ such that:

$$\|x\|_Z \leq \|x\|_{\mathcal{B}} \leq \frac{4+\alpha}{\alpha}\|x\|_Z$$

for every $x \in X$.

(2) If $\varepsilon \in (0,1)$ is fixed, we can select $\alpha > 1$ large enough, then we see that

$$\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B}} \leq (1+\varepsilon)\|\cdot\|_Z.$$

Consequently the quasinorm constructed verifies:

$$\|x+y\|_{\mathcal{B}} \leq (1+\varepsilon)(\|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}})$$

for all $x, y \in X$.

Proof. Only (1) needs some explanation. Recall that θ is 5-Lipschitz with respect to $\|\cdot\|_Z$. That clearly implies $\theta(x) \leq 5$ whenever $\|x\|_Z \leq 1$. Consider the function

$$\theta^\alpha(x) := \theta(x) + (\alpha - 1)\|x\|_Z$$

which can be understood as a modification on the very definition of θ using $\alpha\|\cdot\|$ instead of $\|\cdot\|$. We have $\theta^\alpha(x) \leq 4 + \alpha$ if $\|x\|_Z \leq 1$, and $\theta^\alpha(x) \leq 4 + \alpha$ implies that $\|x\|_Z \leq \frac{4+\alpha}{\alpha}$. Consider the set

$$D^\alpha = \{x \in X \,|\, \theta^\alpha(x) \leq 4 + \alpha\}.$$

The previous inequalities implies that

$$\mathscr{B}_X \subset D^\alpha \subset \frac{4+\alpha}{\alpha} \mathscr{B}_X .$$

Let $\|\cdot\|_{\mathcal{B},\alpha}$ be the Minkowski functional of $\frac{\alpha}{4+\alpha} D^\alpha$. Then $\|\cdot\|_{\mathcal{B},\alpha}$ has all the properties of $\|\cdot\|_{\mathcal{B}}$ and, moreover,

$$\|\cdot\|_Z \leq \|\cdot\|_{\mathcal{B},\alpha} \leq \frac{4+\alpha}{\alpha}\|\cdot\|_Z$$

as we wanted. □

We are able now to prove Theorem 3.2.

Proof of Theorem 3.2. We will prove the equivalence of all the statements.

(3) ⇒ (2) The decomposition Lemma 3.3 says that we have a decomposition of the sets in the family $\mathcal{B}_n$ obtaining families $\mathcal{B}_n^m$, $m = 1, 2, \ldots$, with $\mathcal{B}_n^m$ being $q_{n,m}$-isolated for all $m, n = 1, 2, \ldots$. Therefore it is not a restriction to renumber the sequence and to assume that the given family $\mathcal{B}_n$ is already p_n-isolated for $n = 1, 2, \ldots$. We can now consider the equivalent quasinorms $\|\cdot\|_{\mathcal{B}_n}$ constructed using the p-connection Theorem 3.5 for every one of the families $\mathcal{B}_n$. We shall define now an equivalent quasinorm on X with the expression:

$$|||x||| := \sum_{n\in\mathbb{N}} c_n \|x\|_{\mathcal{B}_n}$$

for every $x \in X$, where the sequence $(c_n)_{n\in\mathbb{N}}$ is chosen accordingly for the convergence of the series. That is possible since we may, and do assume, that the following inequality holds

$$(1-\delta)\|x\|_Z \leq \|x\|_{\mathcal{B}_n} \leq (1+\delta)\|x\|_Z$$

for fixed $\delta > 0$, for all $n \in \mathbb{N}$, after Remark 3.6.

Let us start by proving the Kadec property. Take a net $\{x_\alpha \,|\, \alpha \in (A, \succ)\}$ and x with

$$\lim_{\alpha\in A} |||x_\alpha||| = |||x\,||| \;\text{ and } \sigma(X, Z) - \lim_{\alpha\in A} x_\alpha = x.$$

Then we **CLAIM:**

$$\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}_q} = \|x\|_{\mathcal{B}_q}$$

for every positive integer q.

Indeed, by the $\sigma(X,Z)$-lower semicontinuity of the quasinorms $\|\cdot\|_{\mathcal{B}_q}$, the series definition gives that

$$\sum_{n\in\mathbb{N}} c_n \|x\|_{\mathcal{B}_n} \leq \sum_{n\in\mathbb{N}} \liminf_{\alpha\in A} c_n \|x_\alpha\|_{\mathcal{B}_n}$$

$$\leq \liminf_{\alpha\in A} \sum_{n\in\mathbb{N}} c_n \|x_\alpha\|_{\mathcal{B}_n} = \lim_{\alpha\in A} \sum_{n\in\mathbb{N}} c_n \|x_\alpha\|_{\mathcal{B}_n} = \sum_{n\in\mathbb{N}} c_n \|x\|_{\mathcal{B}_n}$$

and then

$$\liminf_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}_q} = \|x\|_{\mathcal{B}_q}$$

for every $q \in \mathbb{N}$. Since this argument can be performed for every subnet, we easily see that $\lim_{\alpha\in A} \|x_\alpha\|_{\mathcal{B}_q}$ exists and thus our claim is proved.

Now, given $\varepsilon > 0$ let us consider the positive integer q such that for some $B \in \mathcal{B}_q$ we have $x \in B$ and $\|\cdot\|\text{-}\operatorname{diam}(B) < \varepsilon/2$. Theorem 3.5 tells us that there is some $\alpha_{\varepsilon/2}$ such that

$$x_\alpha \in \overline{\operatorname{co}(B \cup \{0\}) + \mathscr{B}(0, \varepsilon/2)}^{\sigma(X,Z)}$$

whenever $\alpha \succ \alpha_{\varepsilon/2}$. We have that $\|\cdot\|\text{-}\operatorname{dist}(x_\alpha, I_x) \leq \varepsilon$ for $\alpha \succ \alpha_{\varepsilon/2}$ where I_x is the segment joining x with the origin, and so there are numbers $r_{(\alpha,\varepsilon)} \in [0,1]$ such that

$$\left\|x_\alpha - r_{(\alpha,\varepsilon)} x\right\| \leq \varepsilon$$

for every $\alpha \succ \alpha_{\varepsilon/2}$. Now we consider the directed set $A \times (0,1]$ with the product order where in the interval $(0,1]$ we consider the order of ε decreasing to 0. Then we can consider the subset $D := \left\{(\alpha,\varepsilon) \in A \times (0,1] \,\middle|\, \alpha \succ \alpha_{\varepsilon/2}\right\}$ which is a directed set with the induced order. Then for the net of numbers $\left\{r_{(\alpha,\varepsilon)} \,\middle|\, (\alpha,\varepsilon) \in D\right\}$ there is a subnet map $\sigma : B \to D$, for some directed set $(B, \succ)$, such that $r := \lim_\beta r_{\sigma(\beta)}$ exists by the compactness of the unit interval $[0,1]$. Let us denote with $\overline{\sigma}$ the composition of the map σ with the projection from $A \times (0,1]$ onto A, which is a subnet map too, and we have:

$$\|\cdot\|\text{-}\lim_{\beta\in B} x_{\overline{\sigma}(\beta)} = rx.$$

The hypothesis $\lim_{\alpha\in A} |||x_\alpha||| = |||x|||$ together the norm continuity of the quasinorm tells us that $|||rx||| = |||x||| \neq 0$ and so $r = 1$, which means that the proof is over because the former reasoning is valid for every subnet of the given net. Then

$$\|\cdot\|\text{-}\lim_{\alpha\in A} x_\alpha = x.$$

Moreover

$$\|x+y\|_{\mathcal{B}_n} \leq \frac{1+\delta}{1-\delta}\big(\|x\|_{\mathcal{B}_n} + \|y\|_{\mathcal{B}_n}\big)$$

we see that

$$|||x+y||| \leq \frac{1+\delta}{1-\delta}(|||x\ ||| + |||\ y|||)$$

and we have constructed the required quasinorm $q_\delta(\cdot) := |||\cdot|||$.

(2) ⇒ (4) Proposition 2.46 in [21] can be applied to the identity map on X and the radial set $\left\{x \in X \,\middle|\, q_{1/2}(x) = 1\right\}$ to get the fact that $id : (X, \sigma(X, Z)) \to (X, \|\cdot\|)$ is σ-continuous.

(4) ⇒ (3) The identity map from (X, d) to $(X, \|\cdot\|)$ is σ-continuous. Indeed, any d-convergent sequence is weakly convergent, so its limit must be in the closed convex hull of the sequence and therefore the hypothesis of Corollary 2.20 of [21] are satisfied, what gives us the σ-continuity. The identity map from $(X, \sigma(X, Z))$ to $(X, \|\cdot\|)$ is σ-continuous as well by the transitive property [21, Corollary 2.41]. Now we apply Proposition 2.7 from [21] to get our conclusion.

(3) ⇔ (1) Propositions 2.7 and 2.38 in [21] show the equivalence.

(4) ⇒ (5) The use of Stone's Theorem will give the proof. Indeed, as described in Proposition 2.7 of [21], our hypothesis implies we will have $\sigma(X, Z)$-isolated families $\mathcal{N}_m$ for $m = 1, 2, \dots$ such that, for every $x \in X$ and every $\varepsilon > 0$ there is some integer p and some set $N \in \mathcal{N}_p$ such that $x \in N \subset B_d(x, \varepsilon)$. Such sequence of families provides a network for the d-topology and thus a network for any coarser topology, and in particular for the weak topology.

(5) ⇒ (4) For every $n \in \mathbb{N}$ let us define, for $(x, y) \in X \times X$, $\rho_n(x, y) = 0$ if both x and y belongs to the same set of $\mathcal{N}_n$, and $\rho_n(x, y) = 1$ otherwise. It follows that ρ_n is a semi-metric on X. We now define

$$\rho(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \rho_n(x, y),$$

for all $(x, y) \in X \times X$, which provides us a metric on X generating a topology finer than the weak topology and such that the identity map from $(X, \sigma(X, Z))$ into (X, ρ) is σ-continuous. Indeed, the family of finite intersections of sets in $\mathcal{N}$ is a basis for the ρ-topology which is σ-isolated in the topology $\sigma(X, Z)$. Proposition 2.7 in [21] concludes that the identity map from $(X, \sigma(X, Z))$ into (X, ρ) is σ-continuous. □

Now we can prove first two equivalences in our main Theorem 1.2:

Corollary 3.7. *Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. Then the following conditions are equivalent:*

1. *There is a norm equivalent $\sigma(X, Z)$-lower semicontinuous and $\sigma(X, Z)$-Kadec* F*-norm $\|\cdot\|_0$ on X, i.e. an* F*-norm $\|\cdot\|_0$ such that the $\sigma(X, Z)$ and norm topologies coincide on the unit "sphere" $\{x \in X \mid \|x\|_0 = 1\}$, and such that the topology determined by the* F*-norm $\|\cdot\|_0$ on X coincides with the topology of the norm $\|\cdot\|$*
2. *There are isolated families $\mathcal{B}_n$ for the $\sigma(X, Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ and a set $B \in \mathcal{B}_n$ with the property that $x \in B$ and that $\|\cdot\|$-$\mathrm{diam}(B) < \varepsilon$.*

Proof. Statement (2) here corresponds with (3) in Theorem 3.2. Observe that if we had taken the p_n-norm $\|\cdot\|_{\mathcal{B}_n}^{p_n}$ instead of the quasinorm $\|\cdot\|_{\mathcal{B}_n}$ in the proof (3) $\Rightarrow$ (2) above , then the function

$$\|x\|_0 := \sum_{n\in\mathbb{N}} a_n \|x\|_{\mathcal{B}_n}^{p_n}$$

where a_n are again chosen accordingly for the uniform convergence of the series on bounded sets, would be a norm equivalent $\sigma(X, Z)$-lower semicontinouos Kadec F-norm and the proof (2) $\Rightarrow$ (1) follows. Indeed, properties of F-norm are derived by the ones of the p-norms together with the uniform convergence of the series on bounded sets. For the reverse implication take $A_q := \{x \in X \,|\, \|x\|_0 \leq q\}$ for every positive rational number q. It follows that this countable family of subsets of X satisfies statement (1) of Theorem 3.2 because $\lim_\alpha \|x_\alpha\|_0 = \|x\|_0$ and $\sigma(X, Z)-\lim_\alpha x_\alpha = x$ implies that $\|\cdot\|-\lim_\alpha x_\alpha = x$ for any net $\{x_\alpha : \alpha \in A\}$. Indeed, it now follows that given $x \in X$ and $\varepsilon > 0$, there are rational numbers $q_1 < \|x\|_0 < q_2$ and some $\sigma(X, Z)$-neighborhood of the origin W such that, the set $\{y \in A_{q_2} \cap (x + W) : \|y\|_0 > q_1\}$ is a relatively $\sigma(X, Z)$-open subset of A_{q_2} containing x with norm diameter at most ε. □

Remark 3.8. Note that we have an alternative argument to Lemma 1.3 of the Introduction leading to uniform continuity. Indeed, both F-norms and quasinorms are norm uniformly continuous functions, as any F-norm is a Lipschitz function as well as the quasinorms constructed in Theorem 3.4.

4. Kadec meets Bing-Nagata-Smirnov-Stone

Acording to Corollary 3.7 we have proved the equivalence between the existence of a Kadec F-norm and the existence of a network for the norm topology which is σ-isolated for the weak topology. What we add in this Section is that it is always possible to do it with a σ-discrete basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ of the norm topology such that every family $\mathcal{B}_n$ is isolated in the weak topology, thus proving the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) of our main Theorem 1.2. This result links Stone's Theorem 4.4.1, Nagata-Smirnov metrization Theorem 4.4.7 and Bing's metrization Theorem 4.4.8 in p.349–353 of [16] with the norm topology of a normed space with Kadec F-norm.

Let us begin with the following fattening lemma. Our Theorem 2.10 gives the tool for the proof. It follows the same arguments as the convex case done in Proposition 2.4 of [25]. We include the proof for completeness.

Lemma 4.1. *Let X be a normed space with a norming subspace $Z \subseteq X^*$. Given a uniformly bounded and $\sigma(X, Z)$-p-isolated family $\mathcal{A} := \{A_i \,|\, i \in I\}$ of subsets in X there exist decompositions $A_i = \bigcup_{n\in\mathbb{N}} A_i^n$ with*

$$A_i^1 \subseteq A_i^2 \subseteq \cdots \subseteq A_i^n \subseteq A_i^{n+1} \subseteq \cdots \subseteq A_i$$

for every $i \in I$ and such that the families

$$\left\{A_i^n + \mathscr{B}_{\|\cdot\|_Z}(0, 1/4n) \,\middle|\, i \in I\right\}$$

are $\sigma(X, Z)$-p-isolated and norm discrete for every $n \in \mathbb{N}$.

Proof. Without loss of generality we may assume that Z is 1-norming. Let us denote by φ_i the Z-distance to the set $\overline{\mathrm{co}_p \bigcup \{A_j \mid j \neq i\}}^{\sigma(X^{**},X^*)}$. Theorem 2.10 gives us the scalpel to split up the sets of the family using these p-convex functions. Indeed, let us define $A_i^n := \{x \in A_i \mid \varphi_i(x) > 1/n\}$ and we have $A_i = \bigcup_{n\in\mathbb{N}} A_i^n$. Recall that φ_i is 1-Lipschitz, therefore if $x \in A_i^n + \mathscr{B}_Z(0, 1/4n)$ then we have $\varphi_i(x) > 3/4n$. On the other hand, if $x \in A_j^n + \mathscr{B}_Z(0, 1/4n)$ with $j \neq i$ then $\varphi_i(x) \leq 1/4n$ again by the Lipschitz property. This means that the family

$$\left\{A_i^n + \mathscr{B}_{\|\cdot\|_Z}(0, 1/4n)\right\}_{i\in I}$$

verifies the condition (3) of Theorem 2.10 with the functions $(\varphi_i)_{i\in I}$ and constants $\alpha = 1/4n$, $\beta = 3/4n$. Thus it is $\sigma(X, Z)$-p isolated as we wanted to prove. Moreover, the former family is discrete for the norm topology. In order to see that, fix $\delta \in (0, 1/4n)$. Then for any $z \in X$ we have that

$$\mathscr{B}_{\|\cdot\|_Z}(z, \delta) \cap \bigcup_{i\in I} \left\{A_i^n + \mathscr{B}_{\|\cdot\|_Z}(0, 1/4n)\right\}$$

has non empty intersection with at most one member of the family, otherwise we will easily arrive to a contradiction with the 1-Lipschitz property of the functions $(\varphi_i)_{i\in I}$. Indeed, if $x \in A_i^n + \mathscr{B}_Z(0, 1/4n)$ and $y \in A_j^n + \mathscr{B}_Z(0, 1/4n)$ with $i \neq j$ then $\varphi_i(x) \leq 1/4n$ and $\varphi_i(y) \geq 3/4n$. If $x, y \in \mathscr{B}_{\|\cdot\|_Z}(z, \delta)$, then we would have $\varphi_i(y) - \varphi_i(x) \geq 1/2n > 2\delta \geq \|y - x\|_Z$, which is a contradiction. □

Now we can prove the following

Proposition 4.2. *Let X be a normed space and Z a norming subspace in the dual space X^*. Let us assume the space X admits an equivalent $\sigma(X, Z)$-lower semi-continuous and $\sigma(X, Z)$-Kadec F-norm (or quasinorm). Then the norm topology admits a network*

$$\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$$

where each one of the families $\mathcal{N}_n$ is $\sigma(X, Z)$-p_n-isolated, for some $p_n \in (0, 1]$, and it consists of sets which are difference of a $\sigma(X, Z)$-closed set and a $\sigma(X, Z)$-closed p_n-convex subset of X. Moreover, there is $\delta_n \searrow 0$ such that $\mathcal{N}_n + \mathscr{B}_{\|\cdot\|_Z}(0, \delta_n)$ is norm discrete and $\sigma(X, Z)$-isolated for every $n \in \mathbb{N}$.

Proof. By Theorem 3.2 we have network $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$ of the norm topology such that every one of the families $\mathcal{M}_r := \{M_{r,i} \mid i \in I_r\}$ are $\sigma(X, Z)$-isolated. The decomposition Lemma 3.3 provide us with a decomposition of the sets in the family $\mathcal{M}_r$ producing families $\mathcal{M}_r^n$, $n = 1, 2, \ldots$ with $\mathcal{M}_r^n$ being $q_{r,n}$-isolated for all $r, n = 1, 2, \ldots$. We see that it is not a restriction to renumber sequences and assume that the given family $\mathcal{M}_r$ is already p_r-isolated for $r = 1, 2, \ldots$ and $0 < p_r \leq 1$. Let us perform another decomposition as follows:

Denote by $\varphi_{r,i}$ the Z-distance to

$$\overline{\mathrm{co}_{p_r} \{M_{r,j} \mid j \neq i, j \in I_r\}}^{\sigma(X^{**},X^*)}$$

and define

$$N_{r,i}^n := \left\{x \in \overline{M_{r,i}}^{\sigma(X,Z)} \,\middle|\, \varphi_{r,i}(x) > \frac{3}{4n}\right\}.$$

The fact that each one of the families $\mathcal{N}_r^n := \{N_{r,i}^n \,|\, i \in I_r\}$ is $\sigma(X,Z)$-p_r-isolated follows from Theorem 2.10. Indeed, the p_r-convexity of the functions $\varphi_{r,i}$ tell us that $\varphi_{r,j}(y) = 0$ for every $y \in \mathrm{co}_{p_r}(M_{r,i})$ and $j \neq i$, $j \in I_r$. The lower semicontinuity finally gives us $\varphi_{r,j}(y) = 0$ for every $y \in \overline{M_{r,i}}^{\sigma(X,Z)}$. Moreover, each one of the sets $N_{r,i}^n$ is the difference of the $\sigma(X,Z)$-closed set $\overline{M_{r,i}}^{\sigma(X,Z)}$ and the $\sigma(X;Z)$-closed and p_r-convex set $\{x \in X \,|\, \varphi_{r,i}(x) \leq 3/4n\}$.

We claim that $\bigcup_{r,n\in\mathbb{N}} \mathcal{N}_r^n$ is the network we are looking for. Indeed, given $x \in X$ there is $r \in \mathbb{N}$ and $i \in I_r$ such that $x \in M_{r,i} \subseteq x + \mathscr{B}_Z(0,\varepsilon)$. Then for $n \in \mathbb{N}$ big enough we have

$$x \in N_{r,i}^n \subset \overline{M_{r,i}}^{\sigma(X,Z)} \subseteq x + \mathscr{B}_Z[0,\varepsilon] \tag{4.1}$$

since $x + \mathscr{B}_Z[0,\varepsilon]$ is $\sigma(X,Z)$-closed set. Moreover, as the function $\varphi_{r,i}$ is 1-Lipschitz, we have here that $\varphi_{r,i}(z) > 3/4n - \mu$ whenever $z \in N_{r,i}^n + \mathscr{B}_Z(0,\mu)$; and $\varphi_{r,i}(z) \leq \mu$ whenever $z \in N_{r,j}^n + \mathscr{B}_Z(0,\mu)$ with $j \neq i, j \in I_r$. Let us choose δ_n such that $0 < 2\delta_n < 3/4n - \delta_n$, then we have that the sets in the family $\{N_{r,i}^n + \mathscr{B}_Z(0,\delta_n) \,|\, i \in I_r\}$ are disjoint norm open sets and they form a norm discrete and $\sigma(X,Z)$-p_r-isolated family by Theorem 2.10 again. □

We are now able to complete proof of equivalences in our main Theorem 1.2:

Proof of (1) ⇔ (2) ⇔ (3) *in Theorem 1.2.* From the network constructed in the previous proposition we continue with the same notations and observe that when we add open balls of small radius the network provided above will become the basis of the norm topology we are looking for in statement (3) of Theorem 1.2. Indeed, we complete inclusion (4.1) arguing that

$$x \in N_{r,i}^n + \mathscr{B}_Z(0,\delta_n) \subset \overline{M_{r,i}}^{\sigma(X,Z)} + \mathscr{B}_Z(0,\delta_n) \subseteq x + \mathscr{B}_Z(0,2\varepsilon)$$

if we take the integer n large enough. So the family

$$\bigcup_{n,r\in\mathbb{N}} \{N_{r,i}^n + \mathscr{B}_Z(0,\delta_n) \,|\, i \in I_r\}$$

is a basis of the norm topology with the required properties. The converse follows from statement (3) in Theorem 3.2 □

5. Some Applications for $\mathscr{C}(K)$ spaces

Let us recall the following definition related to descriptiveness (see [11]):

Definition 5.1. Let (X,τ) be a topological space and let d be a metric on X. It is said that X has *countable cover by sets of small local diameter* (d-SLD, for short) if for every $\varepsilon > 0$ there exists a decomposition

$$X = \bigcup_{n\in\mathbb{N}} X_n^\varepsilon$$

such that for each $n \in \mathbb{N}$ every point of X_n^ε has a relatively non-empty τ neighborhood of d-diameter less than ε.

In other words (X,τ) has d-SLD if, and only if the identity map from (X,τ) into (X,d) is σ-continuous. If (X,τ) is of the kind $\mathscr{C}_p(K)$ or a Banach space endowed with its weak topology, then X has $\|\cdot\|$-SLD if and only if the normed space X is τ-descriptive, see [24] and Chapter 3 in[21].

In the paper [9] it is shown that for every compact totally ordered space K the space $\mathscr{C}(K)$ has a pointwise-Kadec renorming. For an arbitrary product of compact linearly ordered spaces the same is true by [2]. Lexicographic products provide a wide class of examples of compact spaces K such that $C(K)$ has a pointwise Kadec equivalent norm but not equivalent **LUR** norm. It is unknown whether the existence of a pointwise-Kadec renorming for each of $\mathscr{C}(K)$ and $\mathscr{C}(L)$ implies the existence of such a renorming for $\mathscr{C}(K\times L)$. If L belongs to the class of spaces obtained by closing the class of compact metrizable spaces under inverse limits of transfinite continuous sequences of retractions, then $\mathscr{C}(K\times L)$ has a pointwise-Kadec renorming was a main result in [2]. Ribarska and Babev have proved in [29] that the function space $\mathscr{C}(K\times L)$ has an equivalent **LUR** norm provided that both $\mathscr{C}(K)$ and $\mathscr{C}(L)$ are **LUR** renormable. An analogous result holds for **LUR** norms which are pointwise lower semicontinuous. The main result in [30] is the following:

Theorem 5.2. *If K and L are Hausdorff compacta such that $\mathscr{C}_p(K)$ admits a pointwise Kadec norm and $\mathscr{C}_p(L)$ has $\|\cdot\|$-SLD, then $\mathscr{C}_p(K\times L)$ has $\|\cdot\|$-SLD.*

Actually Ribarska observed that the theorem is possible to be proved with the following hypothesis, instead of the existence of a pointwise Kadec norm: there exists a nonnegative, homogeneous, norm continuous and pointwise lower semicontinuous function F on $\mathscr{C}_p(K)$ with $\|h\|\leq F(h)\leq 2\|h\|$, whenever $h\in\mathscr{C}(K)$ and such that the norm and the pointwise topology coincide on the set $S=\{h\in\mathscr{C}(K)\,|\,F(h)=1\}$. Using Lemma 1.3 or Remark 3.8 for Theorem 3.2 we arrive to the following result:

Theorem 5.3. *If K and L are Hausdorff compacta such that both $\mathscr{C}_p(K)$ and $\mathscr{C}_p(L)$ have $\|\cdot\|$-SLD, then $\mathscr{C}_p(K\times L)$ has $\|\cdot\|$-SLD.*

We shall continue proving more permanence results for the class of compact Hausdorff spaces K such that $\mathscr{C}_p(K)$ has $\|\cdot\|$-SLD. In that context, Theorem 5.3 is the starting point. Similar results are going to be valid for the class of compact Hausdorff spaces K such that $\mathscr{C}(K)$ has an equivalent **LUR** norm. Thus by property (R) we shall denote one of the following three properties: “having $\|\cdot\|$-SLD with the pointwise topology”, “having an equivalent **LUR** norm” or “having an equivalent pointwise lower semicontinuous **LUR** norm”. The following generalizes Corollary 8 of [18]:

Theorem 5.4. *Let K be a compact space and let $K_n\subseteq K$ be compact subsets such that every space $\mathscr{C}(K_n)$ has the property (R). If there is a lower semicontinuous metric d on K such that*

$$K=\overline{\bigcup_{n\in\mathbb{N}}K_n}^{d},$$

then $\mathscr{C}(K)$ has the property (R).

Proof. We shall prove the result when the property (R) is the **LUR** renormability of the space and we shall give hints to modify the proof for the other properties. Let $\|\cdot\|_n$ an equivalent **LUR** norm on $\mathscr{C}(K_n)$ bounded by the supremum norm. For every $n \in \mathbb{N}$ define

$$O_n(f) = \sup\left\{ |f(x) - f(y)| \;\middle|\; x, y \in K,\ d(x,y) \leq \frac{1}{n} \right\}$$

and consider the equivalent norm $|||\cdot|||$ on $\mathscr{C}(K)$ defined by the formula

$$|||f|||^2 = \|f\|^2 + \sum_{n\in\mathbb{N}} 2^{-n} \|f_{|K_n}\|_n^2 + \sum_{n\in\mathbb{N}} 2^{-n} O_n(f)^2.$$

If we prove that $|||\cdot|||$ is a w-**LUR** norm, then the result will follow from [19]. To see that, suppose that $|||f_k||| = |||f|||$ and $\lim_k |||f_k + f||| = 2\,|||f|||$. A standard convexity argument [3, Fact II.2.3] gives us that (f_k) converges to f uniformly on every K_n. We claim that $(f_k(x))$ converges to $f(x)$ for every $x \in X$. Fix $\varepsilon > 0$ and take n big enough to have $O_n(f) < \varepsilon/3$ (this is possible because continuous functions on K are d-uniformly continuous by the lower semicontinuity of the metric d over K, see the proof of [27, Theorem 4]). Now take $y \in \bigcup_{m\in\mathbb{N}} K_m$ such that $d(x,y) < 1/n$. If k is big enough, then $O_n(f_k) < \varepsilon/3$ and $|f_k(y) - f(y)| < \varepsilon/3$. We have that

$$|f_k(x) - f(x)| \leq |f_k(x) - f_k(y)| + |f_k(y) - f(y)| + |f(y) - f(x)| < \varepsilon$$

and this end the proof of the claim. Thus we have that (f_k) converges to f weakly by Lebesgue's theorem and $|||\cdot|||$ is w-**LUR** .

For t_p-lower semicontinuous **LUR** renormability, the proof is the same if we notice that the norm $|||\cdot|||$ built above is t_p-lower semicontinuous. For $\|\cdot\|$-SLD consider the formula

$$\Phi(f) = \sum_{n\in\mathbb{N}} 2^{-n} \varphi_n(f_{|K_n}) + \sum_{n\in\mathbb{N}} 2^{-n} O_n(f)$$

where φ_n are Kadec functions on $\mathscr{C}(K_n)$. The convexity argument above con be replaced by an argument of lower semicontinuity in order to obtain that Φ is a Kadec function on $\mathscr{C}(K)$. □

Corollary 5.5. *Let K be a norm fragmented w^*-compact subset of X^* and $H = \overline{\text{co}(K)}^{w^*}$. If $\mathscr{C}(K)$ has the property (R), then $\mathscr{C}(H)$ also has the property (R).*

Proof. First notice that if K is a norm fragmented w^*-compact subset of X^* then

$$\overline{\text{co}(K)}^{w^*} = \overline{\text{co}(K)}^{\|\cdot\|}$$

by a result of Namioka [23]. Also notice that if L is a compact Hausdorff space such that $\mathscr{C}(L)$ has the property (R), then $\mathscr{C}(L')$ has the property (R) for any compact L' which is continuous image of L. Let K_n be the set of convex combinations of at most n points of K. It is easy to see that K_n is compact and continuous image of $L = \Delta \times K^n$, where

$$\Delta = \left\{ (\lambda_i)_{i=1}^n \;\middle|\; \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1 \right\}.$$

By Ribarska's result $\mathscr{C}(L)$ has the property (R), and so $\mathscr{C}(K_n)$ does also. Now we have that $H = \overline{\bigcup_{n\in\mathbb{N}} K_n}$ and the result follows from proposition 5.4. □

Under the hypothesis of the previous corollary the **LUR** norm can be made pointwise lower semicontinuous always. Indeed, for a Radon-Nikodým compact K the space $\mathscr{C}(K)$ has an equivalent pointwise lower semicontinuous norm such that pointwise and weak topologies coincide on the unit sphere by [27, Theorem 4]. Then it is possible to apply [28, Theorem A].

6. Open problems

In relation with descriptive properties let us remind that for a descriptive Banach space the family of weak Borel sets coincides with the norm Borel sets, [7, 24]. Based on a sophisticated construction of Todorcevic [31], Marciszewki and Pol have proved that it is consistent the existence of a compact scattered space K such that in the function space $\mathscr{C}(K)$ each norm open set is an $\mathcal{F}_\sigma$-set with respect to the weak topology but the identity map

$$Id : (\mathscr{C}(K), w) \to (\mathscr{C}(K), \|\cdot\|_\infty)$$

is not σ-continuous, see [17]. Descriptive Banach spaces are weakly Čech analytic and coincide with the ones that can be represented with a Souslin scheme of Borel subsets in their $\sigma(X^{**}, X^*)$ biduals. The fact that every weakly Čech analytic Banach space is σ-fragmented is the main result in [13]. The reverse implications are open questions considered in [11, 12], and we recall here the following:

Problem 6.1. Is there any gap between the classes of descriptive Banach spaces and that of σ-fragmented Banach spaces?

After the seminal paper of R. Hansell [7] we know that a covering property on the weak topology of a Banach space, known as hereditarely weakly θ-refinability, is a necessary and sufficient condition for the coincidence of both classes. Indeed, all known examples of normed spaces which are not weakly θ-refinable are not σ-fragmentable by the norm, see [4, 5]. For spaces of continuous functions on trees Haydon has proved that there is no gap between σ-fragmented and the pointwise Kadec renormability property of the space, see [8]. We can consider a particular case of the former question as follows:

Problem 6.2. Let X be a weakly Čech analytic Banach space where every norm open set is a countable union of sets which are differences of closed sets for the weak topology. Does it follow that the identity map $Id : (X, w) \to (X, \|\cdot\|)$ is σ-continuous?

In the particular case of a Banach space X with the Radon-Nikodým property it is still an open problem to decide if X has even an equivalent strictly convex norm. In that case the **LUR** renormability reduces to the question of Kadec renormability by our results in [20]. So we summarize here:

Problem 6.3. If the Banach space X has the Radon-Nikodým property, does it follow that X has an equivalent Kadec norm? Does it have an equivalent strictly convex norm?

Let us remark here that a result of D. Yost and A. Plicko [26] shows that the Radon-Nikodým property does not imply the separable complementation property. Thus it is not possible any approach to the former question based on the projectional resolution of the identity which works for the dual case, as in [6].

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References

[1] S. Banach, *Theory of Linear Opeations*, Dover Publications, Inc., Mineola, New York, 2009.

[2] M.R. Burke, W. Kubiś and S. Todorčević, *Kadec norms on spaces of continuous functions*, Serdica Math. J. **32** (2006), no. 2-3, 227–258.

[3] R. Deville, G. Godefroy and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 64, Longman, Harlow, 1993.

[4] A. Dow, H. Junnila and J. Pelant, *Weak covering properties of weak topologies*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 349–368.

[5] A. Dow, H. Junnila and J. Pelant, *Coverings, networks and weak topologies*, Mathematika **53** (2006), no. 2, 287–320 (2007).

[6] M. Fabián and G. Godefroy, *The dual of every Asplund space admits a projectional resolution of the identity*, Studia Math. **91** (1988), no. 2, 141–151.

[7] R. W. Hansell, *Descriptive sets and the topology of nonseparable Banach spaces*, Serdica Math. J. **27** (2001), no. 1, 1–66.

[8] R. G. Haydon, *Trees in renorming theory*, Proc. London Math. Soc. (3) **78** (1999), no. 3, 541–584.

[9] R. G. Haydon, J. E. Jayne, I. Namioka and C. A. Rogers, *Continuous functions on totally ordered spaces that are compact in their order topologies*, J. Funct. Anal. **178** (2000), no. 1, 23–63.

[10] H. Jarchow, *Locally Convex Spaces*, Mathematische Leitfden, B. G. Teubner, Stuttgart, 1981.

[11] J. E. Jayne, I. Namioka and C. A. Rogers, *σ-fragmentable Banach spaces*, Mathematika **39** (1992), no. 1, 161–188.

[12] J. E. Jayne, I. Namioka and C. A. Rogers, *σ-fragmentable Banach spaces*, Mathematika **39** (1992), no. 2, 197–215.

[13] J. E. Jayne, I. Namioka and C. A. Rogers, *Topological properties of Banach spaces*, Proc. London Math. Soc. (3) **66** (1993), no. 3, 651–672.

[14] J. L. Kelley, *General Topology*, Graduate Text in Mathematics, Springer-Verlag New York Inc., New York, 1955.

[15] G. Köthe, *Topological vector spaces. I*, Springer-Verlag New York Inc., New York, 1969.

[16] R. Engelking, *General Topology*, Polish Scientific Publishers, Warszawa, 1977.

[17] W. Marciszewski and R.Pol, *On Banach spaces whose norm-open sets are $\mathcal{F}_\sigma$-sets in the weak topology*, J. Math. Anal. Appl. **350** (2009), no. 2, 708–722.

[18] A. Moltó, J. Orihuela and S. Troyanski, *Locally uniformly rotund renorming and fragmentability*, Proc. London Math. Soc. (3) **75** (1997), no. 3, 619–640.

[19] A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *On weakly locally uniformly rotund Banach spaces*, J. Funct. Anal. **163** (1999), no. 2, 252–271.

[20] A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *Kadec and Krein-Milman properties*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), no. 6, 459–464.

[21] A. Moltó, J. Orihuela, S. Troyanski and M. Valdivia, *A nonlinear transfer technique for renorming*, Lecture Notes in Mathematics (1951), Springer-Verlag, Berlin, 2009.

[22] J. Nagata, *Modern general topology*, North-Holland Publishing Co., Amsterdam, 1974.

[23] I. Namioka, *Radon-Nikodým compact spaces and fragmentability*, Mathematika **34** (1987), no. 2, 258–281.

[24] L. Oncina, *The JNR property and the Borel structure of a Banach space*, Serdica Math. J. **26** (2000), no. 1, 13–32.

[25] J. Orihuela and S. Troyanski, *Deville's master lemma and Stone's discreteness in renorming theory*, J. Convex Anal. **16** (2009), no. 3-4, 959–972.

[26] A. M. Plichko and D. Yost, *The Radon-Nikodým property does not imply the separable complementation property*, J. Funct. Anal. **180** (2001), no. 2, 481–487.

[27] M. Raja, *Kadec norms and Borel sets in a Banach space*, Studia Math. **136** (1999), no. 1, 1–16.

[28] M. Raja, *Locally uniformly rotund norms*, Mathematika **46** (1999), no. 2, 343–358.

[29] N. K. Ribarska and V. D. Babev, *A stability property for locally uniformly rotund renorming*, J. Math. Anal. Appl. **350** (2009), no. 2, 811–828.

[30] N. K. Ribarska, *On having a countable cover by sets of small local diameter*, Studia Math. **140** (2000), no. 2, 99–116.

[31] S. Todorčević, *Representing trees as relatively compact subsets of the first Baire class*, Bull. Cl. Sci. Math. Nat. Sci. Math. **30** (2005), 29–45.

[32] S. L. Troyanski, *Locally uniformly convex norms*, C. R. Acad. Bulgare Sci. **32** (1979), no. 9, 1167–1169.

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