Lipschitz subspaces of $C(K)$

Natalia Jonard-Pérez[a], Matías Raja[b,]

[a] *Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México. Circuito Exterior S/N, Cd. Universitaria, Colonia Copilco el Bajo, Delegación Coyoacán, 04510, México D.F., MÉXICO.*
[b] *Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, SPAIN.*

Abstract

Let K be an uncountable metric compact space. It is well known that $C(K)$ is isometrically universal for the separable Banach spaces, but the continuous functions that compose the isometric image of finite dimensional spaces are typically far from being Lipschitz. We prove that the possibility of embedding Euclidean spaces $\mathbb{R}^n \hookrightarrow C(K)$ in such a way that the image in $C(K)$ is made of Lipschitz functions is tightly related to the dimension (topological or Hausdorff) of K.

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1. Introduction

Throughout the paper all the Banach spaces considered are real. We shall denote by K a compact Hausdorff space, and $C(K)$ will be the Banach space of real continuous functions defined on K endowed with the supremum norm. The real unit interval is denoted by $\mathbb{I}$. We shall consider $\mathbb{I}$ and its finite powers with the Euclidean distance. As usual, if X is a Banach space we shall denote by B_X its closed unit ball, and by S_X its unit sphere. For any unexplained concepts or notations about Banach spaces we address the

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Email addresses: `nat@ciencias.unam.mx` (Natalia Jonard-Pérez), `matias@um.es` (Matías Raja)

reader to [6] or [17].

A classical result of Banach and Mazur [6, Theorem 5.8] says that $C(\mathbb{I})$ is isometrically universal for the class of separable Banach spaces. In particular, the Euclidean spaces $(\mathbb{R}^n, \|\cdot\|_2)$ can be found isometrically as subsets of functions defined on $\mathbb{I}$. For $n = 2$ an isometric embedding $J : \mathbb{R}^2 \to C(\mathbb{I})$ can be written explicitly as $J(x_1, x_2)(t) = x_1 \cos(\pi t) + x_2 \sin(\pi t)$, using C^∞ functions. As we see later, an isometric embedding of $\mathbb{R}^3$ cannot be written explicitly using such simple functions. In fact, Peano curves are needed as was first noticed in 1957 by Donoghue [4]. However, $\mathbb{R}^3$ is isometrically embedded into $C(\mathbb{I}^2)$ by means of the formula

$$J(x_1, x_2, x_3)(t, s) = x_1 \cos(\pi t)\cos(\pi s) + x_2 \sin(\pi t)\cos(\pi s) + x_3 \sin(\pi s).$$

We will see that the possibility of finding an "easy formula" for an isometric embedding of $\mathbb{R}^n$ into $C(K)$ is related to the *dimension* of K.

If K_1 and K_2 are uncountable metrizable compacta, then $C(K_1)$ and $C(K_2)$ are isomorphic by Milutin's theorem [17, III.D.19]. These Banach spaces cannot be isometric unless K_1 and K_2 are homeomorphic. On the other hand, $C(K_1)$ and $C(K_2)$ are universal spaces for the class of separable spaces in the isometric category. In particular $C(K_1)$ contains an isometric copy of $C(K_2)$ and vice versa. In particular, that means that it is not possible to distinguish between K_1 and K_2 by isometric embeddings of test spaces.

Our idea is to relate properties of a compact K to the existence of isometric embeddings $J : X \to C(K)$ of finite dimensional linear spaces X such that the set $J(X)$ is composed of "nice" functions. Here nice will mean Lipschitz at least, and the requirement of finite dimension is necessary. Indeed, it is easy to see that if the isometric embedding $J(X)$ is composed of Lipschitz functions, then X must be of finite dimension (Proposition 2.1). The next result shows the relation between K and the existence of nice embeddings of the Euclidean spaces.

Theorem 1.1. *Let (K, d) be an uncountable metric compact space and $n \in \mathbb{N}$. The following are equivalent:*

(i) There is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$.

(ii) *$C(K)$ contains an isometric copy of any $(n+1)$-dimensional Banach space made of Lipschitz functions.*

(iii) *$C(K)$ contains an isometric copy of the Euclidean space $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ made of Lipschitz functions.*

Moreover, if K is a Lipschitz manifold, then statements (i), (ii) and (iii) are also equivalent to

(iv) *The dimension of K is at least n.*

We follow [12] for the definition of Lipschitz manifold (with boundary). A separable metric space is called a *Lipschitz manifold* (of dimension n) if every point has a closed neighborhood which is Lipschitz homeomorphic to $\mathbb{I}^n$, that is, there is a Lipschitz bijective mapping whose inverse is Lipschitz too. We may apply our result as well to topological manifolds. Indeed, Sullivan [15] proved that n-dimensional topological manifolds have a Lipschitz structure for $n \neq 4$. Nevertheless, a fixed metric on K is needed since topologically equivalent metrics on K are in general not Lipschitz equivalent. If K is neither a topological or a Lipschitz manifold, we may still obtain information about K from the previous result using the Hausdorff dimension. Indeed, statement (i) clearly implies that the Hausdorff dimension of K is greater or equal than n (see [7, Corollary 2.4]). On the other hand, a recent result of Keleti, Máthé and Zindulka [9] says that if the Hausdorff dimension of K is strictly greater than n, then statement (i) holds. Unfortunately, the existence of a Lipschitz mapping onto a cube does not characterize the Hausdorff dimension as showed by the example constructed by Vitušhkin, Ivanov and Melnikov [16]. If K is ultrametric, then statement (i) implies that the Hausdorff dimension is at least n by another result of [9]. In the following, $\dim_H(K)$ will denote the Hausdorff dimension of K.

The smooth embedding of a smooth compact manifold into some $\mathbb{R}^N$ (see e.g. [11, Theorem 3.21]) induces on it a metric and a structure of a Lipschitz manifold. This structure is unique because two metrics obtained in the same way are Lipschitz equivalent (indeed, apply the compactness to the fact that both metric spaces are locally Lipschitz homeomorphic) and therefore the expression "Lipschitz function" when referring to a compact smooth manifold makes sense with no need of an explicit metric. For smooth manifolds, the regularity of the functions composing the isometric copy of the Euclidean space is as good as possible.

Theorem 1.2. *Let K be a compact C^r-manifold of dimension $n-1$ for $n \geq 2$ and $r = 1, \ldots, \infty$. Then*

(a) $C(K)$ contains an isometric copy of $(\mathbb{R}^n, \|\cdot\|_2)$ made of C^r-smooth functions;

(b) $C(K)$ contains no isometric copy of $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ made of C^1-smooth functions or Lipschitz functions.

The word *dimension* in Theorems 1.1-(iv) and 1.2 refers to the dimension of K as manifold (topological dimension). However, it is well-known that topological dimension and Hausdorff dimension coincide if the space is a Lipschitz or smooth manifold (see e.g. [7, p. 32]).

We have chosen the Euclidean space as test space because of its easiness, but any finite dimensional space with strictly convex dual will work as a test space. On the other hand, polyhedral spaces can always be isometrically embedded using nice functions. Recall that a finite dimensional Banach space is *polyhedral* if its unit ball is a convex polytope.

Theorem 1.3. *If K is an infinite metric compact space, then $C(K)$ contains isometric copies made of Lipschitz functions of any finite dimensional polyhedral space.*

The proofs of these results depend on some easy facts about Lipschitz mappings, Lipschitz manifolds and isometric embeddings into $C(K)$ spaces that we will develop in the next section. We finish the paper with some remarks about extending the results for Hölder maps and the typical n-dimensional subspaces of $C(K)$.

2. Auxiliary results

We denote by $L(K,d)$ the Lipschitz functions (with respect to d) of $C(K)$. The Lipschitz constant for $f \in L(K,d)$ is the number

$$L(f) = \sup \left\{ \frac{|f(t_1) - f(t_2)|}{d(t_1,t_2)} : t_1, t_2 \in K, t_1 \neq t_2 \right\}.$$

Proposition 2.1. *Let $X \subset C(K)$ be a non-trivial linear subspace. Then either*

(a) *$X \cap L(K,d)$ is of first category in X;*

(b) *or $X \subset L(K,d)$, X is finite dimensional and there exists $\lambda > 0$ such that $L(f) \leq \lambda \|f\|$ for every $f \in X$.*

Proof. Observe that $X \cap L(K,d) = \bigcup_{n=1}^{\infty} \{f \in X : L(f) \leq n\}$ is a decomposition of $X \cap L(K,d)$ into countably many closed balanced convex sets. If $X \cap L(K,d)$ is not of first category in X, then there is $f_0 \in X$ and $\delta > 0$ such that $f_0 + \delta B_X \subset \{f \in X : L(f) \leq n\}$ for some $n \in \mathbb{N}$. The symmetry and convexity of the last set easily imply that $\delta B_X \subset \{f \in X : L(f) \leq n\}$. By homogeneity, we have $L(f) \leq \lambda \|f\|$ with $\lambda = \delta^{-1} n$ for every $f \in X$. In particular $X \subset L(K,d)$. Note that B_X is a complete, bounded and equicontinuous set of functions, and thus it is compact by Ascoli's theorem [10]. Therefore X must be of finite dimension. $\square$

Recall that given a linear operator $T : X \to Y$ between Banach spaces, the *adjoint operator* $T^* : Y^* \to X^*$ is the linear map defined by the rule

$$T^*(y^*)(x) = y^*(T(x)).$$

If T is bounded then T^* is also bounded and $\|T\| = \|T^*\|$. On the other hand, observe that K is naturally embedded in $C(K)^*$ if we define $t : C(K) \to \mathbb{R}$ as $t(f) := f(t)$ for every $t \in K$. In this case, we always have that $K \subset B_{C(K)^*}$.

Proposition 2.2. *Let $J : X \to C(K)$ be an isomorphic embedding. Then $J(X) \subset L(K,d)$ if and only if $J^*|_K$ is Lipschitz from d to the norm of X^*, where J^* denotes the adjoint map from $C(K)^*$ into X^*.*

Proof. If J^* is Lipschitz, then any function $J(x)$ is Lipschitz as well. Indeed, note that $L(J(x)) \leq L(J^*)\|x\|$ since $J(x)(t) = J^*(t)(x)$. Reciprocally, assume that $J(X) \subset L(K,d)$. By Proposition 2.1 there is $\lambda > 0$ such that $L(f) \leq \lambda$ for every $f \in J(X)$. Now, if $x \in B_X$ and $t_1, t_2 \in K$ then

$$|J^*(t_1)(x) - J^*(t_2)(x)| = |J(x)(t_1) - J(x)(t_2)| \leq \lambda\, d(t_1, t_2).$$

Taking supremum on $x \in B_X$ we get $\|J^*(t_1) - J^*(t_2)\| \leq \lambda\, d(t_1, t_2)$. $\square$

Remark 2.3. *A function $f : K \to \mathbb{R}$ is said to be α-Hölder for $\alpha \in (0,1]$ if there is a constant $\lambda > 0$ such that $|f(x) - f(y)| \leq \lambda\, d(x,y)^{\alpha}$ for any $x, y \in K$. It is easy to see that the Lemma 2.1 and Proposition 2.2 can be generalized to the setting of α-Hölder functions.*

The set of extreme points of a convex set C is denoted by $\mathrm{Ext}(C)$.

Proposition 2.4. *Let X be a Banach space and let $J : X \to C(K)$ be a linear operator with $\|J\| \leq 1$. Then J is an isometric embedding if and only if*

$$\mathrm{Ext}(B_{X^*}) \subset J^*(K) \cup (-J^*(K)).$$

Proof. Note that, in general, $J : X \to Y$ is an isometric embedding if and only if $J^*(B_{Y^*}) = B_{X^*}$ (the less easy part relies on the Hahn–Banach theorem). Hence we have just to check that the statement above is equivalent to $B_{X^*} = J^*(B_{C(K)^*})$. Clearly, $\|J^*\| = \|J\| \leq 1$ implies that $J^*(B_{C(K)^*}) \subset B_{X^*}$. On the other hand, suppose that $\mathrm{Ext}(B_{X^*}) \subset J^*(K) \cup (-J^*(K))$. Observe that $J^*(K) \cup (-J^*(K)) \subset J^*(B_{C(K)^*})$, noting that the last set is convex and weak* compact. Now the inclusion $B_{X^*} \subset J^*(B_{C(K)^*})$ follows directly from the Krein–Milman Theorem [6, Theorem 3.65].
For the converse implication, observe that $B_{C(K)^*} = \overline{\mathrm{conv}}^{w^*}(K \cup (-K))$ in general since $\mathrm{Ext}(B_{C(K)^*}) = K \cup (-K)$ [6, Lemma 3.116]. Therefore, if we suppose that $B_{X^*} = J^*(B_{C(K)^*})$, then

$$B_{X^*} = J^*(\overline{\mathrm{conv}}^{w^*}(K \cup (-K))) = \overline{\mathrm{conv}}^{w^*}(J^*(K \cup (-K)))$$

where the last equality comes from the linearity and weak*-continuity of J^*. Finally, Milman's Theorem [6, Theorem 3.66] implies that $\mathrm{Ext}(B_{X^*}) \subset J^*(K \cup (-K))$, as desired. $\square$

Corollary 2.5. *Let X be a Banach space. There exists an isometric embedding of X into $C(K)$ if and only if there exists a continuous mapping $\Psi : K \to B_{X^*}$ such that*

$$\mathrm{Ext}(B_{X^*}) \subset \Psi(K) \cup (-\Psi(K)).$$

Proof. If such an isometric embedding $J : X \to C(K)$ exists, then $\Psi = J^*|_K$. For the other implication, define $J(x)(t) = \Psi(t)(x)$. Evidently, J is a linear operator with $\|J\| \leq 1$ that satisfies $J^*|_K = \Psi$. So, by Proposition 2.4, it is an isometric embedding. $\square$

The first part of the following result is due to Donoghue [4] who used it for the construction of Peano-type filling curves.

Corollary 2.6. *Let X be a Banach space such that X^* is strictly convex and let $J : X \to C(K)$ be an isometric embedding. Then*

$$S_{X^*} \subset J^*(K) \cup (-J^*(K)).$$

Moreover, there exists $t_1, t_2 \in K$ such that $J(x)(t_1) = -J(x)(t_2)$ for every $x \in X$.

Recall that a Banach space X is *strictly convex* if given $x, y \in S_X$, with $x \neq y$ then $\|\frac{x+y}{2}\| < 1$.

Proof. In this case $\mathrm{Ext}(B_{X^*}) = S_{X^*}$. Therefore $S_{X^*} \subset J^*(K) \cup (-J^*(K))$. But the connectedness of S_{X^*} implies that $J^*(K) \cap (-J^*(K)) \neq \emptyset$. Take $x^* \in J^*(K) \cap (-J^*(K))$ and $t_1, t_2 \in K$ such that $J^*(t_1) = x^*$ and $J^*(t_2) = -x^*$. Thus, for every $x \in X$ we have that $J(x)(t_1) = J^*(t_1)(x) = -J^*(t_2)(x) = -J(x)(t_2)$, as desired. $\square$

A metric space M is called an absolute *Lipschitz retract* if for any isometric embedding $\phi : M \to \tilde{M}$ into another metric space $\tilde{M}$, there exists a Lipschitz map $\psi : \tilde{M} \to M$ such that $\psi \circ \phi$ is the identity map on M. The notion of Lipschitz retract will be useful in what follows.

Lemma 2.7. *Let K be a k-dimensional Lipschitz manifold. Then $k \geq n$ if and only if there exists a Lipschitz onto map $\psi : K \to \mathbb{I}^n$.*

Proof. Since K is a k-dimensional Lipschitz manifold, we can find a finite cover $\{O_1, O_2, \ldots, O_m\}$ of K each member of which is bi-Lipschitz homeomorphic to $\mathbb{I}^k$. In particular, there exists a Lipschitz onto map $\phi : O_1 \to \mathbb{I}^k$. Since $\mathbb{I}^k$ is an absolute Lipschitz retract, by [2, Proposition 1.2] we can find a Lipschitz (onto) extension $\Phi : K \to \mathbb{I}^k$ of ϕ. Now, if $k \geq n$, the projection in the first n-coordinates $p : \mathbb{I}^k \to \mathbb{I}^n$ is a Lipschitz map with Lipschitz constant 1. Therefore, the composition $p \circ \Phi : K \to \mathbb{I}^n$ is a Lipschitz onto map, which proves the first implication.

Now assume that $\psi : K \to \mathbb{I}^n$ is a Lipschitz onto map. Since $\{\psi(O_i)\}_{i=1}^m$ is a finite compact cover of $\mathbb{I}^n$, by Baire's category theorem there must exist $i \in \{1, \ldots, m\}$ such that $\psi(O_i)$ has non-empty interior in $\mathbb{I}^n$. Since ψ cannot increase the Hausdorff dimension of O_i, we thus have that

$$k = \dim_H(O_i) \geq n.$$

$\square$

We finish this section with the following two lemmas that may seem intuitively obvious. For the sake of completeness we include their proofs.

Lemma 2.8. *Let X be a vector space of dimension $n+1$.*

1. *Two spheres for different norms on X are Lipschitz homeomorphic.*

2. *The unit sphere, for any norm on X, is an n-dimensional Lipschitz manifold.*

Proof. Clearly it is enough to consider unit spheres centered at the origin. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on X, and S_1 and S_2 be their respective unit spheres. The map $\phi : S_1 \to S_2$ defined by $\phi(x) = \|x\|_2^{-1}x$ is Lipschitz. Indeed, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, that is $\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$ for some constants $\alpha, \beta > 0$ and every $x \in X$. Suppose now that $x, y \in S_1$. Then

$$\begin{aligned}\|\phi(x) - \phi(y)\|_2 &= \left\|\|x\|_2^{-1}x - \|x\|_2^{-1}y + \|x\|_2^{-1}y - \|y\|_2^{-1}y\right\|_2 \\ &\leq \|x\|_2^{-1}\|x-y\|_2 + \left|\|x\|_2^{-1} - \|y\|_2^{-1}\right| \|y\|_2 \\ &= \|x\|_2^{-1}\|x-y\|_2 + \left|\frac{\|y\|_2 - \|x\|_2}{\|x\|_2\|y\|_2}\right| \|y\|_2 \\ &= \|x\|_2^{-1}\|x-y\|_2 + |\|x\|_2 - \|y\|_2| \, \|x\|_2^{-1} \\ &\leq 2\|x\|_2^{-1}\|x-y\|_2 \leq 2\alpha^{-1}\beta\|x-y\|_1.\end{aligned}$$

The symmetry of the argument ensures that ϕ^{-1} is Lipschitz as well, and so ϕ is a Lipschitz homeomorphism.

Given a norm $\|\cdot\|$ on X there is a Lipschitz homeomorphism $\phi : \mathbb{S}^n \to S_X$ by the previous statement, where $\mathbb{S}^n$ is the n-dimensional Euclidean sphere. That homeomorphism endows S_X with a structure of a Lipschitz manifold. □

Actually, the homeomorphism described in the proof carries the structure of a C^∞-manifold from the sphere $\mathbb{S}^n$ into S_X, although S_X is not a smooth submanifold of X in general.

Lemma 2.9. *Let X be a normed space of dimension $n+1$. For any non-empty open subset $O \subset S_X$ there is an injective Lipschitz mapping $\psi : \mathbb{I}^n \to S_X$ such that $S_X = \psi(\mathbb{I}^n) \cup O$.*

Proof. The mapping ψ can be easily obtained by combining a Lipschitz homeomorphism from S_X to $\mathbb{S}^n$ and the inverse of a suitable stereographic projection of $\mathbb{S}^n$ into $\mathbb{R}^n$, see for instance [11, Example 1.39]. □

3. Proofs of the main results and final remarks

Proof of Theorem 1.1. (i)⇒(ii) If X is $(n+1)$-dimensional, then the dual sphere S_{X^*} is a Lipschitz manifold of dimension n. Using Lemma 2.9, we can find a Lipschitz mapping $\psi : \mathbb{I}^n \to S_{X^*}$ such that $S_{X^*} \subset \psi(\mathbb{I}^n) \cup (-\psi(\mathbb{I}^n))$. For $x \in X$, consider the function $J(x) \in C(K)$ given by $J(x)(t) = \psi(\phi(t))(x)$. Clearly, $J^*(K) = \psi(\mathbb{I}^n)$, and so J is an isometric embedding by Corollary 2.5.

(ii)⇒(iii) is trivial. (iii)⇒(i) Assume that $X = (\mathbb{R}^{n+1}, \|\cdot\|)$ embeds into $C(K)$ with an isometric embedding J. If $J(X) \subset L(K,d)$, then $J^*|_K : K \to B_{X^*}$ is Lipschitz by Proposition 2.2. Now, Corollary 2.6 implies that

$$S_{X^*} \subset J^*(K) \cup (-J^*(K)).$$

Using Baire's category theorem, we can find a closed neighborhood $O \subset S_{X^*}$ such that $O \subset J^*(K)$. Without loss of generality, we may assume that there is a Lipschitz homeomorphism to $\psi : O \to \mathbb{I}^n$. By [2, Proposition 1.2], there is a Lipschitz map $\Psi : B_{X^*} \to \mathbb{I}^n$ extending ψ. Therefore $\Psi \circ J^*$ is a Lipschitz mapping from K onto $\mathbb{I}^n$. This proves the desired implication.

Finally if K is a Lipschitz manifold, implication $(i) \Leftrightarrow (iv)$ follows immediately from Lemma 2.7. □

Proof of Theorem 1.2. For the sake of simplicity, let X be $(\mathbb{R}^n, \|\cdot\|_2)$. If K is an $(n-1)$-dimensional C^r-manifold, then there is a compact set $H \subset K$ which is C^r-homeomorphic to $\mathbb{I}^{n-1}$ and such that there exists a C^r-smooth retraction $\psi : K \to H$. With the help of the stereographic projection we can find a C^r-smooth mapping $\phi : H \to S_{X^*}$ such that $S_{X^*} \subset \phi(H) \cup (-\phi(H))$. Define $J : X \to C(K)$ by $J(x)(t) = \phi(\psi(t))(x)$. Clearly $J(x)$ is a C^r-smooth function and J is an isometric embedding by Corollary 2.5, and this completes the proof of (a).
If $C(K)$ contains an isometric copy of $(\mathbb{R}^{n+1}, \|\cdot\|_2)$ made of Lipschitz functions (in particular, if they are C^1-smooth), then there is a Lipschitz mapping of K onto $\mathbb{I}^n$, and so $\dim_H(K) \geq n$. □

Proof of Theorem 1.3. If X is polyhedral and finite dimensional, its dual X^* is also polyhedral and so $\mathrm{Ext}(B_{X^*}) = \{x_1^*, \ldots, x_N^*\}$ is a finite set. Take different points $\{t_n\}_{n=1}^N \subset K$ and disjointly supported Lipschitz functions $\psi_n : K \to [0,1]$ such that $\psi_n(t_m) = 0$ if $n \neq m$ and $\psi_n(t_n) = 1$. The map defined by $\Psi(t) = \sum_{n=1}^N \psi_n(t) x_n^*$ is Lipschitz and $\|\Psi(t)\| \leq 1$ for every $t \in K$, that is $\Psi(K) \subset B_{X^*}$. Since $\mathrm{Ext}(B_{X^*}) \subset \Psi(K)$, Corollary 2.5 implies that the linear operator $J : X \to C(K)$ defined by $J(x)(t) = \Psi(t)(x)$ is an isometric embedding. $\square$

We will finish with some remarks:

(1) There exist Peano's filling curves in the Hölder class, see [14, Theorem 3.1] for instance, where it is shown that there is an $1/2$-Hölder surjection from $\mathbb{I}$ onto $\mathbb{I}^2$. Note that such a map with small modifications and the help of Corollary 2.5 provides an embedding of $(\mathbb{R}^3, \|\cdot\|_2)$ into $C(\mathbb{I})$ made of $1/2$-Hölder functions. Indeed, if $\phi : \mathbb{I} \to \mathbb{I}^2$ is onto and $1/2$-Hölder, write $\phi = (\tau, \sigma)$ and note that $J : \mathbb{R}^3 \hookrightarrow C(\mathbb{I})$ defined by

$$J(x_1, x_2, x_3) = x_1 \cos(\pi\tau)\cos(\pi\sigma) + x_2 \sin(\pi\tau)\cos(\pi\sigma) + x_3 \sin(\pi\sigma)$$

is an isometric embedding made of $1/2$-Hölder functions.

(2) Hausdorff dimension has a good behavior under Hölder maps, [7, Proposition 2.3]. Therefore it would be possible to obtain information about the Hausdorff dimension of a metric compact K from the dimension of the Euclidean subspaces of $C(K)$ made of Hölder functions, as in Theorem 1.1. This observation combined with Remark 2.3 and the ideas from the previous remark can be used to generalize Theorem 1.1 in the Hölder case.

(3) In the remaining remarks we will use the techniques about isometric embeddings into $C(K)$ spaces to understand how a "typical" n-dimensional subspace of $C(K)$ looks like. Let us start by recalling that the strictly convex norms are generic in the following sense: the set of strictly convex norms on a separable Banach space is a dense $\mathcal{G}_\delta$-set in the metric space of equivalent norms endowed with the Banach-Mazur distance. In particular, the "generic norm" on $\mathbb{R}^n$ is strictly convex and smooth. Baire's category theorem allows us to blend generic properties of norms, see the Asplund averaging technique [3, p. 52].

(4) However, a "typical" n-dimensional subspace of $C(\mathbb{I})$ is far from being smooth. A subspace of $C(K)$ of dimension equal or less than n is determined by n "random" functions $\{f_1, \ldots, f_n\} \subset C(K)$. Putting $F = (f_1, \ldots, f_n)$ this is an element $F \in C(K, \mathbb{R}^n)$, and name $X_F = \mathrm{span}\{f_1, \ldots, f_n\}$. Let J_F

be the mapping from $\mathbb{R}^n$ into $C(K)$ given by

$$J_F(x_1, \ldots, x_n) = x_1 f_1 + \cdots + x_n f_n$$

and endow $\mathbb{R}^n$ with the seminorm $p_F(x_1, \ldots, x_n) = \|x_1 f_1 + \cdots + x_n f_n\|_\infty$. If X_F has dimension n, then J_F is the isometric embedding of $(\mathbb{R}^n, p_F)$ into $C(K)$. Clearly, we have $J_F^*|_K = F$. Suppose that X_F has strictly convex dual, then the radial boundary of $F(K) \cup (-F(K))$ should be a $(n-1)$-dimensional sphere, by Corollary 2.6. This seems to be "highly unlikely". Indeed, in [1] the authors proved that from a generic point of view the Hausdorff dimension of $F(K)$ for $F \in C(K, \mathbb{R}^n)$ is the minimum of n and the topological dimension of K. In particular, if $K = \mathbb{I}$, the set $F(K)$ has generic Hausdorff dimension 1. That implies for $n > 2$ that X_F^* is generically far from being strictly convex.

(5) Finally, we may compare two n-dimensional subspaces of $C(K)$ by measuring the Hausdorff distance $d_{\mathcal{H}}$ between their unit balls (or spheres). This way of measuring the distance between subspaces of a Banach space was introduced long ago in [8] and sometimes it is called the *Gokhberg-Markus gap*. Following the notation above, the next two observations show that the relation between F and X_F is continuous back and forth (we omit the elementary proofs)

(a) Given $F \in C(K, \mathbb{R}^n)$ such that X_F is n-dimensional and $\varepsilon > 0$, there is $\delta > 0$ such that for any $G \in C(K, \mathbb{R}^n)$ with $\|F - G\|_\infty < \delta$, then $d_{\mathcal{H}}(X_F, X_G) < \varepsilon$.

(b) Given $F \in C(K, \mathbb{R}^n)$ and $\varepsilon > 0$, there is $\delta > 0$ such that if $X \subset C(K)$ satisfies that $d_{\mathcal{H}}(X_F, X) < \delta$, we can then find $G \in C(K, \mathbb{R}^n)$ with $X = X_G$ and $\|F - G\|_\infty < \varepsilon$.

Now, consider the space of n-dimensional subspaces of $C(\mathbb{I})$, equipped with the topology induced by $d_{\mathcal{H}}$. In this case, polyhedral subspaces are dense, however smooth subspaces are not dense for $n \geq 2$. Indeed, given $\varepsilon > 0$ and X_F a n-dimensional subspace of $C(\mathbb{I})$, by observation (a) we can find $G \in C(\mathbb{I}, \mathbb{R}^n)$ close enough to F in order that $d_{\mathcal{H}}(X_F, X_G) < \varepsilon$ and such that $\overline{conv}(G(\mathbb{I}))$ has finitely many extreme points. This implies that X_G^* is polyhedral and thus X_G is polyhedral too. To prove the other statement consider a subspace X_F such that $F(\mathbb{I})$ is far from its convex hull (for example if $F(\mathbb{I})$ is a star). Fix $\varepsilon > 0$ such that $(F(\mathbb{I}) \cup (-F(\mathbb{I}))) + \varepsilon B_{\mathbb{R}^n}$ is not convex. Any subspace $X \subset C(\mathbb{I})$ close enough to X_F is of the form $X = X_G$ with $\|G - F\|_\infty < \varepsilon$ by observation (b). With such a choice, $G(\mathbb{I}) \cup (-G(\mathbb{I})) \subset (F(\mathbb{I}) \cup (-F(\mathbb{I}))) + \varepsilon B_{\mathbb{R}^n}$ is far from containing the boundary of a large convex body, and so $B_{X^*} = \overline{conv}(G(\mathbb{I}) \cup (-G(\mathbb{I})))$ cannot be strictly convex. Therefore X is not smooth by Šmulyan's duality [3, Proposition 1.6] or [6, Corollary 7.23].

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