A BOURGAIN-LIKE PROPERTY OF BANACH SPACES WITH NO COPIES OF c_0

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ABSTRACT. We give a characterization of the existence of copies of c_0 in Banach spaces in terms of indexes. As an application, we deduce new proofs of James Distortion theorem and Bessaga-Pełczynski theorem about weakly unconditionally Cauchy series.

1. Introduction

The aim of this paper is to study the existence of copies of c_0 in Banach spaces in terms of indexes and by purely geometrical methods. Our motivation for this is the beautiful characterization given by Bourgain [1, Lemma 3.7, p. 39] of Banach spaces not containing ℓ^1 , as those satisfying that for every bounded subset C of X^* and each $\varepsilon > 0$ there exist relatively weak*-open subsets U_1, \ldots, U_m of C such that $\frac{1}{m}(U_1 + \ldots + U_m)$ has diameter less than ε . Several results concerning this type of spaces follow from this, like the fact that their dual unit ball (B_{X^*}, ω^*) is convex block compact [1, Proposition 3.11, p. 43].

We prove that a Banach space X does not contain an isomorphic copy of c_0 if and only if for every bounded subset A of X and each $\varepsilon > 0$ there are x_1, \ldots, x_m in A such that $\bigcap_{j=1}^m (A-x_j) \cap (x_j-A)$ has diameter less than ε . Actually, we give a quantitative version of this fact. We first associate to any bounded set $A \subset X$ a sequence of indexes $\delta_m(A)$ ($m \ge 0$), being $\delta_m(A)$ half of the infimum of all diameters of sets $\bigcap_{j=1}^m (A-x_j) \cap (x_j-A)$ where $x_1, \ldots, x_m \in A$. Then, we prove in Theorem 2.5 that for each $\varepsilon > 0$ we can find a sequence $(x_n)_{n \in \mathbb{N}}$ in the absolute convex hull of A such that

$$(1) \qquad (\delta_{2^{N}}(A) - \varepsilon) \cdot \max_{1 \le n \le N} |\lambda_{n}| \le \left\| \sum_{n=1}^{N} \lambda_{n} x_{n} \right\| \le \delta_{0}(A) \cdot \max_{1 \le n \le N} |\lambda_{n}|$$

for every $\lambda_1, \dots \lambda_N \in \mathbb{R}$ and $N \in \mathbb{N}$.

From the previous result we deduce the characterization of Banach spaces containing an isomorphic copy of c_0 mentioned above (Theorem 3.1), as well as the known theorems of James (Theorem 3.2) and Bessaga-Pełczynski (Theorem 3.3) without using basic sequences.

Our notation is standard and follows [5]. We denote by X a real Banach space with the norm $\|\cdot\|$. Its topological dual will be denoted by X^* , and for any $x^* \in X^*$ and $x \in X$ the evaluation of x^* at x is written as $x^*(x) = \langle x^*, x \rangle = \langle x, x^* \rangle$. The closed

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unit ball (resp. unit sphere) of X is denoted by B_X (resp. S_X). If $D \subset X$ then we write $\operatorname{co}(D)$, $\operatorname{aco}(D)$ and $\operatorname{span}(D)$ to denote the convex hull, the absolutely convex hull and the linear hull of D. The supremum of $x^* \in X$ on D is denoted by $\sup(x^*,D)$. A *slice* of D is a set of the form $S(D,x^*,\delta):=\{x\in D:x^*(x)>\sup(x^*,D)-\delta\}$ for some $x^*\in X^*$ and $\delta>0$. Recall that the *diameter* of D is defined as $\operatorname{diam}(D):=\sup\{\|x-y\|:x,y\in D\}$.

2. Indexes of symmetrization

Definition 2.1. Given $A \subset X$ bounded, the symmetrized of A with respect to $x_1, \ldots, x_N \in A$ is defined as $\bigcap_{n=1}^N (A - x_n) \cap (x_n - A)$. For each $N \in \mathbb{N}$ we also define:

$$\Delta_N(A) := \left\{ \bigcap_{n=1}^N (A - x_n) \cap (x_n - A) : x_1, \dots, x_N \in A \right\}.$$

$$\delta_0(A) := \operatorname{diam}(A)/2, \quad \delta_N(A) := \inf \left\{ \delta_0(D) : D \in \Delta_N(A) \right\}.$$

It is clear from the definition that $\{\Delta_N(A): N \in \mathbb{N}\}$ is an increasing sequence of sets, and hence $\{\delta_N(A): N \in \mathbb{N}\}$ is decreasing. We will write $\delta_\infty(A) := \lim_N \delta_N(A)$. Let us point out that if $x \in A$, then $d \in (A-x) \cap (x-A)$ is equivalent to $x \pm d \in A$. With this in mind, the following (useful) observations are direct:

- (I) If $D \in \Delta_N(A)$ is the symmetrized of A with respect to $x_1, \ldots, x_N \in A$, then for every $d \in D$ the set $(D-d) \cap (d-D)$ is the symmetrized of A with respect to $x_1 \pm d, \ldots, x_N \pm d \in A$. In particular, $(D-d) \cap (d-D) \in \Delta_{2N}(A)$.
- (II) Given $x^* \in X^*$, $\delta > 0$ and $x \in S(A, x^*, \delta)$, every $d \in D := (A x) \cap (x A)$ satisfies $|x^*(x)| + |x^*(d)| < \sup(x^*, A)$, so that $|x^*(d)| < \delta$. In particular, $x \pm D \subset S(A, x^*, 2\delta)$.

Recall that the *Kuratowski measure of non-compactness* of a set $S \subset X$ is

 $\alpha(S) := \inf \{ \varepsilon > 0 : \text{ there are finitely many balls of radius } \varepsilon \text{ which cover } S \}.$

Lemma 2.2. If
$$A \subset X$$
 is bounded and $D \in \Delta_N(A)$, then $\alpha(D) \geq \delta_{2N}(A)$.

Proof. Suppose that $\alpha(D) < \varepsilon$, and let D_1, \ldots, D_n be a finite family of subsets of D whose union is equal to D and such that each D_k is contained in a ball of radius less than ε . If $D \subset \overline{\operatorname{co}}(D_1)$, then $\operatorname{diam}(D) < 2\varepsilon$ and so $\delta_{2N}(A) \le \delta_N(A) \le \delta_0(D) < \varepsilon$. Otherwise, we can assume that there is $2 \le m \le n$ such that

(2)
$$D \subset \overline{\operatorname{co}}(D_1 \cup \ldots \cup D_m)$$
 and $D \not\subset \overline{\operatorname{co}}(D_1 \cup \ldots \cup D_{m-1})$.

We can take $x_0^* \in S_{X^*}$ and $\delta > 0$ such that the slice $S(D, x_0^*, \delta)$ has empty intersection with $\overline{\operatorname{co}}(B_1 \cup \ldots \cup B_{m-1})$. We claim that for every $0 < \eta < 1$ it holds that

(3)
$$S(D, x_0^*, \eta \delta) \subset D_m + \eta (1 + \operatorname{diam} D) B_X.$$

If $d \in S(D, x_0^*, \eta \delta)$, then by (2) we can find $d' := \lambda d_m + (1 - \lambda)c_m$ where $0 \le \lambda \le 1$, $d_m \in D_m$ and $c_m \in \operatorname{co}(D_1 \cup \ldots \cup D_{m-1})$ such that $||d - d'|| < \eta$ and $d' \in S(D, x_0^*, \delta \eta)$. Since $x_0^*(c_m) \le \sup(x_0^*) - \delta$, we deduce that

$$\sup(x_0^*, D) - \eta \, \delta < x_0^*(d') = \lambda x_0^*(d_m) + (1 - \lambda) x_0^*(c_m) \le \sup(x_0^*, D) - (1 - \lambda) \, \delta.$$

This yields $1 - \lambda < \eta$, and so

$$||d-d_m|| \le ||d-d'|| + ||d'-d_m|| < \eta + (1-\lambda)||d_m-c_m|| < \eta(1+\operatorname{diam} D).$$

This proves the claim. By observations (I) and (II), for every $d_0 \in S(D, x_0^*, \eta \delta/2)$ the set $D_0 := (D - d_0) \cap (d_0 - D)$ belongs to $\Delta_{2N}(A)$ and $d_0 \pm D_0 \subset S(D, x_0^*, \eta \delta)$. Hence, we get by (3) that

$$\delta_{2N}(A) \leq \delta_0(D_0) \leq \frac{1}{2}\operatorname{diam} S(D, x_0^*, \eta \delta) \leq \varepsilon + \eta (1 + \operatorname{diam} D).$$

Since $\eta > 0$ is arbitrary, we conclude that $\delta_{2N}(A) < \varepsilon$.

Remark 2.3. We are thankful to an anonymous referee for pointing out to us that Lemma 2.2 can be obtained as a corollary of the so-called "Superlemma" of Namioka and Bourgain [3, Chapter IX, p. 157]. Indeed, under the assumption (2) we can apply this result to the closed convex hull of D to obtain a slice $S = S(D, x_0^*, \delta)$ of D with diameter smaller than the diameter of $\overline{\operatorname{co}}(D_m)$, which is less than 2ε . Taking $d_0 \in S(D, x_0^*, \delta/2)$ we can argue as in the last part of the proof of Lemma 2.2 to conclude the result.

Lemma 2.4. Let $F \subset X$ be a finite-dimensional subspace and $D \subset X$ bounded. If $\alpha(D) > \lambda > 0$, then there exists $x_0^* \in S_{F^{\perp}}$ such that $\sup (x_0^*, D) > \lambda$.

Proof. Suppose that every $x_0^* \in S_{F^{\perp}}$ satisfies that $\sup(x_0^*, D) \leq \lambda$. By Hahn-Banach Theorem we have that $D \subset F + \lambda B_X$. But then $D \subset \mu B_F + \lambda B_X$ for some $\mu > 0$, which implies that $\alpha(D) \leq \lambda$ by the compactness of B_F .

Theorem 2.5. Let $A \subset X$ be bounded. For every $\varepsilon > 0$ there is a sequence $(x_n)_{n \in \mathbb{N}}$ in aco (A) such that

$$(4) \qquad (\delta_{2^{N}}(A) - \varepsilon) \cdot \max_{1 \le n \le N} |\lambda_{n}| \le \left\| \sum_{n=1}^{N} \lambda_{n} x_{n} \right\| \le \delta_{0}(A) \cdot \max_{1 \le n \le N} |\lambda_{n}|$$

for every $\lambda_1, \ldots, \lambda_N$ *in* \mathbb{R} *and* $N \in \mathbb{N}$.

Proof. Write $\eta = \varepsilon/3$. Fix $x_0 \in A_0 := A$ and put $A_1 := (A - x_0) \cap (x_0 - A)$. By Lemma 2.2 we have that $\alpha(A_1) > \delta_2(A) - \eta$, so Lemma 2.4 yields that there are $x_1 \in A_1$ and $x_1^* \in S_{X^*}$ with $x_1^*(x_1) > \sup(x_1^*, A_1) - \eta > \delta_2(A) - 2\eta$. Suppose that $N \ge 1$ and we have constructed $(x_n^*)_{n=1}^N$ in S_{X^*} , $(x_n)_{n=1}^N$ in aco (A) and $(A_n)_{n=1}^N$ subsets of X satisfying for each $1 \le n \le N$:

- (a) $x_{n-1} \pm A_n \subset A_{n-1}$ and $A_n \in \Delta_{2^{n-1}}(A)$.
- (b) $\{x_k : 1 \le k < n\} \subset \ker x_n^*$.
- (c) $x_n^*(x_n) > \sup(x_n^*, A_n) \eta > \delta_{2^n}(A) 2\eta$.

Put $A_{N+1}:=(A_N-x_N)\cap(x_N-A_N)\in\Delta_{2^N}(A)$. By Lemma 2.2 we have that $\alpha(A_{N+1})>\delta_{2^{N+1}}(A)-\eta_{N+1}$, so using Lemma 2.4 we obtain $x_{N+1}\in A_{N+1}$ and $x_{N+1}^*\in S_{X^*}$ such that $\{x_k:1\leq k\leq N\}\subset\ker x_{N+1}^*$ and $x_{N+1}^*(x_{N+1})>\sup(x_{N+1}^*,A_{N+1})-\eta>\delta_{2^{N+1}}(A)-2\eta$. This finishes the inductive construction. Notice that conditions (a) and (c) imply that

(d)
$$|x_n^*(z)| < \eta$$
 whenever $z \in A_{n+1}$.

Given $N \in \mathbb{N}$, we show now that the sequence $(x_n)_{n \in \mathbb{N}}$ satisfies (4). For every $0 \neq (\lambda_n)_{n=1}^N \in \mathbb{R}^N$ we can write

$$\left\| \sum_{n=1}^{N} \lambda_n x_n \right\| = |\lambda_m| \cdot \left\| \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_m} x_n \right\| \le |\lambda_m| \cdot \delta_0(A)$$

being m such that $|\lambda_m| = \max\{|\lambda_n| : 1 \le n \le N\}$, since $x_0 + \sum_{n=1}^N \pm x_n \in A$. Furthermore

$$\left\| \sum_{n=1}^{N} \frac{\lambda_n}{\lambda_m} x_n \right\| \ge \langle x_m^*, x_m \rangle + \langle x_m^*, \sum_{m < n \le N} \frac{\lambda_n}{\lambda_m} x_n \rangle \ge \delta_{2^m}(A) - 3\eta \ge \delta_{2^N}(A) - 3\eta,$$

where we have used (b), (c), (d) and the fact that

$$\sum_{m < n \le N} \frac{\lambda_n}{\lambda_m} x_n \in \text{co}(A_{m+1}), \text{ which is a consequence of (a)}.$$

Corollary 2.6. Let $A \subset X$ be bounded. For every $\varepsilon > 0$ there is a sequence in $(x_n)_{n \in \mathbb{N}}$ in aco(A) such that

$$(\delta_{\infty}(A) - \varepsilon) \max_{n \in \mathbb{N}} |\lambda_n| \le \left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| \le \delta_0(A) \cdot \max_{n \in \mathbb{N}} |\lambda_n|$$

for every finitely supported sequence $(\lambda_n)_{n\in\mathbb{N}}$ *in* \mathbb{R} .

3. Copies of c_0 in Banach spaces

Theorem 3.1. *Let X be a Banach space. The following assertions are equivalent:*

- (i) c_0 is not isomorphic to a subspace of X.
- (ii) $\delta_{\infty}(C) = 0$ for every bounded set $C \subset X$.
- (iii) $\delta_{\infty}(C) = 0$ for every bounded, convex and closed set $C \subset X$.

Proof. Implication (i) \Rightarrow (ii) is a consequence of Corollary 2.6, while (ii) \Rightarrow (iii) is obvious. We just have to check that (iii) \Rightarrow (i). Let $T: c_0 \to X$ be an isomorphism, and consider $A:=T(B_{c_0})$. Given $a_1,\ldots,a_N\in A$ and $0<\varepsilon<1$ we can find $m\in\mathbb{N}$ such that $a_n\pm(1-\varepsilon)T(e_m)\in A$ for every $1\leq n\leq N$. This shows that $\delta_N(A)\geq (1-\varepsilon)/\|T^{-1}\|$ for each $N\in\mathbb{N}$.

When c_0 is isomorphic to a subspace of X, it is also said that X has a copy of c_0 . It turns out that these spaces have indeed almost isometric copies of c_0 , which means that for every $\varepsilon > 0$ we can find a closed subspace $Y \subset X$ and an isomorphism $T: c_0 \to Y$ such that $||T|| ||T^{-1}|| \le 1 + \varepsilon$.

Theorem 3.2 (James). If X has a copy of c_0 , then it has almost isometric copies of c_0 .

Proof. If c_0 embedds into X then there exists a bounded set $A \subset X$ with $\delta_{\infty}(A) > 0$ by Theorem 3.1. It follows from the definition of $\delta_{\infty}(A)$ that for every $\varepsilon > 0$ there exists an element $D \in \bigcup_{N \in \mathbb{N}} \Delta_N(A)$ such that

$$\delta_{\infty}(A) \leq \delta_0(D) \leq (1+\varepsilon)\delta_{\infty}(A).$$

Since $\delta_{\infty}(A) \leq \delta_{\infty}(D)$, we deduce that $\delta_0(D) \leq (1+\varepsilon)\delta_{\infty}(D)$, so an application of Corollary 2.6 with D leads to the desired copy of c_0 .

Another easy consequence is the Bessaga-Pełzcynski criterion for the existence of copies of c_0 . Recall that a series $\sum_n x_n$ in a Banach space X is said to be wuC if $\sum_n |x^*(x_n)|$ converges for every $x^* \in X^*$, which by the Uniform Boundedness Principle implies that $\sum_n |x^*(x_n)|$ is uniformly bounded for $x^* \in B_{X^*}$.

Theorem 3.3 (Bessaga-Pełczynski). *If* $c_0 \nsubseteq X$ *and* $\sum_n x_n$ *is wuC, then the series is unconditionally convergent.*

Proof. Consider the uniformly bounded sets given by

$$A_m = \left\{ \sum_{n=1}^m \theta_n x_n : \theta_n \in \{-1,1\} \text{ for each } 1 \le n \le m \right\}, \quad A = \bigcup_{m \in \mathbb{N}} A_m.$$

If X does not contain a copy of c_0 , then $\delta_{\infty}(A) = 0$, so given $\varepsilon > 0$ we can find $a_1, \ldots, a_N \in A$ with

$$\operatorname{diam}\left(\bigcap_{j=1}^{N}\left(A-a_{j}\right)\cap\left(a_{j}-A\right)\right)<\varepsilon.$$

There is $M \in \mathbb{N}$ such that $\{a_n : 1 \le n \le N\} \subset \bigcup_{m \le M} A_m$, so $\left\| \sum_{n=M}^{M'} \theta_n x_n \right\| \le \varepsilon$ for every $\theta_n \in \{-1, 1\}$ and $M' \ge M$.

We finish with a non-symmetrized characterization of Banach spaces with no copies of c_0 .

Proposition 3.4. A Banach space X does not contain an isomorphic copy of c_0 if and only if for every bounded set $A \subset X$ and each $\varepsilon > 0$ there are $x_1, \ldots, x_N \in A$ such that

$$\operatorname{diam}\left(\bigcap_{j=1}^{N}\left(A-x_{j}\right)\right)<\varepsilon.$$

Proof. The sufficiency of the condition is consequence of Theorem 3.1. To see the converse, assume that there exists $A \subset X$ and $\varepsilon > 0$ such that any intersection like in the statement has diameter greater or equal than ε . Fix an arbitrary $x_0 \in A$ and then pick $x_1 \in (A - x_0)$ such that $||x_1|| \ge \varepsilon$. Consider the set $A_1 := \{x_0, x_0 + x_1\} \subset A$. Now we take

$$x_2 \in \bigcap_{x \in A_1} (A - x) \text{ with } ||x_2|| \ge \varepsilon \text{ and } A_2 := A_1 \cup (A_1 + x_2).$$

Following in this way, we will have a sequence $(x_n)_{n\in\mathbb{N}}$ of vectors of norm greater or equal to ε for $n \ge 1$ and sets $A_n \subset A$ of cardinality 2^n . Then consider

$$x_{n+1} \in \bigcap_{x \in A_n} (A - x)$$
 with $||x_{n+1}|| \ge \varepsilon$ and $A_{n+1} := A_n \cup (x_n + A_n)$.

Notice that the sums $\sum_{n=1}^{N} \theta_n x_n$ are uniformly bounded independently of N and the choice of $\theta_n \in \{-1, 1\}$, since they are difference of two elements of $A_N \subset A$. Now Theorem 3.3 implies that X contains a copy of c_0 .

4. Remarks

Let A be a subset of X. Recall that an ε -tree in A is a a sequence $\{x_n \colon n \in \mathbb{N}\}$ such that $x_n = (x_{2n} + x_{2n+1})/2$ and $||x_{2n} - x_{2n+1}|| \ge \varepsilon$ for every $n \in \mathbb{N}$. The index $\delta_1(A)$ is directly related to existence of ε -trees inside A. In fact, if $\delta_1(A) > \varepsilon$, then we can construct a 2ε -tree inside of A in the following way: fix any $x_1 \in A$. Since diam $((A - x_1) \cap (x_1 - A)) > 2\varepsilon$, we can find $u_1 \in X$ such that $||u_1|| \ge \varepsilon$ and $x_1 \pm u_1 \in A$. Put $x_2 := x_1 - u_1$ and $x_3 := x_1 + u_1$. Repeating this process with x_2, x_3 and the subsequent constructed points, we obtain the desired 2ε -tree. On the other hand, it is clear that every ε -tree A' satisfies that $\delta_1(A') \ge \varepsilon/2$. As a consequence, we can conclude that a set $A \subset X$ contains no ε -trees (for any $\varepsilon > 0$) if and only if $\delta_1(A') = 0$ for each $A' \subset A$. In particular, if C is a closed and convex set having the Radon-Nikodým Property (RNP), then $\delta_1(A) = 0$ for every $A \subset C$.

We say that $x_0 \in A$ is an ε -extreme point of A if $\operatorname{diam}((A-x_0) \cap (x_0-A))$ is less than 2ε . It is not difficult to see that x_0 is an extreme point of A if and only if it is ε -extreme for every $\varepsilon > 0$. As a consequence, if $K \subset X$ is a bounded, closed and convex set having the Krein-Milman Property (KMP), then $\delta_1(C) = 0$ for every closed and convex set $C \subset K$.

The previous notion reminds of the following concept introduced by Kunen and Rosenthal [6]: $x_0 \in A$ is an ε -strong extreme point of A if there is $\delta > 0$ such that whenever $a_1, a_2 \in A$ and there exists a point $u = \lambda a_1 + (1 - \lambda)a_2$ ($0 < \lambda < 1$) with $||x_0 - u|| < \delta$, then $||u - a_1|| < \varepsilon$ or $||u - a_2|| < \varepsilon$. If x_0 is ε -strong extreme for every $\varepsilon > 0$, then we simply say that it is a strong extreme point. It is not difficult to see that every ε -strong extreme point of A is an ε -extreme point of the same set. The converse is not true, since as it is pointed out in [6, Remark 3, p. 173] every strong extreme point of a bounded, closed and convex set is also an extreme point of its $\sigma(X^{**}, X^*)$ -closure (in the terminology of [4] we might say that these are preserved extreme points), while there are, for instance, Banach spaces where B_X has extreme points that are not extreme points of $B_{X^{**}}$ (see [4]). With this formulation we have that if K is a bounded, closed and convex set such that every $A \subset K$ has ε -extreme points for every $\varepsilon > 0$ (i.e. $\delta_1(A) = 0$), then each closed and convex set $C \subset K$ has an ε -strong extreme point for every $\varepsilon > 0$ (see [6, Proposition 3.2, p. 170]).

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