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# CONVEX COMPACT SETS THAT ADMIT A LOWER SEMICONTINUOUS STRICTLY CONVEX FUNCTION

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Abstract. We study the class of compact convex subsets of a topological vector space which admits a strictly convex and lower semicontinuous function. We prove that such a compact set is embeddable in a strictly convex dual Banach space endowed with its weak* topology. In addition, we find exposed points where a strictly convex lower semicontinuous function is continuous.

## 1. Introduction

A well-known result of Hervé [6] says that a compact convex subset $K \subset X$ of a locally convex space is metrizable if and only if there exists $f\colon K \to \mathbb{R}$ which is both continuous and strictly convex. It happens that lower semicontinuity is a very natural hypothesis for a convex function, so it is natural to wonder if the existence of a strictly convex lower semicontinuous function on compact convex subset $K \subset X$ of a locally convex space enforces special topological properties on $K$. Ribarska proved [16, 17] that such a compact is *fragmentable* by a finer metric, and in particular it contains a completely metrizable dense subset. The third named author proved [14] that the same is true for the set of its extreme points $\text{ext}(K)$. On the other hand, Talagrand's argument in [2, Theorem 5.2.(ii)] shows that $[0, \omega_1]$ is not embeddable in such a compact set. In addition, Godefroy and Li showed [5] that if the set of probabilities on a compact group $K$ admits a strictly convex lower semicontinuous function then $K$ is metrizable.

Our purpose here is to continue with the study of the class of compact convex subsets which admits a strictly convex lower semicontinuous function. We shall denote this class by $\mathcal{SC}$. The first remarkable fact that we have got is a Banach representation result.

**Theorem 1.1.** *Let $X$ be a locally convex topological vector space and let $K \subset X$ be convex compact subset. Then there exists a function $f\colon K \to \mathbb{R}$ which is both lower semicontinuous and strictly convex if and only if $K$ imbeds linearly into a strictly convex dual Banach space $Z$ endowed with its weak* topology.*

Notice that the strictly convex norm of the dual Banach space in the statement is weak* lower semicontinuous, which is a stronger condition that just being a strictly convex Banach space isomorphic to a dual space.

If $f\colon K \to \mathbb{R}$ is a strictly convex function, then the *symmetric* defined by

$$\rho(x, y) = \frac{f(x) + f(y)}{2} - f(\frac{x + y}{2})$$

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provides a consistent way to measure *diameters* of subsets of $K$. This idea was successfully applied in renorming theory [9]. We will prove that every nonempty subset of $K$ has slices of arbitrarily small $\rho$-diameter, we can mimic some arguments of the geometric study of the Radon–Nikodým property which leads to results as the following one.

**Theorem 1.2.** *Let $X$ be a locally convex topological vector space and let $f\colon X \to \mathbb{R}$ be lower semicontinuous, strictly convex and bounded on compact sets. Then for every $K \subset X$ compact and convex, the set of points in $K$ which are both exposed and continuity points of $f|_K$ is dense in $ext(K)$.*

The organization of the paper is as follows. In the second section we present stability properties of the class $\mathcal{SC}$, which allow us to prove the embedding Theorem 1.1. The third and fourth sections are devoted to the search of faces and exposed points of continuity, respectively. Finally, a characterization of the class $\mathcal{SC}$ in terms of the existence of a symmetric with countable dentability index is given in Section 5.

## 2. Embedding into a dual space

Along this section $X$ will denote a locally convex topological vector space. Our first goal is to study the properties of following class of compact sets.

**Definition 2.1.** The class $\mathcal{SC}(X)$ consists of all the nonempty compact convex subsets $K$ of $X$ such that there exists a function $f\colon K \to \mathbb{R}$ which is lower semicontinuous and strictly convex. In addition, $\mathcal{SC}$ denotes the class composed of all the families $\mathcal{SC}(X)$ for any locally convex space $X$.

Since a lower semicontinuous function on a compact space attains its minimum, the function $f$ is bounded below. Later we shall show that we may always take $f$ to be bounded. Notice that metrizable convex compacts admits continuous strictly convex functions, so they are in the class. In particular, if $X$ is metrizable then $\mathcal{SC}(X)$ contains all the convex compact subsets of $X$. If $X$ is a Banach space endowed with its weak topology, then $\mathcal{SC}(X)$ is made up of all convex weakly compact subsets as a consequence of the strictly convex renorming results for WCG spaces.

**Proposition 2.2.** *The class $\mathcal{SC}$ satisfies the following stability properties:*

a) *$\mathcal{SC}(X)$ is stable by translations and homothetics;*
b) *$\mathcal{SC}$ is stable by Cartesian products;*
c) *$\mathcal{SC}$ is stable by linear continuous images;*
d) *If $A, B \in \mathcal{SC}(X)$, then $A + B \in \mathcal{SC}(X)$.*

*Proof.* Statement *a*) is obvious. To prove *b*) suppose that $f_i$ witnesses $A_i \in \mathcal{SC}(X_i)$ for $i = 1, \ldots, n$. Then $\sum_{i=1}^n f_i \circ \pi_i$, where $\pi_i\colon \bigotimes_{i=1}^n X_i \to X_i$ is the coordinate projection, witnesses that $A_1 \times \cdots \times A_n \in \mathcal{SC}(\bigotimes_{i=1}^n X_i)$.

To prove *c*) assume that $A \in \mathcal{SC}(X)$ and $T\colon X \to Y$ is linear and continuous. Obviously $T(A)$ is convex and compact. Let $f\colon A \to \mathbb{R}$ be lower semicontinuous and strictly convex. It is straightforward to check that the function $g\colon T(A) \to \mathbb{R}$ defined by

$$g(y) = \inf \left\{ f(x) : x \in T^{-1}(y) \right\}$$

does the work. Finally, *d*) follows by a combination of *b*) and *c*). □

We will need a kind of external convex sum of convex compact sets.

**Definition 2.3.** Given $A, B \subset X$ convex compact define a subset of $X \times X \times \mathbb{R}$ by

$$A \oplus B = \{(\lambda x, (1 - \lambda)y, \lambda) : x \in A, y \in B, \lambda \in [0, 1]\} .$$

**Lemma 2.4.** *Let $A, B \subset X$ be convex compact subsets. Then*

a) *$A \oplus B$ is a convex compact subset of $X \times X \times \mathbb{R}$;*

b) *if $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ are convex, then $h\colon A \oplus B \to \mathbb{R}$ defined by*
$$h\big((\lambda x, (1-\lambda)y, \lambda)\big) = \lambda f(x) + (1-\lambda)g(y)$$
*is convex as well;*

c) *if $A, B \in \mathcal{SC}(X)$, then $A \oplus B \in \mathcal{SC}(X \times X \times \mathbb{R})$.*

*Proof.* Compactness is clear in statement $a$). Given $(\lambda_i x_i, (1-\lambda_i)y_i, \lambda_i) \in A \oplus B$ for $i = 1, 2$, just observe that
$$\begin{aligned}&\left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{2}, \frac{\lambda_1 + \lambda_2}{2}\right)\\ &= \left(\frac{\lambda_1+\lambda_2}{2}\,\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1+\lambda_2}, (1 - \frac{\lambda_1+\lambda_2}{2})\frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{(1-\lambda_1)+(1-\lambda_2)}, \frac{\lambda_1+\lambda_2}{2}\right)\end{aligned}$$
(the case where $\lambda_1 = \lambda_2 = 0, 1$ can be handed in a different way). Thus, $A \oplus B$ is convex. For the convexity of function $h$ notice that
$$\begin{aligned}&h\big((\frac{\lambda_1 x_1 + \lambda_2 x_2}{2}, \frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{2}, \frac{\lambda_1+\lambda_2}{2})\big)\\ &= \frac{\lambda_1+\lambda_2}{2} f\big(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1+\lambda_2}\big) + \left(1 - \frac{\lambda_1+\lambda_2}{2}\right) g\big(\frac{(1-\lambda_1)y_1 + (1-\lambda_2)y_2}{(1-\lambda_1)+(1-\lambda_2)}\big)\\ &\leq \frac{\lambda_1+\lambda_2}{2}\,\frac{\lambda_1 f(x_1) + \lambda_2 f(x_2)}{\lambda_1+\lambda_2} + \left(1 - \frac{\lambda_1+\lambda_2}{2}\right)\frac{(1-\lambda_1)g(y_1) + (1-\lambda_2)g(y_2)}{(1-\lambda_1)+(1-\lambda_2)}\\ &= \frac{1}{2}\left(h\big((\lambda_1 x_1, (1-\lambda_1)y_1, \lambda_1)\big) + h\big((\lambda_2 x_2, (1-\lambda_2)y_2, \lambda_2)\big)\right).\end{aligned}$$
If $f$ and $g$ were strictly convex, the above inequality for $h$ would become strict if $x_1 \neq x_2$ or $y_1 \neq y_2$. To overcome this difficulty consider the function
$$k\big((\lambda x, (1-\lambda)y, \lambda)\big) = h\big((\lambda x, (1-\lambda)y, \lambda)\big) + \lambda^2$$
and notice that $\lambda^2$ provides the strict inequality when $x_1 = x_2$ and $y_1 = y_2$. □

**Proposition 2.5.** *Suppose that $A, B \in \mathcal{SC}(X)$. Then conv$(A \cup B) \in \mathcal{SC}(X)$ and aconv$(A) \in \mathcal{SC}(X)$.*

*Proof.* Consider the map $T\colon X \times X \times \mathbb{R} \to X$ defined by $T\big((x, y, t)\big) = x + y$ and observe that $T(A \oplus B) = \operatorname{conv}(A \cup B)$. Since $T$ is linear and continuous, the combination of the previous results gives us that $\operatorname{conv}(A \cup B) \in \mathcal{SC}(X)$. The application to the symmetric convex hull follows by applying it with $B = -A$. □

**Lemma 2.6.** *Let $B \subset X$ be a symmetric compact convex set and let $Z = span(B)$. Then the following hold:*

a) *$Z$, with the norm given by the Minkowski functional of $B$, is isometric to a dual Banach space;*

b) *$B$ imbeds linearly into $(Z, w^*)$;*

c) *if $f\colon X \to \mathbb{R}$ is convex and lower semicontinuous, then $f|_Z$ is weak* lower semicontinuous.*

*Proof.* Notice that $Z = \bigcup_{n=1}^{\infty} nB$, and thus the Minkowski functional of $B$ is a norm on $Z$. Of course, $B$ is the unit ball of $Z$ endowed with this norm. By a result of Dixmier-Ng, see for instance [10], the space $Z$ is isometric to the dual of the Banach space $W$ of all linear functionals $f$ on $Z$ such that $f|_B$ is $\tau$-continuous. If $f\colon X \to \mathbb{R}$ is convex and lower semicontinuous, then the sets $\{f \leq a\}$ are convex and closed for any $a \in \mathbb{R}$. We have $\{f|_Z \leq a\} = \{f \leq a\} \cap Z$, and thus $\{f|_Z \leq a\} \cap nB = \{f \leq a\} \cap nB$ is compact, and so it is weak* compact as subset of $Z$ for every $n \in \mathbb{N}$. By the Banach-Dieudonné theorem, $\{f|_Z \leq a\}$ is a weak* closed subset of $Z$. □

*Proof of Theorem 1.1.* Let $B = \operatorname{aconv}(K)$ which is in $\mathcal{SC}(X)$. The function $f$ witnessing that $B \in \mathcal{SC}(X)$ is weak* lower semicontinuous and strictly convex. By Lemma 2.6 we only need to renorm the dual space $Z$. Notice that the function $f$ can be taken symmetric and bounded. Indeed, for the symmetry just take $g(x) = f(x) + f(-x)$. Now apply the Baire theorem to the $B = \bigcup_{n=1}^{\infty} g^{-1}((-\infty, n])$ to obtain a set of the form $\lambda B$ with $\lambda > 0$ where $g$ is bounded. Then redefine $f$ as $f(x) = g(\lambda x)$.

Without loss of generality we may assume that $f$ takes values in $[0, 1]$. Consider the function defined on $B_Z$ by

$$h(x) = \frac{1}{2}(3\|x\| + f(x))$$

and consider the set $C = \{x \in B_Z : h(x) \leq 1\}$. Clearly $\frac{1}{3}B_Z \subset C \subset \frac{2}{3}B_Z$, and $C$ is convex, symmetric and weak* closed. Moreover, if $h(x) = h(y) = 1$, then $h\left(\frac{x+y}{2}\right) < 1$. Therefore, $C$ is the unit ball of an equivalent strictly convex dual norm on $Z$. □

**Corollary 2.7.** *If $K \in \mathcal{SC}(X)$, then it is witnessed by the square of a lower semicontinuous strictly convex norm defined on span$(K)$.*

We shall finish this section by showing the connection between the class $\mathcal{SC}$ and $(*)$ property. The following notion was introduced in [11] in order to characterize dual Banach spaces that admit a dual strictly convex norm:

**Definition 2.8.** A compact space $K$ is said to have $(*)$ if there exists a sequence $(\mathcal{U}_n)_{n=1}^{\infty}$ of families of open subsets of $K$ such that, given any $x, y \in K$, there exists $n \in \mathbb{N}$ such that:

a) $\{x, y\} \cap \bigcup \mathcal{U}_n$ is non-empty;
b) $\{x, y\} \cap U$ is at most a singleton for every $U \in \mathcal{U}_n$.

Here we are using the agreement that $\bigcup \mathcal{U}_n = \bigcup \{U : U \in \mathcal{U}_n\}$. Recall that if $K$ is a subset of a locally convex topological vector space then a slice of $K$ is an intersection of $K$ with an open halfspace. If the elements of $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ can be taken to be slices of $K$, then $K$ is said to have $(*)$ *with slices.* It is shown in [11, Theorem 2.7] that if $Z$ is a dual Banach space then $(B_Z, w^*)$ has $(*)$ with slices if and only if $Z$ admits a dual strictly convex norm.

**Corollary 2.9.** *Let $(X, \tau)$ be locally convex topological vector space and $K \subset X$ be compact and convex. Then $K \in \mathcal{SC}(X)$ if and only if $K$ has $(*)$ with slices.*

*Proof.* By Lemma 2.6 we may assume that $K \subset Z = \operatorname{span}(K)$ has $(*)$ with weak* slices. It follows from [11, Proposition 2.2] that then there is a lower semicontinuous strictly convex function defined on $K$. On the other hand, assume that $\phi$ witnesses $K \in \mathcal{SC}(X)$. For $f \in (X, \tau)^*$ and $r \in \mathbb{R}$, denote $S(f, r) = \{x \in K : f(x) > r\}$. Consider the families $\{\mathcal{U}_{qr}\}_{q,r \in \mathbb{Q}}$ of open subsets given by

$$\mathcal{U}_{qr} = \{S(f, r) : f \in (X, \tau)^*, S(f, r) \cap \{x : \phi(x) \leq q\} = \emptyset\} .$$

Let $x \neq y$ be in $K$. We may assume that $\phi(x) \leq \phi(y)$. Since $\phi$ is strictly convex, there exists $q \in \mathbb{Q}$ such that $\phi(\frac{x+y}{2}) < q < \phi(y)$. By the Hahn–Banach theorem, there is $f \in (X, \tau)^*$ and $r \in \mathbb{Q}$ such that $\sup\{f(z) : \phi(z) \leq q\} < r < f(y)$. Therefore, $S(f, r) \cap \{z : \phi(z) \leq q\} = \emptyset$ and $\{x, y\} \cap \bigcup \mathcal{U}_{qr} \neq \emptyset$.

Suppose that $x, y \in S(g, r) \in \mathcal{U}_{qr}$. Then $g(x), g(y) > r$ implies $g\left(\frac{x+y}{2}\right) > r$. Hence $\frac{x+y}{2} \notin \{z : \phi(z) \leq q\}$, a contradiction. So $\{x, y\} \cap S(g, q)$ is at most a singleton for each $S(g, q) \in \mathcal{U}_{qr}$. □

## 3. Faces of continuity

We will assume along the section that $Z = W^*$ is a dual Banach space endowed with the weak$^*$ topology. Therefore any unspecified topological concept (compact, open, ...) is always referred to the weak$^*$ topology. The elements of $W$ will be considered as functionals on $Z$. Other topological ingredient that we will use is a symmetric $\rho\colon Z \times Z \to [0, +\infty)$. Recall that a symmetric satisfies $\rho(x, y) = \rho(y, x)$ and $\rho(x, y) = 0$ if and only if $x = y$. Since a symmetric does not satisfy the triangle inequality, its associated topology is complicated to handle. Nevertheless we have a natural notion of diameter associated to $\rho$ defined by

$$\rho\text{-diam}(A) = \sup\{\rho(x, y) : x, y \in A\}$$

Let us recall the definition of face of a convex set.

**Definition 3.1.** Let $C \subset Z$ be closed and convex. We say that a closed subset $F \subset C$ is a face if there is a continuous affine function $w\colon C \to \mathbb{R}$ such that

$$F = \{x \in C : w(x) = \sup\{w, C\}\} .$$

In that case we say that the face is produced by $w$. In addition, we say that a point $x \in C$ is a exposed point of $C$ if $\{x\}$ is a face of $C$.

Sometimes the face is produced by an element of the dual. Nevertheless, there may exist continuous affine functions on $C$ that are not the restriction of an element of the dual.

We shall need the following lemma.

**Lemma 3.2** (Lemma 3.3.3 of [1]). *Suppose that $w \in W$ and $\|w\| = 1$. For $r > 0$ denote by $V_r$ the set $rB_Z \cap w^{-1}(0)$. Assume that $x_0$ and $y$ are points of $Z$ such that $w(x_0) > w(y)$ and $\|x_0 - y\| \leq r/2$. If $u \in W$ satisfies that $\|u\| = 1$ and $u(x_0) > \sup\{u, y + V_r\}$, then $\|w - u\| \leq \frac{2}{r}\|x_0 - y\|$.*

First we shall discuss the dual Banach case.

**Proposition 3.3.** *Let $f\colon Z \to \mathbb{R}$ be a convex lower semicontinuous function which is bounded on compact subsets. If $K \subset Z$ is compact convex, then there exists a $G_\delta$ dense set of elements of $W$ producing faces where $f|_K$ is constant and continuous.*

*Proof.* Define the pseudo-symmetric $\rho$ by the formula

$$\rho(x, y) = \frac{f(x)^2 + f(y)^2}{2} - f\big(\frac{x + y}{2}\big)^2 .$$

We claim that $\rho(x, y) = 0$ implies $f(x) = f(y) = f(\frac{x+y}{2})$ (in particular, if $f$ were strictly convex, $\rho$ would be a symmetric). Indeed, it follows easily from this observation

$$\rho(x, y) \geq \frac{f(x)^2 + f(y)^2}{2} - \left(\frac{f(x) + f(y)}{2}\right)^2 = \left(\frac{f(x) - f(y)}{2}\right)^2 \geq 0 .$$

Now we claim that the set $G(K, \varepsilon)$ is open and dense in $W$ for $K \subset Z$ compact convex and $\varepsilon > 0$, where

$$G(K, \varepsilon) = \{w \in W : \exists a < \sup\{w, K\}, \rho\text{-diam}(K \cap \{w > a\}) < \varepsilon\} .$$

Suppose that $w \in G(K, \varepsilon)$. If $w' \in W$ is close enough to $w$ to fulfill that

$$\sup\{w', K\} > \sup\{w', K \cap \{w \leq a\}\}$$

then $w' \in G(K, \varepsilon)$ as well. Thus $G(K, \varepsilon)$ is open. In order to see that it is also dense, fix $w \in W$ and $\delta \leq 1/4$. Take $x \in K$ and $y \in Z$ with $w(x) > a > w(y)$ for some $a \in \mathbb{R}$. Take $r = \sup\{\|x' - y\|, x' \in K\}/2\delta$, consider the set $V_r$ given by Lemma 3.2 and define the set $C = \text{conv}(K \cup (y + V_r))$. By [14, Theorem 1.1], the halfspace $\{w > a\}$ contains a point $x_0 \in \text{ext}(C)$ where $f|_C$ is continuous. Notice that $x_0 \in \text{ext}(K)$ and $\|x_0 - y\| \leq r/2$. There exists $u \in W$ and $b \in \mathbb{R}$ such that $u(x_0) > b$, $C \cap \{u > b\} \subset C \cap \{w > a\}$ and $\rho$-diam$(C \cap \{u > b\}) < \varepsilon$. In particular $\rho$-diam$(K \cap \{u > b\}) < \varepsilon$. Since $C \cap \{u > b\}$ does not meet $y + V_r$, we have $u(x_0) > \sup\{u, y + V_r\}$. Thus, $\|w - u\| \leq \frac{2}{r}\|x_0 - y\| \leq \delta$. That completes the proof of the density of $G(K, \varepsilon)$ in $W$.

By the Baire theorem, the set $G(K) = \bigcap_{n=1}^{\infty} G(K, 1/n)$ is dense. If $w \in G(K)$ and $s = \sup\{w, K\}$ then

$$\lim_{t \to s^-} \rho\text{-diam}(K \cap \{w > t\}) = 0\,.$$

In particular, the face $F = K \cap \{w = s\}$ satisfies that $\rho$-diam$(F) = 0$. That implies that $f$ is constant on $F$. Moreover, we claim that any point $x \in F$ is a point of continuity of $f|_K$. If $(x_\alpha) \subset K$ is a net with limit $x$, then $\lim_\alpha w(x_\alpha) = w(x)$. Therefore $\lim_\alpha \rho(x_\alpha, x) = 0$. It follows that $\lim_\alpha f(x_\alpha) = f(x)$, so $f|_K$ is continuous at $x$. □

Now the above result can be translated into a more general setting.

**Proposition 3.4.** *Let $f\colon X \to \mathbb{R}$ be a convex lower semicontinuous function which is bounded on compact subsets. Then for every compact convex subset $K \subset X$ and every open slice $S \subset K$, there is a face $F \subset S$ of $K$ such that $f|_K$ is constant and continuous on $F$.*

*Proof.* By Lemma 2.6, $Z = \bigcup_{n=1}^{\infty} n\,\text{aconv}(K)$ is a dual Banach space and $f|_Z$ is weak* lower semicontinuous. Then we can apply the previous proposition. □

It is clear that the last two results are true for countably many functions simultaneously.

*Remark* 3.5. We do not know if the function $f$ in Proposition 3.3 and 3.4 can be assumed to be defined only on $K$. Notice that if $\|\ \|$ is a strictly convex norm on $Z$ then $f(x) = -\sqrt{1 - ||x||^2}$ is a strictly convex weak* lower semicontinuous function on $(B_Z, w^*)$ that cannot be extended to a convex function on $Z$.

## 4. Exposed points

Notice that if a strictly convex function is constant on a face of a compact $K$, then necessarily that face should be an exposed point of $K$. Having this in mind, Propositions 3.3 and 3.4 can be rewritten. As in the previous section $Z = W^*$ is a dual Banach space endowed with the weak$^*$ topology and we understood all the topological notions referred to that topology.

**Proposition 4.1.** *Let $f\colon Z \to \mathbb{R}$ be a strictly convex lower semicontinuous function which is bounded on compact subsets. If $K \subset Z$ is compact convex, then there exists a $G_\delta$ dense set of elements of $W$ exposing points of $K$ at which $f|_K$ is continuous.*

*Proof.* It follows straightforward from Proposition 3.3. □

In particular, we retrieve the following result, which is usually proved in the frame of Gâteaux Differentiability Spaces [13, Corollary 2.39 and Theorem 6.2].

**Corollary 4.2** (Asplund, Larman–Phelps)**.** *Let $Z$ be a strictly convex dual Banach space. Then every convex compact is the closed convex hull of its exposed points.*

*Proof of Theorem 1.2.* It follows straightforward from Proposition 3.4. □

**Corollary 4.3.** *Assume that $K \in \mathcal{SC}(X)$. Then $K$ is the closed convex hull of its exposed points.*

*Proof.* Thanks to Theorem 1.1 it can be reduced to the previous corollary. □

Notice that the previous result is far from being a characterization. For instance, consider $X = C([0,\omega_1])^*$ and $K = (B_X, w^*)$. Then $X$ has the Radon–Nikodým Property and thus there exist *strongly exposed points* of $K$ [1, Theorem 3.5.4]. Nevertheless, Talagrand's argument in [2, Theorem 5.2.(ii)] shows that $K \notin \mathcal{SC}(X, w^*)$. Indeed, the result of Larman and Phelps mentioned aboved states that Banach spaces for which each weak* compact convex subset has an exposed point are exactly dual spaces of a Gâteaux Differentiability Space.

*Remark* 4.4. A point $x$ in a subset $C$ of a normed space $(Z, \|\ \|)$ is said to be a *farthest point* in $C$ if there exists $y \in Z$ such that $\|y - x\| \geq \sup\{\|y - c\| : c \in C\}$. If $\|\ \|$ is strictly convex then every farthest point of $C$ is exposed by a functional in $Z^*$. In addition, it was shown in [3] that there exists a weak* compact subset of $\ell_1$ that has no farthest points, so the existence of exposed points does not imply the existence of farthest points. On the other hand, suppose that $Z$ is a strictly convex dual Banach space, $C$ is a compact subset of $Z$ and $x$ is a farthest point in $C$ with respect to $y \in Z$. Consider the symmetric $\rho(u,v) = \frac{\|u-y\|^2+\|v-y\|^2}{2} - \|\frac{u+v}{2} - y\|^2$. Then $x$ is a $\rho-$denting point of $C$, that is, admits slices with arbitrarily small $\rho$-diameter. Indeed, if $\delta = \frac{\varepsilon}{1+2\|x-y\|+2\|y\|}$ then every slice of $C$ that does not meet $B(y, \|y - x\| - \delta)$ has $\rho$-diameter less than $\varepsilon$.

Typically a variational principle provides strong minimum for certain functions after a small perturbation. But in the compact setting, a lower semicontinuous function already attains its minimum. Nevertheless, inspired by Stegall's variational principle [4, Theorem 11.6], we have obtained the following result.

**Proposition 4.5.** *Suppose that $K \in \mathcal{SC}(X)$ and let $f : K \to \mathbb{R}$ be a lower semicontinuous function. Given $\varepsilon > 0$, there exists an affine continuous function $w$ on $K$ with oscillation less than $\varepsilon$ such that $f + w$ attains its minimum exactly at one point. Moreover, if $X$ is a dual Banach space then $w$ can be taken from the predual with norm less than $\varepsilon$.*

*Proof.* By the embedding it is enough to consider the Banach case. Let $m$ be the minimum of $f$ and take $M > 0$ such that $K \subset MB_X$. Consider the compact set

$$H = \{(x,t) : f(x) \leq t \leq m + \varepsilon M\}$$

and take its convex closed envelop $A$. By Proposition 2.2, $A \in \mathcal{SC}(X \times \mathbb{R})$. The functional on $X \times \mathbb{R}$ given by $(0, 1)$ attains its minimum on $A$. Proposition 4.1 provides a small perturbation of the form $(w, 1)$, with $\|w\| < \varepsilon$, attaining its minimum on $A$ at one single point $(x_0, t_0)$. Notice that $t_0 = f(x_0)$ and $f(x_0) + w(x_0) \leq m + \varepsilon M$. If $y \in K$, then either $f(y) \leq m + \varepsilon M$ and $(y, f(y)) \in A$, or $f(y) > m + \varepsilon M \geq f(x_0) + w(x_0)$. □

## 5. Ordinal indices

Let $K$ be a convex and compact subset of a locally convex topological vector space and $\rho$ a symmetric on $K$. We consider the following set derivations:

$$[K]'_\varepsilon = \{x \in K : x \in S \text{ slice of } K \Rightarrow \rho\text{-diam}(S) \geq \varepsilon\}\,;$$
$$\langle K\rangle'_\varepsilon = \{x \in K : x \in U \text{ open } \Rightarrow \rho\text{-diam}(S) \geq \varepsilon\}\ .$$

The iterated derived sets are defined as $[K]^{\alpha+1}_\varepsilon = [[K]^\alpha_\varepsilon]'_\varepsilon$, $\langle K\rangle^{\alpha+1}_\varepsilon = \langle\langle K\rangle^\alpha_\varepsilon\rangle'_\varepsilon$ and intersection in case of limit ordinals. If there exists some ordinal such that $[K]^\alpha_\varepsilon = \emptyset$, then we set $Dz_\rho(K,\varepsilon) = \min\{\alpha : [K]^\alpha_\varepsilon = \emptyset\}$. Otherwise, we take $Dz_\rho(K,\varepsilon) = \infty$, which is beyond the ordinals. The $\rho$-dentability index of $K$ is defined by $Dz_\rho(K) = \sup_{\varepsilon>0} Dz(K,\varepsilon)$. The $\rho$-Szlenk index of $K$, $Sz_\rho(K)$, is defined the same way. Obviously $Sz_\rho(K) \leq Dz_\rho(K)$. Set derivations with respect to a symmetric were introduced in [7] in order to characterize dual Banach spaces admitting a dual strictly convex norm.

**Proposition 5.1.** *Let $K$ be a convex compact subset of a locally convex space. Then the following assertions are equivalent:*

a) *$K \in \mathcal{SC}$;*
b) *there exists a symmetric $\rho$ on $K$ such that $Dz_\rho(K) \leq \omega$;*
c) *there exists a symmetric $\rho$ on $K$ such that $Dz_\rho(K) \leq \omega_1$.*

*Proof.* Let $f$ be a bounded function witnessing that $K \in \mathcal{SC}$ and assume that $f$ takes values in $[0,1]$. For a fixed $\varepsilon > 0$, take $N > 1/\varepsilon$ and define the closed convex subsets $F_n = \{x \in K : f(x) \leq 1 - n/N\}$ for $n = 0, \ldots N$. Take

$$\rho(x,y) = \frac{f(x)+f(y)}{2} - f\big(\frac{x+y}{2}\big)\,.$$

We claim that $[K]'_\varepsilon \subset F_1$. Let $x_0 \in K \setminus F_1$. By the Hahn–Banach theorem, there exists a slice $S$ of $K$ such that $x_0 \in S$ and $S \cap F_1 = \emptyset$. If $x, y \in S$, then $\frac{x+y}{2} \in S$ and $\rho(x,y) \leq 1 - (1 - 1/N) = 1/N$. Thus, $\rho$-diam$(S) < \varepsilon$ and $x_0 \notin [K]'_\varepsilon$. By iteration, we get that $[K]^N_\varepsilon \subset F_N$ and hence $[K]^{N+1}_\varepsilon = \emptyset$. Therefore, $Dz_\rho(K,\varepsilon) < \omega$ for each $\varepsilon > 0$.

Now suppose that $Dz_\rho(K) \leq \omega_1$. Notice that indeed $Dz_\rho(K) < \omega_1$. By Corollary 2.9, it suffices to show that $K$ has $(*)$ with slices. For each $n \in \mathbb{N}$ and $\alpha < Dz_\rho(K, 1/n)$ consider the family

$$\mathcal{U}_{n,\alpha} = \Big\{S : S \text{ slice of } K, [K]^{\alpha+1}_{1/n} \cap S = \emptyset, \rho\text{-diam}([K]^\alpha_{1/n} \cap S) < 1/n\Big\}\,.$$

Given distinct $x, y \in K$, take $n$ so that $\rho(x,y) > 1/n$ and let $\alpha$ be the least ordinal such that $\{x,y\} \cap [K]^{\alpha+1}_{1/n}$ is at most a singleton. Then it is clear that there is a slice in $\mathcal{U}_{n,\alpha}$ containing either $x$ or $y$, and no slice in $\mathcal{U}_{n,\alpha}$ contains both points. □

*Remark* 5.2. By using deep results of descriptive set theory, Lancien proved in [8] that there exists an universal function $\psi\colon [0,\omega_1) \to [0,\omega_1)$ such that $Dz_{\|\ \|}(B_{X^*}) \leq \psi(Sz_{\|\ \|}(B_{X^*}))$ whenever $X$ is a Banach space such that $Sz_{\|\ \|}(B_{X^*}) < \omega_1$. We do not know if a similar statement holds when the norm is replaced by a symmetric.

We shall show that we cannot change symmetric by metric in Proposition 5.1. That would imply that $K$ is a Gruenhage compact, which is a strictly stronger condition that being in $\mathcal{SC}$ [18, Theorem 2.4]. By [19, Lemma 7.1 and Proposition 7.4], a compact space $K$ is Gruenhage if and only if there exist a countable set $D$, a family of closed sets $\{A_d : d \in D\}$ and families $(\mathcal{U}_d)_{d\in D}$ of open sets such that the family $\{A_d \cap U : U \in \mathcal{U}_d\}$ is pairwise disjoint for each $d \in D$ and the family $\{A_d \cap U : U \in \mathcal{U}_d\}$ separates the points of $K$.

**Proposition 5.3.** *Let $K$ be a compact space. Then the following assertions are equivalent:*

a) *$K$ is Gruenhage;*
b) *there exists a metric $d$ on $K$ such that $Sz_d(K) \leq \omega$;*
c) *there exists a metric $d$ on $K$ such that $Sz_d(K) \leq \omega_1$.*

*Proof.* If $K$ is a Gruenhage compact space, then the same construction used in the proof of [15, Theorem 2.8] provides a metric on $K$ such that $Sz_d(K) \leq \omega$.

Now assume that $d$ is a metric on $K$ with countable Szlenk index. Let $\mathcal{B} = \bigcup_{m\in\mathbb{N}} \mathcal{B}_m$ be a basis of the metric topology such that every $\mathcal{B}_m$ is discrete. Consider the open sets $U^{n,\alpha}_V = \bigcup\big\{U : U \text{ open}, \langle K\rangle^\alpha_{2^{-n}} \cap U \subset V\big\}$ and the families $\mathcal{U}^{n,\alpha}_m =$

$\{U_V^{n,\alpha} : V \in \mathcal{B}_m\}$. Then $\{\langle K\rangle_{2^{-n}}^{\alpha} \cap U : U \in \mathcal{U}_m^{n,\alpha}\}$ is pairwise disjoint for each $n, m \in \mathbb{N}$ and $\alpha < Dz(K, 2^{-n})$. Given distinct $x, y \in K$ take $V \in \mathcal{B}_m$ such that $x \in V$ and $y \notin V$. Fix $n$ such that $B_d(x, 2^{-n+1}) \subset V$. Let $\alpha$ be the least ordinal so that $x \notin \langle K\rangle_{2^{-n}}^{\alpha+1}$. Then there is an open subset $U$ of $K$ such that $x \in \langle K\rangle_{2^{-n}}^{\alpha} \cap U$ and $\mathrm{diam}(\langle K\rangle_{2^{-n}}^{\alpha} \cap U) \leq 2^{-n}$. Thus $x \in \langle K\rangle_{2^{-n}}^{\alpha} \cap U_V^{n,\alpha} \subset V$, so $y \notin \langle K\rangle_{2^{-n}}^{\alpha} \cap U_V^{n,\alpha}$. □

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