Subspaces of Hilbert-generated Banach spaces and the quantification of super weak compactness[1]

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Abstract

We introduce a measure of super weak noncompactness Γ defined for bounded subsets and bounded linear operators in Banach spaces that allows to state and prove a characterization of the Banach spaces which are subspaces of a Hilbert-generated space. The use of super weak compactness and Γ casts light on the structure of these Banach spaces and complements the work of Argyros, Fabian, Farmaki, Godefroy, Hájek, Montesinos, Troyanski and Zizler on this subject. A particular kind of relatively super weakly compact sets, namely uniformly weakly null sets, plays an important role and exhibits connections with Banach-Saks type properties.

Keywords: super weak compactness, uniformly weakly null sets, Hilbert-generated spaces, uniformly Eberlein compact sets.

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1. Introduction

Along the paper X is a real Banach space, its unit ball is denoted B_X and X^* stands for the dual. In general, our notation is quite standard and the knowledge requirements minimal, however we can address the reader to [21, 32] for any unexplained notation or concept. Ultrapowers are a powerful tool to provide brief equivalent definitions of the main notions here (see [29] for an account of that method in Banach space theory). Here we will consider only ultrafilters on $\mathbb{N}$, although the theory is much richer allowing arbitrary cardinals. Given a free ultrafilter $\mathcal{U}$, recall that $X^{\mathcal{U}}$ is the quotient of $\ell_\infty(X)$ by the subspace of those $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n,\mathcal{U}} \|x_n\| = 0$. A Banach space is said to be *super-reflexive* if for some

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[1]Dedicated to our friend Gilles Godefroy with admiration and gratitude.

(or, equivalently, any) nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, its ultrapower $X^{\mathcal{U}}$ is reflexive. The most representative results on super-reflexive Banach spaces are James' characterizations [31], Enflo's uniformly convex renorming [17] and Pisier's applications to Banach valued martingales and renormings with power type modulus [38]. See the books [7, 21] for an account on the theory of super-reflexive spaces.

A localized version of super-reflexivity was introduced by the second named author in [39] for convex sets (and, somehow more generally, for non-linear maps) with the name of finitely dentable sets. The more natural name super weakly compactness was introduced in [11]. Given a bounded set $A \subset X$ we will denote $A^{\mathcal{U}}$ the subset of $X^{\mathcal{U}}$ whose elements have a representative in $A^{\mathbb{N}}$. A set $A \subset X$ is said to be *relatively super weakly compact* (relatively SWC) if $A^{\mathcal{U}}$ is a relatively weakly compact subset of $X^{\mathcal{U}}$ for some (or, equivalently, any) free ultrafilter $\mathcal{U}$. Moreover, $A \subset X$ is said to be *super weakly compact* (SWC, of course) if it is relatively super weakly compact and weakly closed. The class of SWC sets lies strictly between the norm compact and the weakly compact subsets. The theory of SWC sets has been developed during the last 15 years in a series of papers [39, 11, 12, 40, 45, 13, 42, 43, 34, 27].

Super weak compactness is more widespread than it may appear and some results in Banach space theory could be understood in terms of hidden super weak compactness. For instance, any weakly compact subset of $L_1(\mu)$ (μ any measure) is super weakly compact, see [34, Proposition 6.1]. The classic Szlenk result establishing that a weakly convergent sequence in $L_1(\mu)$ has a subsequence whose *Cesàro means* converge (to the same limit) is a consequence of two facts: the weakly compact subsets of $L_1(\mu)$ are SWC; and the SWC sets have the *Banach-Saks property* [34, Corollary 6.3].

Recall that Banach space is said *weakly compactly generated* (WCG) if it that contains a weakly compact subset whose linear span is dense. Thanks to the celebrated interpolation result of Davis, Figiel, Johnson and Pełczyński [14] (see also [21, Theorem 13.22]), a Banach space X is WCG if and only if there exists a reflexive space Z and an operator $T : Z \to X$ with dense range. Moreover, if the space Z can be taken a Hilbert space, we say that X is *Hilbert-generated*. The name *Eberlein* applies to the compact spaces that are homeomorphic to a weakly compact set of a Banach space. It is well known after Amir and Lindenstrauss (see [21, Corollary 13.17], for instance) that an Eberlein compact embeds as a weakly (equivalent, bounded and pointwise) compact subset of $c_0(I)$ for I large enough. If such an embedding can be done into a Hilbert space $\ell_2(I)$, then the compact is said to be *uniformly Eberlein*.

The aim of this paper is to show that, actually, super weak compactness and, particularly, its quantification, may cast light on the structure of the subspaces of Hilbert-generated Banach spaces. Indeed, we have realized that several "technical hypotheses" in papers of Troyanski [41], Argyros and Farmaki [3], and the series by Fabian, Godefroy, Hájek, Montesinos and Zizler [24, 20, 19, 23] on the structure of Hilbert-generated spaces and uniformly Gâteaux renorming, can be understood in terms of a quantified version of super weak compactness. For a better comprehension of our main result, we will state firstly the "non uniform version" with the help of a measure of weak noncompactness. Let $A \subset X$ be a bounded set, then take

$$\gamma(A) = \inf\{\varepsilon > 0 : \overline{A}^{w^*} \subset X + \varepsilon B_{X^{**}}\}.$$

We have that a set A is relatively weakly compact if and only if $\gamma(A) = 0$. This measure has been studied in [22, 26, 9], see also [30, Section 3.6], and there are several measures of weak noncompactness that turn out to be equivalent [1].

Theorem 1.1 ([8, 25])**.** *For a Banach space X the following statements are equivalent:*

(i) X is a subspace of a WCG space;

(ii) (B_{X^*}, w^*) is an Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets (A_n^ε) such that $B_X = \bigcup_{n=1}^{\infty} A_n^\varepsilon$ and $\gamma(A_n^\varepsilon) < \varepsilon$.

The equivalence (i)$\Leftrightarrow$(ii) is due to Benyamini, Rudin and Wage [8]. The inner characterization (iii) was obtained by Fabian, Montesinos and Zizler [25]. Note that the third statement in Theorem 1.1 is actually an internal characterization as it is written in terms of the space X, not an over-space or its dual. A different matter is if the computation of γ requieres the use of the over-space X^{**}, as we will see later there are equivalent definitions of γ that does not appeal to the bidual space.

Let us prepare the way to state the uniform analogue of Theorem 1.1. We will requiere the following *measure of super weak noncompactness*: for a bounded set $A \subset X$ take

$$\Gamma(A) := \gamma(A^{\mathcal{U}})$$

where $\mathcal{U}$ is a free ultrafilter and γ is computed in $X^{\mathcal{U}}$. Later we will see that Γ does not depend, essentially, on the choice of the ultrafilter $\mathcal{U}$. Obviously, A is relatively SWC if and only if $\Gamma(A) = 0$. Now we are ready to state our main result. Please, note the parallelism with Theorem 1.1.

Theorem 1.2. *For a Banach space X the following statements are equivalent:*

(i) X is a subspace of a Hilbert-generated space;

(ii) (B_{X^*}, w^*) is a uniform Eberlein compact;

(iii) For every $\varepsilon > 0$ there are sets (A_n^ε) such that $B_X = \bigcup_{n=1}^\infty A_n^\varepsilon$ and $\Gamma(A_n^\varepsilon) < \varepsilon$.

Again, the equivalence (i)⇔(ii) goes back to Benyamini, Rudin and Wage [8]. Subspaces of a Hilbert-generated spaces have been investigated in a series of papers [24, 20, 19, 23] (see also [30]), where there are many more characterizations that we do not include in order to stress the analogy with Theorem 1.1. Also, in relation with our Theorem 1.2, we will prove that we can change B_X in (iii) by any linearly dense subset of X, see Theorem 5.2. Also, in order to apply statement (iii) is quite relevant the fact that Γ can be computed in several fashions, some of them without ultrapowers neither over-spaces, see Proposition 3.2.

Previous works on uniformly Gâteaux renorming by Fabian, Godefroy, Hájek and Zizler [19], as well as early results by Troyanski [41], unawarely contain estimations of Γ. The explanation will come through the following result.

Proposition 1.3. *Let $A \subset X$ a bounded subset and consider the two following numbers:*

(ε_1) *is the infimum of the $\varepsilon > 0$ such that there is $n_1 \in \mathbb{N}$ such that for every $x^* \in B_{X^*}$ then*

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n_1;$$

(ε_2) *is the infimum of the $\varepsilon > 0$ such that there is $n_2 \in \mathbb{N}$ such that for any finite set $B \subset A$ with $|B| \geq n_2$ then*

$$\left\| \frac{1}{|B|} \sum_{x \in B} x \right\| < \varepsilon.$$

Then $\varepsilon_1 = \varepsilon_2$ and in such a case $\Gamma(A) \leq \varepsilon_1$.

The sets satisfying the statements of Proposition 1.3 with $\varepsilon_1 = \varepsilon_2 = 0$ will be called *uniformly weakly null* sets. Note that a uniformly weakly null set becomes SWC by adding $\{0\}$. Together with unit balls of super-reflexive spaces, uniformly weakly null sets are the most prototypical examples of SWC sets. It can be shown that an unconditional Schauder basis that does not behave as the basis of ℓ_1 (in a very precise sense) is a uniformly weakly null set. Note that the second statement (ε_2) is a sort of uniform Banach-Saks property (with unique limit 0). That will allow us to apply results of infinite combinatorics, such as the Erdös-Magidor [18] and Mercourakis [36] selections.

The structure of the paper is as follows. Section 2 deals with uniformly weakly null sets and the proof of Proposition 1.3. There, we show that the Eberlein-Šmulian theorem fails for super weak compactness. In section 3, we study the properties of the measure Γ and we show several different ways to compute or estimate it. Our results depends on some equivalent forms for γ, that may be of independent interest. Section 4 deals with the application of Γ to operators between Banach spaces. Particularly, we will prove quantified versions of some properties enjoyed by the *super weakly compact* operators. Section 5 is devoted to the proof of Theorem 1.2, actually a more general version.

2. Uniformly weakly null sets

Let us recall that a subset $A \subset X$ is *uniformly weakly null* if for every $\varepsilon > 0$ there is $n(\varepsilon) \in \mathbb{N}$ such that, for every $x^* \in B_{X^*}$,

$$|\{x \in A : |x^*(x)| > \varepsilon\}| \leq n(\varepsilon).$$

Note that any sequence made of different points of a uniformly weakly null set is a weakly null sequence. Therefore, uniformly weakly null sets are relatively weakly compact (and become weakly compact just by adding 0). We have something better.

Theorem 2.1. *Let $A \subset X$ be a uniformly weakly null set and let $\mathcal{U}$ be any free ultrafilter. Then $A^{\mathcal{U}}$ is uniformly weakly null in $X^{\mathcal{U}}$ and, therefore, A is relatively super weakly compact in X.*

Proof. Let $\overline{x}_1, \dots, \overline{x}_n \in A^{\mathcal{U}}$ be different vectors, $\overline{x}^* \in B_{(X^{\mathcal{U}})^*}$ and $\varepsilon > 0$ such that $|\overline{x}^*(\overline{x}_k)| > \varepsilon$ for every $1 \leq k \leq n$. We claim that for $\varepsilon' < \varepsilon$, there are different elements $x_1, \dots, x_n \in A$ and $x^* \in B_{X^*}$ with $|x^*(x_k)| > \varepsilon'$. Indeed, the proof that $X^{\mathcal{U}}$ is finitely representable in X (see [7, p. 222] for instance), provides those $x_1, \dots, x_n \in X$ in such a way that $Z = \operatorname{span}\{\overline{x}_1, \dots, \overline{x}_n\}$ and $Y = \operatorname{span}\{x_1, \dots, x_n\}$ are ε/ε'-isomorphic. Moreover, the vector x_k is found on the "coordinates" of $\overline{x}_k$, so we may assume $x_k \in A$ for all k. Then $T : Y \to Z$ be the isomorphism. Let x^* be the Hahn-Banach extension of $(\varepsilon'/\varepsilon)\overline{x}^* \circ T$.

Then, $x^* \in B_{X^*}$ and $|x^*(x_k)| > \varepsilon'$ for all $1 \leq k \leq n$ as desired. That claim shows that $A^{\mathcal{U}}$ have to be uniformly weakly null. Now we have $A^{\mathcal{U}}$ is weakly compact in $X^{\mathcal{U}}$ and thus A is SWC. ■

A sequence (x_n) that is a uniformly weakly null set is called a *uniformly weakly null sequence*. A sequence (x_n) is *uniformly weakly convergent* to x if $(x_n - x)$ is a uniformly weakly null sequence. The fact that a uniformly weakly convergent sequence together its limit is a super weakly compact set was noted in [13]. Uniformly weakly convergent sequences are closely related to the Banach-Saks property. A sequence (x_n) is said to be *Cesàro convergent* if the sequence of its arithmetic means

$$n^{-1}\sum_{k=1}^{n} x_k$$

converges (in norm) to some $x \in X$. A set $A \subset X$ is said to have the *Banach-Saks property* if every sequence $(x_n) \subset A$ has a Cesàro convergent subsequence. Recall that relatively SWC sets are Banach-Saks [34, Corollary 2.4]. The relations between both properties is an interesting topic, although we will not deal here with the Banach-Saks property in general. Let us introduce the following "ephemeral" definition. A set $A \subset X$ is said to be *uniformly Banach-Saks null* if for every $\varepsilon > 0$ there is $n(\varepsilon)$ such that whenever $B \subset A$ is finite with $|B| \geq n(\varepsilon)$ then

$$|B|^{-1}\left\|\sum_{x\in B} x\right\| < \varepsilon.$$

Proposition 1.3 has the following consequence.

Corollary 2.2. *Let $A \subset X$ be a bounded subset. Then A is uniformly Banach-Saks null if and only if it is uniformly weakly null.*

Proof of Proposition 1.3. Let $r > 0$ such that $A \subset rB_X$. Take $\varepsilon > \varepsilon_1$ and fix the corresponding number n_1. For $n > n_1$ and any $B \subset A$ with $|B| = n$ we have

$$|x^*(\sum_{x\in B} x)| < n_1 r + (n - n_1)\varepsilon$$

for every $x^* \in B_{X^*}$. Therefore

$$n^{-1}\left\|\sum_{x\in B} x\right\| < \frac{n_1 r}{n} + \left(1 - \frac{n_1}{n}\right)\varepsilon$$

Since the bound can be taken arbitrarily closed to ε independently from B if n is large enough, we have that $\varepsilon_2 \leq \varepsilon_1$. That proves the equality $\varepsilon_1 = \varepsilon_2$ in case $\varepsilon_1 = 0$. Assume now that $\varepsilon_1 > 0$ and take

$0 < \varepsilon < \varepsilon_1$. Then, for every $n \in \mathbb{N}$ we can find $C \subset A$ with $|C| = 2n$ and $x^* \in B_{X^*}$ such that $x^*(x) > \varepsilon$ or $x^*(x) < -\varepsilon$ for all $x \in C$. Since at least one half of the elements satisfies the same inequality, we may find $B \subset C$ such that $|B| = n$ and

$$|x^*(\sum_{x \in B} x)| > n\varepsilon.$$

Therefore, we have

$$n^{-1} \left\| \sum_{x \in B} x \right\| > \varepsilon,$$

that implies $\varepsilon_2 \geq \varepsilon_1$. Now, note that the first statement implies

$$\overline{A}^{w^*} \subset A \cup \varepsilon_1 B_{X^{**}} \subset X + \varepsilon_1 B_{X^{**}}$$

and so $\gamma(A) \leq \varepsilon_1$. In order to pass to Γ, just follow the ideas in the proof of Theorem 2.1 or check that the property of the second statement is stable by ultrapowers. In any case, we get that $\Gamma(A) \leq \varepsilon_1$. ■

Remark 2.3. *The proof of the equivalence shows that it is enough to check condition (ε_1) for x^* from a norming subset of B_{X^*}.*

Mercourakis [36] improvement of the Erdös-Magidor [18] dichotomy for bounded sequences can be stated in this way (see also [35] for related results and references).

Theorem 2.4 (Mercourakis)**.** *Let $(x_n) \subset X$ a bounded sequence. Then there exists a subsequence (x_{n_k}) of (x_n) for which one of the following statements holds:*

(i) either, (x_{n_k}) is uniformly weakly convergent;

(ii) or, no subsequence of (x_{n_k}) is Cesàro convergent.

The celebrated Eberlein-Šmulian theorem, see [21] for instance, says that weak compactness is determined by sequences. As an application, we get that there is no Eberlein-Šmulian for super weak compactness. That is, the fact that every sequence has a relatively SWC subsequence does not imply that the set is relatively SWC.

Corollary 2.5. *Let $A \subset X$ be a relatively super weakly compact set. Then every sequence $(x_n) \subset A$ contains a uniformly weakly convergent subsequence. However, this property does not characterize the super weakly compactness. Actually, it characterizes the Banach-Saks property.*

Proof. For a Banach-Saks set the dichotomy 2.4 always produces a uniformly weakly convergent subsequence. On the other hand, every uniformly convergent sequence is Cesàro convergent. Therefore, the Banach-Saks property is characterized by sequences. The other statements follow from the fact that relatively SWC sets are Banach-Saks and there exist Banach-Saks sets which are not relatively SWC [34, Corollary 2.5]. ■

3. Different ways to quantify SWC

Measures of noncompactness can be defined in very general settings. Here we will restrict ourselves to the frame of topological vector spaces. Let X be a topological vector space and let $\mathfrak{K}$ be a vector bornology of compact subsets (that just means the class is stable under some elementary operations). A *measure of noncompactness* associated to $\mathfrak{K}$ is a nonnegative function μ defined on the bounded subsets of X that satisfies the following properties:

1. $\mu(\overline{A}) = \mu(A)$
2. $\mu(A) = 0$ if and only if $\overline{A} \in \mathfrak{K}$
3. $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$
4. $\mu(\lambda A) = |\lambda|\mu(A)$ for all $\lambda \in \mathbb{R}$
5. $\mu(A + B) \leq \mu(A) + \mu(B)$
6. there exists $k > 0$ such that $\mu(\text{conv}(A)) \leq k\,\mu(A)$

This list of conditions comes from the usual requirements in literature [2] and some properties enjoyed by several measures that are interesting for Banach space geometry, such as γ or the family of measures introduced in [33] in relation with the Szlenk index. Condition (6) is usually the most tricky and necessarily requieres that the class $\mathfrak{K}$ be stable by closed convex hulls (Krein-type theorem).

The quantification of the super weak non-compactness is linked to the quantification of the weak non-compactness. De Blasi (see [1], for instance) introduced a measure of weak noncompactness ω as follows

$$\omega(A) = \inf\{\varepsilon > 0 : \exists K \subset X \text{ weakly compact}, A \subset K + \varepsilon B_X\}.$$

It is not hard to check that ω enjoys all the properties above. In particular, we have

$$\omega(\text{conv}(A)) = \omega(A),$$

that is, its "convexifiability constant" is 1. Another quite natural way to measure weak noncompactness, is the function γ mentioned in the introduction

$$\gamma(A)=\inf\{\varepsilon>0:\overline{A}^{w^*}\subset X+\varepsilon B_{X^{**}}\}=\sup\{d(X,x^{**}):x^{**}\in\overline{A}^{w^*}\}.$$

It is easy to check that $\gamma(A)\leq\omega(A)$ for any bounded set $A\subset X$. However, there is no constant $c>0$ such that $\omega(A)\leq c\,\gamma(A)$ in general, see [1, Corollary 3.4]. That fact says that ω and γ are not *equivalent*. The measure γ was introduced in [22] where the authors also proved ([26] independently, see also [30, Theorem 3.64]) that

$$\gamma(\mathrm{conv}(A))\leq 2\,\gamma(A)$$

for any bounded $A\subset X$. Notably, there are many different equivalent ways to deal with γ which are interesting to us because they have a "super" version. The following contains the quantified version of two classic James' characterizations of relative weak compactness together with the quantified version of Grothendieck's commutation of limits criterion.

Proposition 3.1. *Let $A\subset X$ be a bounded set. Consider the following numbers:*

(γ_1) $=\gamma(A)$;

(γ_2) *the supremum of the numbers $\varepsilon\geq 0$ such that there are sequences $(x_n)\subset A$ and $(x_n^*)\subset B_{X^*}$ such that $x_n^*(x_m)=0$ if $m<n$ and $x_n^*(x_m)\geq\varepsilon$ if $m\geq n$;*

(γ_3) *the supremum of the numbers $\varepsilon>0$ such that there exists a sequence $(x_n)\subset C$ such that*

$$\mathrm{d}(\mathrm{conv}\{x_1,\dots,x_n\},\mathrm{conv}\{x_{n+1},x_{n+2},\dots\})\geq\varepsilon$$

for all $n\in\mathbb{N}$;

(γ_4) *the infimum of the numbers $\varepsilon\geq 0$ such that*

$$|\lim_n\lim_m x_n^*(x_m)-\lim_m\lim_n x_n^*(x_m)|\leq\varepsilon$$

whenever $(x_n)\subset A$, $(x_n^*)\subset B_{X^*}$ and the iterated limits exist.

Then $\gamma_1\leq\gamma_2\leq\gamma_3\leq\gamma_4\leq 2\gamma_1$.

Proof. Take $\varepsilon < \gamma(A)$ and let $x^{**} \in \overline{A}^{w^*}$ with $d(X, x^{**}) > \varepsilon$. We will build sequences satisfying the second statement for such an ε. Indeed, there exists $x_1^* \in B_{X^*}$ with $|x^{**}(x_1^*)| > \varepsilon$. Now take $x_1 \in A$ such that $x_1^*(x_1) \geq \varepsilon$. Assume we have x_k and x_k^* already built for $1 \leq k < n$ and it is satisfied $x^{**}(x_k^*) > \varepsilon$. An elementary application of Helly's theorem [21, p. 159] to X^{**} allows us to choose $x_n^* \in B_{X^*}$ such that $x_n^*(x_k) = 0$ for $1 \leq k < n$ and $x^{**}(x_n^*) > \varepsilon$. Now we take

$$x_n \in A \cap \{x \in X : x_k^*(x) > \varepsilon, 1 \leq k \leq n\}$$

since the set is nonempty. That finishes the construction of the sequence and proves $\gamma_1 \leq \gamma_2$. The inequality $\gamma_2 \leq \gamma_3$ follows straight. In order to prove $\gamma_3 \leq \gamma_4$, take $\varepsilon < \gamma_3$, and sequence (x_n) as in the statement (γ_3). For every $n \in \mathbb{N}$, take $x_n^* \in B_{X^*}$ such that $x_n^*(y) \leq \varepsilon + x_n^*(z)$ for every $y \in \operatorname{conv}\{x_1, \dots, x_n\}$ and $z \in \operatorname{conv}\{x_{n+1}, x_{n+2}, \dots\}$. The sequences satisfies the following property

$$x_n^*(x_p) \leq \varepsilon + x_n^*(x_q)$$

whenever $p \leq n < q$. Passing to a subsequence, we may assume the existence of the limits $\lim_n x_n^*(x_m)$ and $\lim_m x_n^*(x_m)$, as well as the existence of the iterated limits. In such a case we will get

$$\lim_m \lim_n x_n^*(x_m) \leq \varepsilon + \lim_n \lim_m x_n^*(x_m)$$

which implies $\varepsilon \leq \gamma_4$, and therefore $\gamma_3 \leq \gamma_4$. Finally, $\gamma_4 \leq 2\gamma_1$ is proved in [1]. ■

Now we will state the "super" version of Proposition 3.1, for which we prefer to avoid a uniform version of *Grothendieck's commutation of limits* (fourth statement).

Proposition 3.2. *Let $A \subset X$. Consider the following numbers:*

(Γ_1) $= \gamma(A^{\mathcal{U}})$ *measured in $X^{\mathcal{U}}$ for $\mathcal{U}$ a free ultrafilter;*

(Γ_2) *the infimum of the numbers $\varepsilon > 0$ such that there are no arbitrarily long sequences $(x_k)_1^n \subset A$, $(x_k^*)_1^n \subset B_{X^*}$ with $x_k^*(x_j) = 0$ if $j < k$ and $x_k^*(x_j) > \varepsilon$ if $j \geq k$;*

(Γ_3) *the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \dots, x_n \in C$ such that $\mathrm{d}(\operatorname{conv}\{x_1, \dots, x_k\}, \operatorname{conv}\{x_{k+1}, \dots, x_n\}) \geq \varepsilon$ for all $k = 1, \dots, n-1$;*

Then $\Gamma_1 \leq \Gamma_2 \leq \Gamma_3 \leq 2\Gamma_1$.

Proof. The fact $\Gamma_1 \leq \Gamma_2$ follows straight by applying finite representatibity to inequality $\varepsilon_1 \leq \varepsilon_2$ in Proposition 3.1. It is quite easy to get $\Gamma_2 \leq \Gamma_3$, and $\Gamma_3 \leq 2\Gamma_1$ follows using the standard ultrapower technique, (see also Theorem 3.4 below where the convex case is considered). ∎

Recall that Γ_1 is the measure introduced at the introduction

$$\Gamma(A) := \gamma(A^{\mathcal{U}})$$

that depends on the choice of $\mathcal{U}$. From now on, we will assume the free ultrafilter $\mathcal{U}$ is fixed when speaking of Γ or dealing with the ultrapowers. Note that the equivalent measures Γ_2 and Γ_3 does not depend on any ultrafilter. Moreover, Γ_3 does not involves explicitly the dual space. In next section we will use $\Gamma_s(A) = \Gamma_2$ as an alternative to $\Gamma(A)$.

Proposition 3.3. *Let $T : X \to Y$ be an operator and let $A \subset X$ be a bounded set. Then* $\Gamma(T(A)) \leq \|T\| \, \Gamma(A)$.

Proof. Firstly, we will prove a similar statement for γ. Consider $T^{**} : X^{**} \to Y^{**}$ which is weak* to weak* continuous. For any bounded set $A \subset X$ we have

$$\overline{T(A)}^{w^*} = T^{**}(\overline{A}^{w^*}) \subset T(X) + \varepsilon T^{**}(B_{X^{**}}) \subset Y + \varepsilon \|T\| B_{Y^{**}}$$

where $\varepsilon > \gamma(A)$. Therefore $\gamma(T(A)) \leq \|T\| \, \gamma(A)$. In order to prove the statement for Γ, consider the induced operator $T^{\mathcal{U}} : X^{\mathcal{U}} \to Y^{\mathcal{U}}$. Then we have

$$\Gamma(T(A)) = \gamma(T^{\mathcal{U}}(A^{\mathcal{U}})) \leq \|T^{\mathcal{U}}\| \gamma(A^{\mathcal{U}}) = \|T\| \, \Gamma(A),$$

as we wished. ∎

In order to state the results from our paper [27] that we will need later, it is necessary to introduce a certain number of quantities related to sets in Banach spaces. Let us denote by $\mathbb{H}$ the set of all the *open half-spaces* of X, that is, all the sets of the form $H = \{x \in X, \ x^*(x) > \alpha\}$, with $x^* \in X^*$ and $\alpha \in \mathbb{R}$. A *slice of* of $D \subset X$ is a set of the form $D \cap H \neq \emptyset$, where $H \in \mathbb{H}$. We say that a bounded closed convex set $C \subset X$ is *dentable* if for any nonempty closed convex subset $D \subset C$ has (nonempty) slices of arbitrarily small diameter. If C is dentable we may consider the following *set derivation*:

$$[D]'_\varepsilon = \{x \in D : \operatorname{diam}(D \cap H) > \varepsilon, \ \text{for any } H \in \mathbb{H} \text{ s.t. } x \in H\}.$$

Clearly, $[D]'_\varepsilon$ is what remains of D after removing all the slices of D of diameter at most ε. Consider the sequence of sets defined by $[C]^0_\varepsilon = C$ and, for every $n \in \mathbb{N}$, inductively by

$$[C]^n_\varepsilon = [[C]^{n-1}_\varepsilon]'_\varepsilon.$$

If there is an n in $\mathbb{N}$ such that $[C]^{n-1}_\varepsilon \neq \emptyset$ and $[C]^n_\varepsilon = \emptyset$ we set $Dz(C, \varepsilon) = n$. We say that C is *finitely dentable* if $Dz(C, \varepsilon)$ is finite for every $\varepsilon > 0$. Given a convex set $C \subset X$, let us denote by $\mathrm{Dent}(C)$ the infimum of the numbers $\varepsilon > 0$ such that C has nonempty slices contained in balls of radius less than ε, and take $\Delta(C) = \sup\{\mathrm{Dent}(B) : B \subset C\}$. The measure Δ was introduced in [10] as a way to quantify the lack of *Radon-Nikodym property* (RNP). Let $\varepsilon > 0$. A function $f : X \to \overline{\mathbb{R}}$ is said to be *ε-uniformly convex* with respect to some metric d if there is $\delta > 0$ such that whenever $d(x, y) \geq \varepsilon$, then

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2} - \delta.$$

No mention to an explicit metric d means that we are using the norm metric. The function is said to be just *uniformly convex* if it is ε-uniformly convex for all $\varepsilon > 0$.

Theorem 3.4 ([27]). *Let $C \subset X$ be a bounded closed convex subset. Consider the following numbers:*

(η_1) $= \Gamma(C)$;

(η_2) *the supremum of the numbers $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there are $x_1, \ldots, x_n \in C$ such that* $\mathrm{d}(\mathrm{conv}\{x_1, \ldots, x_k\}, \mathrm{conv}\{x_{k+1}, \ldots, x_n\}) \geq \varepsilon$ *for all $k = 1, \ldots, n-1$;*

(η_3) *the supremum of the $\varepsilon > 0$ such that there are ε-separated dyadic trees in C of arbitrary height;*

(η_4) $= \Delta(C^{\mathcal{U}})$;

(η_5) *the infimum of the $\varepsilon > 0$ such that* $\mathrm{Dz}(C, \varepsilon) < \omega$;

(η_6) *the infimum of the $\varepsilon > 0$ such that C supports a convex bounded ε-uniformly convex function.*

Then $\eta_1 \leq \eta_2 \leq 2\eta_3 \leq 2\eta_4 \leq 2\eta_1$ and $\eta_4 \leq 2\eta_5 \leq 2\eta_6 \leq 2\eta_2$.

Let us finish this section by showing that Γ fulfils the all requirements for a genuine measure of noncompactness listed at the beginning.

Proposition 3.5. *The function Γ defined on bounded subsets of X has the following properties:*

1. $\Gamma(\overline{A}) = \Gamma(A)$*;*
2. $\Gamma(A) = 0$ *if and only if* $\overline{A}$ *is SWC;*
3. $\Gamma(A \cup B) = \max\{\Gamma(A), \Gamma(B)\}$*;*
4. $\Gamma(\lambda A) = |\lambda|\Gamma(A)$ *for all* $\lambda \in \mathbb{R}$*;*
5. $\Gamma(A + B) \leq \Gamma(A) + \Gamma(B)$*;*
6. $\Gamma(\operatorname{conv}(A)) \leq 4\,\Gamma(A)$*.*

Proof. (1) and (2) follow straightly from the definition of Γ. (3), (4) and (5) follow from set identities: $(A \cup B)^{\mathcal{U}} = A^{\mathcal{U}} \cup B^{\mathcal{U}}$, $(\lambda A)^{\mathcal{U}} = \lambda A^{\mathcal{U}}$ and $(A + B)^{\mathcal{U}} = A^{\mathcal{U}} + B^{\mathcal{U}}$. Statement (6), the most tricky, was proved in [27, Theorem 6.7]. ■

4. Quantifying uniform convexity for operators

In this section we will discuss the application of the measure of weak noncompactness to operators. Firstly, let us recall a few facts about super weak compactness for operators. An operator $T : X \to Y$ is said to *super weakly compact* (SWC) if the induced operator $T^{\mathcal{U}} : X^{\mathcal{U}} \to Y^{\mathcal{U}}$ is weakly compact for any ultrafilter $\mathcal{U}$ (equivalently, a free ultrafilter on $\mathbb{N}$). Note that we can think of taking ultrapowers for a fixed ultrafilter $\mathcal{U}$ as a functor on the category of Banach spaces. The definition goes back to Beauzamy [5], although he introduced it in a different but equivalent fashion, namely *uniformly convexifying operators*. The set of super weakly compact operators is an operator ideal denoted by $\mathfrak{W}^{super}$. Notably, $\mathfrak{W}^{super}$ is a symmetric ideal, that is, $T \in \mathfrak{W}^{super}$ if and only if $T^* \in \mathfrak{W}^{super}$. See [5, 6, 16, 28] for more properties of $\mathfrak{W}^{super}$ and its relation with other operator ideals, and see also [44] for characterizations in terms of martingale type and cotype. All the operators considered in this paper are supposed to be linear and bounded.

For an operator $T : X \to Y$, we will write $\Gamma(T) := \Gamma(T(B_X))$. Obviously, an operator $T : X \to Y$ is SWC if and only if $\Gamma(T(B_X)) = 0$. We have the following.

Proposition 4.1. *Let* $A \subset X$ *be a convex symmetric bounded set with* $\Gamma(A) < \varepsilon$*. Then there exists a Banach space* Z *and an operator* $T : Z \to X$ *such that* $\|T\| = 1$*,* $A \subset T(B_Z)$ *and* $\Gamma(T) < \varepsilon$*.*

Proof. Without loss of generality we may assume that A is closed. Then, just take $Z = \operatorname{span}(A)$, endow it with the norm given by the Minkowski functional of A and take T the identity operator. ■

If we consider the alternative measure of weak noncompactness Γ_s introduced after Proposition 3.2, we have the following quantified version of the symmetry of the operator ideal $\mathfrak{W}^{super}$.

Theorem 4.2. *Let $T : X \to Y$ and operator. Then $\Gamma_s(T^*) = \Gamma_s(T)$.*

Proof. We will assume firstly that $\Gamma_s(T) > 0$. Take $0 < \varepsilon < \Gamma_s(T)$. Then, for every $N \in \mathbb{N}$ there are elements $(x_n)_{n=1}^N \subset B_X$ and $(x_n^*)_{n=1}^N \subset B_{X^*}$ such that

$$\langle x_n^*, T(x_m)\rangle = 0 \text{ for } m < n,$$

$$\langle x_n^*, T(x_m)\rangle \geq \varepsilon \text{ for } m \geq n.$$

But this is exactly the same that

$$\langle T^*(x_n^*), x_m\rangle = 0 \text{ for } m < n,$$

$$\langle T^*(x_n^*), x_m\rangle \geq \varepsilon \text{ for } m \geq n.$$

By reversing the order of $1, \dots, N$, we get $\Gamma_s(T^*) \geq \varepsilon$. That gives $\Gamma_s(T^*) \geq \Gamma_s(T)$. Suppose now that $\Gamma_s(T^*) > 0$ and take $0 < \varepsilon < \Gamma_s(T^*)$. Then, for every $N \in \mathbb{N}$ there are elements $(x_n^{**})_{n=1}^N \subset B_{X^{**}}$ and $(x_n^*)_{n=1}^N \subset B_{X^*}$ such that

$$\langle x_n^{**}, T^*(x_m^*)\rangle = 0 \text{ for } m < n,$$

$$\langle x_n^{**}, T^*(x_m)\rangle \geq \varepsilon \text{ for } m \geq n.$$

Fix $\lambda > 1$. Helly's theorem [21, p. 159] allows us to find $(x_n)_{n=1}^N \subset \lambda B_X$ such that

$$\langle x_n^{**}, T^*(x_m^*)\rangle = \langle x_n, T^*(x_m^*)\rangle$$

for every $1 \leq n, m \leq N$. That implies $\Gamma_s(T) \geq \lambda^{-1}\varepsilon$, after reversing the order of $1, \dots, N$. By the arbitrarily choice of constants, we get $\Gamma_s(T) \geq \Gamma_s(T^*)$.
So far we have proved that $\Gamma_s(T) > 0$ if and only if $\Gamma_s(T^*) > 0$ and, in such a case, $\Gamma_s(T) = \Gamma_s(T^*)$. That also implies $\Gamma_s(T) = 0$ if and only if $\Gamma_s(T^*) = 0$, therefore the proof is complete. ■

Corollary 4.3. *Let $T : X \to Y$ be an operator. Then $2^{-1}\Gamma(T) \leq \Gamma(T^*) \leq 2\Gamma(T)$.*

Remark 4.4. *Using γ_2 as a measure of weak noncompactness for sets and operators, the quantified version of Gantmacher theorem [1] would become an equality.*

De Blasi's measure applied to operators does not satisfy a similar quantified Gantmacher result, as observed in [1] after an example from [4], neither does the measure on super weak noncompactness introduced by Tu [43], inspired by De Blasi's definition, as

$$\sigma(T) = \inf\{\varepsilon > 0 : \exists K \subset Y, K \text{ is SWC}, T(B_X) \subset K + \varepsilon B_Y\}$$

Indeed, Tu shows provides a sequence of operators T_n such that and $\sigma(T_n^*) = 1$ for all $n \in \mathbb{N}$ and $\lim_n \sigma(T_n) = 0$.

Now we will consider a notion of uniform convexity for operators. In order to make notation shorter, for a convex function f we will write

$$\Delta_f(x, y) = \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right).$$

An operator $T : X \to Y$ is called uniformly convex if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in B_X$ are such that $\Delta_{\|\cdot\|^2}(x, y) < \delta$. An operator $T : X \to Y$ is called uniformly convexifying if it becomes uniformly convex after a suitable renorming of X. It turns out that the class of uniformly convexifying operators agrees with $\mathfrak{W}^{super}$.

We will say that T is ε-uniformly convex (ε-UC) if there is $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in B_X$ are such that $\Delta_{\|\cdot\|^2}(x, y) < \delta$. The following result contains two alternative forms of the ε-UC property that we will need later.

Lemma 4.5. *For an operator $T : X \to Y$ and $\varepsilon > 0$, the following statements are equivalent:*

(i) T is ε-UC;

(ii) $\limsup_n \|T(x_n) - T(y_n)\| \leq \varepsilon$ whenever $x_n, y_n \in B_X$ are such

$$\lim_n \Delta_{\|\cdot\|^2}(x_n, y_n) = 0;$$

(iii) there is $\delta > 0$ such that $\|T(x) - T(y)\| \leq \varepsilon$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| > 2(1 - \delta)$.

Proof. The proof is left to the reader. ■

For the construction of a quantified uniformly convex norm we will use this result.

Theorem 4.6 ([27])**.** *Let $(X,\|\cdot\|)$ be a Banach space, let $f : X \to [0,+\infty]$ be a proper convex function and let $C \subset \operatorname{dom}(f)$ be a bounded convex set. Assume f is Lipschitz on C. Then given $\delta > 0$ there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X and $\zeta > 0$ such that $\Delta_f(x,y) < \delta$ whenever $x,y \in C$ satisfy $\Delta_{|\!|\!|\cdot|\!|\!|^2}(x,y) < \zeta$. Therefore, if f was moreover ε-uniformly convex for some $\varepsilon > 0$ (with respect to a pseudo-metric) on C, then $|\!|\!|\cdot|\!|\!|^2$ would be ε-uniformly convex on C (with respect to the same pseudo-metric).*

We are ready to prove the quantified Beauzamy's renorming result.

Theorem 4.7. *Let $(X,\|\cdot\|)$ be a Banach space, and let $T : X \to Y$ be an operator such that $\Gamma(T) < \varepsilon$. Then there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that $|\!|\!|\cdot|\!|\!| \le \|\cdot\|$ and such that T is ε-UC on $(X,|\!|\!|\cdot|\!|\!|)$. Moreover, in case X and Y are dual Banach spaces and T is an adjoint operator, then the norm $|\!|\!|\cdot|\!|\!|$ making T is ε-UC can be taken to be a dual one.*

Proof. Take $\Gamma(T) = \varepsilon' < \varepsilon$ and $1 < \lambda < \varepsilon/\varepsilon'$. By Theorem 3.4, the set $B = \lambda\overline{T(B_X)}$ supports a convex bounded ε-uniformly convex function f that we may assume also Lipschitz, see [27, Proposition 5.4]. The function $f \circ T$ is ε-uniformly convex with respect to the pseudo-metric $d(x,y) = \|T(x) - T(y)\|$ on λB_X. By Theorem 4.6, there is an equivalent norm $\|\cdot\|_u$ on X such that $\|\cdot\|_u^2$ is ε-uniformly convex with respect to d on the set λB_X. All the norms defined by the formula

$$|\!|\!|\cdot|\!|\!|^2 = \lambda^{-2}\|\cdot\|^2 + \xi\|\cdot\|_u^2$$

are ε-uniformly convex with respect to d on the set λB_X. By taking $\xi > 0$ small enough we may assume that

$$\lambda^{-1}\|\cdot\| \le |\!|\!|\cdot|\!|\!| \le \|\cdot\|.$$

Since the unit ball of $|\!|\!|\cdot|\!|\!|$ contains λB_X, we get that T becomes ε-UC when X is endowed with $|\!|\!|\cdot|\!|\!|$.

Assume now that X and Y are dual spaces and T is an adjoint operator, and therefore it is weak* to weak* continuous. By the first part, we may assume that X is already endowed with a (non dual) norm such that T is ε-UC. We claim that the norm $|\!|\!|.|\!|\!|$ on X having $\overline{B_X}^{w^*}$ as the unit ball makes T ε-UC too. By Lemma 4.5 there is $\delta > 0$ such that $x,y \in B_X$ and $\|x+y\| > 2(1-\delta)$ implies $\|T(x) - T(y)\| \le \varepsilon$. Therefore, $\operatorname{diam}(T(H \cap B_X)) \le \varepsilon$ whenever H is a halfspace such that $H \cap (1-\delta)B_X = \emptyset$. Take $x,y \in X$ with $|\!|\!|x|\!|\!| = |\!|\!|y|\!|\!| = 1$ and $|\!|\!|x+y|\!|\!| > 2(1-\delta/2)$. Note that the condition implies that the segment $[x,y]$ does not meet $(1-\delta)\overline{B_X}^{w^*}$ Take H a weak*-open halfspace such that $[x,y] \subset H$ and $H \cap (1-\delta)\overline{B_X}^{w^*} = \emptyset$. By the weak* to weak*-continuity of T we have

$$T(H \cap \overline{B_X}^{w^*}) \subset \overline{T(H \cap B_X)}^{w^*}.$$

As $\mathrm{diam}(\overline{T(H\cap B_X)}^{w^*}) = \mathrm{diam}(T(H\cap B_X)) \leq \varepsilon$ by the weak* semicontinuity of the norm of Y and the previous observation, we get that $\|T(x)-T(y)\| \leq \varepsilon$ as wished. ■

5. Proof of the main result and final remarks

The norm of the Banach space $(X, \|\cdot\|)$ is said to be *uniformly Gateaux smooth* if for every $h \in X$

$$\sup\{\|x+th\| + \|x-th\| - 2 : x \in S_X\} = o(t) \text{ when } t \to 0.$$

It is well known [15, Theorem 6.7] that the norm on X is uniformly Gâteaux smooth if and only if the dual norm on X^* is weak* uniformly rotund (W*UR), that is, weak*-$\lim_n(x_n^* - y_n^*) = 0$ whenever $x_n^*, y_n^* \in B_{X^*}$ are such that $\lim_n \Delta_{\|\cdot\|^2}(x_n^*, y_n^*) = 0$.

Lemma 5.1. *Let $A \subset X$ be a subset and let $\varepsilon > 0$. Assume that $A = \bigcup_{k=1}^\infty A_k$ with A_k bounded and $\Gamma(A_k) < \varepsilon$ for every $k \in \mathbb{N}$. Then, there exists an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that the dual norm on X has the following property: whenever $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{|\!|\!|\cdot|\!|\!|^2}(x_n^*, y_n^*) = 0$, then*

$$\limsup_n |x_n^*(x) - y_n^*(x)| \leq 8\varepsilon$$

for every $x \in A$.

Proof. Let B_k be the symmetric convex hull of A_k. By Proposition 3.5, we have $\Gamma(B_k) < 4\varepsilon$. Let $T_k : Z_k \to X$ the operator given by Proposition 4.1 such that $\Gamma(T_k) < 4\varepsilon$ and $A_k \subset B_k \subset T_k(B_{Z_k})$. Now, by Corollary 4.3 $\Gamma(T_k^*) < 8\varepsilon$, and, by Theorem 4.7, T_k^* became 8ε-UC with an equivalent dual norm $\|\cdot\|_k \leq \|\cdot\|$. Consider the equivalent dual norm on X^* defined by the formula

$$|\!|\!|\cdot|\!|\!|^2 = \sum_{k=1}^\infty 2^{-k}\|\cdot\|_k^2.$$

Suppose given $(x_n^*), (y_n^*) \subset B_{X^*}$ with $\lim_n \Delta_{|\!|\!|\cdot|\!|\!|^2}(x_n^*, y_n^*) = 0$. Then, for every $k \in \mathbb{N}$, we have $\lim_n \Delta_{\|\cdot\|_k^2}(x_n^*, y_n^*) = 0$ and therefore $\limsup_n \|T_k^*(x_n^*) - T_k^*(y_n^*)\| \leq 8\varepsilon$ on Z_k^*. In particular, for every $z \in Z_k$, we get

$$\limsup_n |\langle T_k(z), x_n^*\rangle - \langle T_k(z), y_n^*\rangle| = \limsup_n |\langle z, T_k^*(x_n^*)\rangle - \langle z, T_k^*(y_n^*)\rangle| \leq 8\varepsilon.$$

Having in mind that $A_k \subset T(B_{Z_k})$, we obtain $\limsup_n |x_n^*(x) - y_n^*(x)| \leq 8\varepsilon$ for every $x \in A_k$. Since this is true for every $k \in \mathbb{N}$, the lemma is proved. ■

By [8], the following statement is equivalent to Theorem 1.2.

Theorem 5.2. *Let X be a Banach space. The following statements are equivalent:*

(i) X is a subspace of a Hilbert-generated space;

(ii) For every $\varepsilon > 0$ there are sets (B_n^ε) such that $B_X = \bigcup_{n=1}^\infty B_n^\varepsilon$ and $\Gamma(B_n^\varepsilon) < \varepsilon$;

(iii) There exists a linearly dense set $A \subset X$ such that for every $\varepsilon > 0$ it can be decomposed as $A = \bigcup_{n=1}^\infty A_n^\varepsilon$ where each A_n^ε is bounded and $\Gamma(A_n^\varepsilon) < \varepsilon$;

(iv) X admits an equivalent uniformly Gâteaux norm.

Proof. (i)⇒(ii) It is enough to prove statement (ii) for a Hilbert-generated space since that property is clearly inherited by subspaces. Let H be a Hilbert space and $T : H \to X$ an operator with dense range. For every $0 < \varepsilon' < \varepsilon$ we have

$$B_X \subset \bigcup_{n=1}^{\infty} (nT(B_H) + \varepsilon' B_X).$$

We have $\Gamma(nT(B_H) + \varepsilon' B_X) \le \varepsilon'$ and we can take $B_n^\varepsilon = B_X \cap (nT(B_H) + \varepsilon' B_X)$.
(ii)⇒(iii) It is obvious.
(iii)⇒(iv) By Lemma 5.1, for every $k \in \mathbb{N}$ there exists an equivalent dual norm $\|\cdot\|_k$ on X^* such that: whenever $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{\|\cdot\|_k^2}(x_n^*, y_n^*) = 0$, then

$$\limsup_n |x_n^*(x) - y_n^*(x)| \le 1/k$$

for every $x \in A$. The dual norm defined by

$$|\!|\!|\cdot|\!|\!|^2 = \sum_{k=1}^{\infty} 2^{-k} \|\cdot\|_k^2$$

satisfies then $\limsup_n |x_n^*(x) - y_n^*(x)| = 0$ whenever $x \in \operatorname{span}(A)$ and $(x_n^*), (y_n^*) \subset B_{X^*}$ are such that $\lim_n \Delta_{|\!|\!|\cdot|\!|\!|^2}(x_n^*, y_n^*) = 0$. As the sequences $(x_n^*), (y_n^*)$ are bounded and $\operatorname{span}(A)$ is dense, we have $\limsup_n |x_n^*(x) - y_n^*(x)| = 0$ for every $x \in X$. Therefore, the norm $|\!|\!|\cdot|\!|\!|$ is W*UR and its predual norm on X is uniformly Gâteaux.
(iv)⇔(i) It was proved in [20] (see also [30, Theorem 6.30]). ■

The result of Fabian, Godefroy and Zizler [20] (see also [30, Theorem 6.30]) gives actually more information: a subspace of a Hilbert-generated Banach space is generated by a linearly dense set which can be decomposed, for every $\varepsilon > 0$, in countably many pieces which are uniformly weakly null up to ε

in the sense of Proposition 1.3. It is natural, therefore, to wonder if a SWC subset can be replaced by a uniformly weakly null set spanning the same subspace. However, we do not know the general answer to this problem.

Problem 5.3. *Is every super WCG Banach space generated by a uniformly weakly null set?*

We can provide an answer within the Banach spaces of density up to ω_1 as a consequence of a result of Fabian, Godefroy, Hájek and Zizler.

Proposition 5.4. *Let X be a super WCG Banach space of density at most ω_1. Then X is generated by a uniformly weakly null set.*

Proof. According to [40], X is strongly UG renormable. By [19, Theorem 4], there is an injective weak* to weak continuous linear operator $T : X^* \to c_0(\omega_1)$ such that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ satisfying

$$|\{\gamma \in \omega_1 : |Tx^*(\gamma)| > \varepsilon\|x^*\|\}| < n \text{ for all } x^* \in X^*.$$

Note that $Tx^*(\gamma) = x^*(x_\gamma)$ for some $x_\gamma \in X$ and the set $K = \{x_\gamma : \gamma \in \omega_1\}$ is uniformly weakly null. ■

Note as well that any Hilbert-generated Banach spaces is generated by a uniformly weakly null set (the image of the orthonormal basis), however the converse is false. Indeed, the space $\ell_{3/2}(I)$ for I uncontable is generated by a uniformly weakly null set (just take its canonical basis) but it is not Hilbert-generated after [19, p. 316].

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