Representation in $C(K)$ by Lipschitz functions

M. Raja*

January, 2024

Abstract

The isometric universality of the spaces $C(K)$ for K a non scattered Hausdorff compact does not take into account the "quality" of the representation. Indeed, the existence of an isometric copy of a separable Banach space X into $C(K)$ made of regular enough functions, say Lipschitz with respect to a lower semicontinuous metric defined on K, imposes severe restrictions to both X and K. In this paper, we present a systematic treatment of the representation of Banach spaces into $C(K)$ by Lipschitz functions improving previous results of the author.

1 Introduction

A celebrated result of Banach and Mazur says that any separable Banach space X can be found linearly isometrically embedded into $C[0,1]$, the space of continuous functions over the unit interval endowed with the supremum norm. A different matter is to find explicit representations simple Banach spaces. For instance,

$$(x, y) \to x\cos(\pi t) + y\sin(\pi t), \quad t \in [0,1]$$

gives a representation of the Euclidean plane $(\mathbb{R}^2, \|\cdot\|_2)$ into $C[0,1]$ by fairly nice functions. However, it is not easy at all to the same with $(\mathbb{R}^3, \|\cdot\|_2)$. Indeed, the functions witnessing the isometric embedding cannot be even Lipschitz. The reason, first noticed by W. F. Donoghue [4], is related to gap between the topological dimensions of the interval $[0,1]$ and of the sphere $\mathbb{S}^2 = \partial B_{\mathbb{R}^3}$. We may consider a more general problem: given a Hausdorff compact space K and a finer metric d, study the Banach spaces that isometrically embed into $C(K)$ as a subset of d-Lipschitz functions.

Some results about this topic appeared scattered in several papers [16, 17, 18, 9]. The aim of this note is to bring together the main ideas on *Lipschitz subspaces of* $C(K)$ with improvements and some new results. Despite the quite

*This research has been supported by: Fundación Séneca – ACyT Región de Murcia, project 21955/PI/22; and grant PID2021-122126NB-C32 funded by MCIN/AEI/ 10.13039/501100011033 and by "ERDF A way of making Europe", by the EU.

mathematical insignificance of our original problem, we have found very interesting the connections between Topology and Banach space theory motivated by this research. Moreover, there are some ties with other trending topics, such as *lineability* (or *spaceability*) and *Lipschitz-free Banach spaces*.

The paper is organised as follows. The next section gathers some tools and generalities on compact spaces endowed with a lower semicontinuous metric. The third section deals with properties of Lipschitz subspaces both from the isomorphic and isometric points of view, however the results in the isometric theory are much more satisfactory. The fourth section covers the case when the metric actually metrizes K. The fifth section is devoted to the use of fragmentability, the main property making a difference for strictly finer lower semicontinuous metrics. So far in our paper, the only infinite-dimensional Lipschitz subspaces identified into $C(K)$ are copies of c_0 or ℓ_1, thus the last section deals with Lipschitz embeddings and obstructions for other Banach spaces. In order to make this note more independent of our previous work on the subject, we have included some sketched proofs of the quoted key results, or complete proofs in case they are simpler here or lead to a statement more general than in the original paper.

All the Banach spaces considered are real. Our notation is totally standard and we address to generic references for any unexplained definition [10, 5].

2 The bitopological setting

In Banach space theory is usual to deal with more than one topology: the norm and weak topologies, the weak* in case of dual Banach spaces, or the pointwise for function spaces. That bitopological setting, actually one topology and a metric generating a finer topology, can be discussed more generally. We will assume compactness for the coarse topology, so we will mainly use K to denote the space, and lower semicontinuity (lsc) for the metric as a function $d : K \times K \to [0, +\infty)$.

Theorem 2.1. *Let K be a compact Hausdorff space and d be a lower semicontinuous metric defined on K. Then:*

(a) the topology generated on K by d is finer;

(b) K is complete when endowed with d;

(c) for every closed subset $H \subset K$ and $r > 0$, the set

$$B[H, r] = \{x \in K : d(H, x) \leq r\}$$

is closed;

(d) every $f \in C(K)$ is d-uniformly continuous.

Proof. Statements (a), (b) and (c) are proved in [8]. We left (d), which is an easy exercise, to the reader. □

From now on K will be a compact Hausdorff space together with a lower semi-continuous metric denoted d. In case K is a weak* compact of a dual Banach space (notably, the dual unit ball), the metric d will be the one induced by the norm, unless otherwise stated.

The Lipschitz constant of a real function $f : K \to \mathbb{R}$ is defined as follows

$$L(f) = \sup \left\{ \frac{|f(t_1) - f(t_2)|}{d(t_1, t_2)} : t_1, t_2 \in K, t_1 \neq t_2 \right\}.$$

A real function f defined on K is said to be Lipschitz if $L(f) < +\infty$. The set of Lipschitz functions defined on K will be denoted $\mathrm{Lip}(K)$. We will also consider Lipschitz mappings between metric spaces, for whom the Lipschitz constant is defined likewise.

The following important result due to Eva Kopecká [13] implies, among other things, the great availability of functions that are both continuous and d-Lipschitz.

Theorem 2.2. *Let $H \subset K$ be closed and let $f : H \to [a, b]$ be continuous and Lipschitz. Then there exists $\tilde{f} : K \to [a, b]$ being continuous and Lipschitz with $L(\tilde{f}) = L(f)$ such that $\tilde{f}|_H = f$.*

Now we will see two ways to linearize compacta with lower semicontinuous metrics. Fix a point $t_0 \in K$ and let

$$\mathrm{Lip}_0(K) = \{f \in \mathrm{Lip}(K) : f(t_0) = 0\}.$$

The following result of Jayne, Namioka and Rogers appears in [8]. We include a simpler proof using Theorem 2.2.

Theorem 2.3. *Let K be a compact Hausdorff space and d be a lower semicontinuous metric. There exists a Banach space X such that K imbeds as w^*-compact subset of X^* and d coincides with the metric induced by the norm of X^*.*

Proof. Take $X = \mathrm{Lip}_0(K) \cap C(K)$ with the Lipschitz seminorm L, that is an actual norm here. Denote by $\hat{t}(f) = f(t)$ the evaluation. Obviously, the assignment $t \to \hat{t}$ is an homeomorphism. Note that for any $t_1, t_2 \in K$ with $t_1 \neq t_2$ there is $f \in C(K) \cap \mathrm{Lip}(K)$ with $L(f) = 1$ such that $f(t_2) - f(t_1) = d(t_1, t_2)$. Indeed, apply Theorem 2.2 to f defined on $\{t_1, t_2\}$ by $f(t_1) = 0, f(t_1) = d(t_1, t_2)$. Adding a constant we may even get that $f(t_0) = 0$ and thus $f \in B_X$. Now, for any two points $t_1, t_2 \in K$, we have

$$d(t_1, t_2) = \sup\{|f(t_1) - f(t_2)| : f \in B_X\}$$

$$= \sup\{|\hat{t}_1(f) - \hat{t}_2(f)| : f \in B_X\} = \|\hat{t}_1 - \hat{t}_2\|,$$

as wished. ☐

The previous construction allows the linearization of mappings in the following sense: a continuous mapping between compacts that is also Lipschitz extend to a linear weak* continuous mapping, obviously Lipschitz for the norms. Let us mention the relation to the *Lipschitz free spaces*. The seminorm L is a norm when restricted to $\mathrm{Lip}_0(K)$. It is possible to prove that $(\mathrm{Lip}_0(K), L)$ is isometric to a dual Banach space. The Lipschitz free space generated by K is a predual for $\mathrm{Lip}_0(K)$ that can be identified by the completed linear span of K into $\mathrm{Lip}_0(K)^*$. The Lipschitz free spaces have been studied over the last two decades, however we can not benefit from the research as the topology of K does not play a role.

We finish this section with an interesting result of Benyamini [1].

Theorem 2.4. *Let K be a metrizable compact space. There is weak* continuous retraction from $C(K)^*$ onto $B_{C(K)^*}$ that is also Lipschitz with constant one.*

3 General results

As said before, the compact Hausdorff space K is given together with a lower semicontinuous metric d. Let us stress that $C(K)$ refers to the continuous real functions with respect to the compact topology, meanwhile $\mathrm{Lip}(K)$ stands for the Lipschitz real functions with respect to the metric d. In general, those sets are not contained one another, however the intersection is rich enough to recover either the topology or the metric. The following is our main definition.

Definition 3.1. *A closed subspace $X \subset C(K)$ is said Lipschitz if $X \subset \mathrm{Lip}(K)$.*

Despite the definition is quiete clear, we will illustrate it with an example.

Example 3.2. *Let $K = B_{\ell_2}$ be endowed with the weak topology of ℓ_2 and let d be the norm (Hilbert) metric. Consider the functions $f_n : K \to [0,1]$ defined by*

$$f_n((x_k)_{k\in\mathbb{N}}) = x_n^2.$$

Then the sequence (f_n) spans a Lipschitz subspace of $C(K)$ isometric to c_0.

Proof. We will perform the computations although we can get the statement from the general results we will establish in this section. Note that for any bounded sequence (a_k) we have

$$\sup\{|a_k| : k \in \mathbb{N}\} \leq \sup\{\sum_{k=1}^{\infty} a_k x_k^2 : (x_k) \in K\},$$

but

$$\sum_{k=1}^{\infty} a_k x_k^2 \leq \sup\{|a_k| : k \in \mathbb{N}\} \sum_{k=1}^{\infty} x_k^2 \leq \sup\{|a_k| : k \in \mathbb{N}\}$$

for $(x_k) \in K$. Therefore $\| \sum_{k=1}^{\infty} a_k f_k \|_\infty = \|(a_k)\|_\infty$. If $(a_k) \in c_0$ we get also the uniform convergence of the series, so it defines an element of $C(K)$. As to the Lipschitzness, let $(x_k), (y_k) \in K$. Then

$$\left| \sum_{k=1}^{\infty} a_k x_k^2 - \sum_{k=1}^{\infty} a_k y_k^2 \right| \leq \sum_{k=1}^{\infty} |a_k|\, |x_k + y_k|\, |x_k - y_k|$$

$$\leq \left(\sum_{k=1}^{\infty} |a_k|^2 |x_k + y_k|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |x_k - y_k|^2\right)^{1/2} \leq 2\|(a_k)\|_\infty \, d((x_k),(y_k))$$

as wished. $\square$

The following simple application of Baire's theorem was written in similar terms in [9], although it was only applied in the case d metrizes K.

Proposition 3.3. *Let $X \subset C(K)$ be a non trivial linear subspace. Then either*

(a) $X \cap \mathrm{Lip}(K)$ is of first category in X;

(b) or $X \subset \mathrm{Lip}(K)$, that is, X is a Lipschitz subspace of $C(K)$, and there exists $\lambda > 0$ such that $L(f) \leq \lambda \|f\|$ for every $f \in X$.

Proof. Observe that $X \cap \mathrm{Lip}(K) = \bigcup_{n=1}^{\infty} \{f \in X : L(f) \leq n\}$ is a decomposition into countably many closed balanced convex sets. If $X \cap \mathrm{Lip}(K)$ is not of first category in X, then there is $\delta > 0$ such that $\delta B_X \subset \{f \in X : L(f) \leq n\}$ for some $n \in \mathbb{N}$. By homogeneity, we have $L(f) \leq \lambda \|f\|$ with $\lambda = \delta^{-1} n$ for every $f \in X$. In particular $X \subset \mathrm{Lip}(K)$. $\square$

Proposition 3.4. *Let $J : X \to C(K)$ be an isomorphic embedding. Then $J(X) \subset \mathrm{Lip}(K)$ if and only if $J^*|_K$ is Lipschitz from d to the norm of X^*, where J^* denotes the adjoint mapping from $C(K)^*$ into X^*. In such a case, there is $\delta > 0$ such that*

$$\delta B_{X^*} \subset \overline{\mathrm{conv}}(J^*(K) \cup (-J^*(K))).$$

Proof. If $J^*|_K$ is Lipschitz, then any function $J(x)$ is Lipschitz as well, since $J(x)(t) = J^*(t)(x)$. Reciprocally, assume that $J(X) \subset \mathrm{Lip}(K)$. By Proposition 3.3 there is $\lambda > 0$ such that $L(f) \leq \lambda$ for every $f \in J(X)$. Now, if $x \in B_X$ and $t_1, t_2 \in K$ then

$$|J^*(t_1)(x) - J^*(t_2)(x)| = |J(x)(t_1) - J(x)(t_2)| \leq \lambda \, d(t_1, t_2).$$

Taking supremum on $x \in B_X$ we get $\|J^*(t_1) - J^*(t_2)\| \leq \lambda \, d(t_1, t_2)$.
The last statement is just an equivalent expression of the fact that $J^*(K)$ is a norming set. $\square$

Now we will turn our attention to isometric embeddings. The set of extreme points of a convex subset C is denoted by $\mathrm{Ext}(C)$.

Lemma 3.5. *Let X, Y be Banach spaces and let $J : X \to Y$ be a linear operator. Then J is an isometric embedding if and only if*

$$\mathrm{Ext}(B_{X^*}) \subset J^*(\mathrm{Ext}(B_{Y^*})).$$

Proof. Note that, in general, $J : X \to Y$ is an isometric embedding if and only if $J^*(B_{Y^*}) = B_{X^*}$ (the less easy part relies on the Hahn–Banach theorem). Now, it is an easy exercise to prove that, for any $x^* \in \mathrm{Ext}(B_{X^*})$, any extreme point of the convex w^*-compact set $(J^*)^{-1}(x^*) \cap B_{Y^*}$ must be an extreme point in B_{Y^*}. $\square$

Theorem 3.6. *There is an isometric embedding of $J : X \to C(K)$ as a Lipschitz subspace if and only if there exists a mapping $\Psi : K \to B_{X^*}$ which is continuous for the weak* topology, Lipschitz for the metrics d-$\|\cdot\|$ and such that*

$$\mathrm{Ext}(B_{X^*}) \subset \Psi(K) \cup (-\Psi(K)).$$

In such a case, $\Psi = J^*|_K$ and $J(x)(t) = \Psi(t)(x)$.

Proof. Let us call $J : X \to C(K)$ the isometric embedding. By Proposition 3.4 we already know that $\Psi = J^*|_K$ is Lipschitz, it is obviously continuous from K to the w^*-topology and satisfies the required condition by Lemma 3.5 since $\mathrm{Ext}(B_{C(K)^*}) = K \cup (-K)$. For the other implication, consider the proposed formula $J(x)(t) = \Psi(t)(x)$. Evidently, J is a linear operator with $\|J\| \leq 1$ that satisfies $J^*|_K = \Psi$. Lemma 3.5 implies that J is an isometric embedding of X into $C(K)$ as a Lipschitz subspace. □

The idea behind the following result was used by Donoghue [4] for the construction of Peano-type filling curves.

Corollary 3.7. *Let X be a Gâteaux smooth Banach space and let $J : X \to C(K)$ be an isometric embedding. Then*

$$S_{X^*} \subset J^*(K) \cup (-J^*(K)).$$

Moreover, if X is infinite-dimensional, then

$$B_{X^*} = J^*(K) \cup (-J^*(K)).$$

Proof. The smoothness hypothesis implies that $\mathrm{Ext}(B_{X^*})$ is norm-dense into S_{X^*}, see [5]. Since $J^*(K) \cup (-J^*(K))$ is norm-closed, we have $S_{X^*} \subset J^*(K) \cup (-J^*(K))$. For the second part, just observe that S_{X^*} is weak*-dense in B_{X^*} when X is infinite-dimensional. □

4 When d metrizes K

The following result is essentially a folklore, although with different variations (see [5, Exercise 2.59], for instance). An interesting version with vector-valued functions, as well as some other beautiful applications of Baire theorem to subspaces of of $C(K)$ made up of regular functions see [6]

Theorem 4.1. *If d metrizes K, then all the Lipschitz subspaces of $C(K)$ are finite dimensional.*

Proof. If X is a Lipschitz subspace, then B_X is a bounded and complete set of functions of $C(K)$. By Proposition 3.3 we also know that B_X is equicontinuous. Therefore, by Ascoli's theorem, B_X is norm compact and so X is finite dimensional. □

On the other hand, if d does not metrizes K, there $C(K)$ contains infinite dimensional Lipschitz subspaces.

Theorem 4.2. *If the topology generated by d is strictly finer than the topology of K, then $C(K)$ contains a Lipschitz subspace isometric to c_0.*

Restricted proof. We will assume here that K is metrizable, or more generally, sequentially compact. The full proof will be provided in the next section. Since the topology generated by d cannot be compact, there is $\varepsilon > 0$ and a sequence $(t_n) \subset K$ such that $d(t_n, t_m) > 6\varepsilon$ for every $n \neq m$. We may assume that the sequence is converging to some $t_0 \in K$, and removing one more element if necessary, we may assume that $d(t_n, t_0) \geq 3\varepsilon$. Suppose we pick $s_n \in B[t_n, \varepsilon]$ for infinitely many $n \in \mathbb{N}$ and let $s \in K$ a cluster point of (s_n). Lower semicontinuity of d implies $d(t_0, s) \leq \varepsilon$ and thus any set of the form

$$B[t_0, \varepsilon] \cup \bigcup_{k=n}^{\infty} B[t_k, \varepsilon]$$

is closed for every $n \in \mathbb{N}$. Take disjoint open sets U_1 and V_1 such that $B[t_1, \varepsilon] \subset U_1$ and

$$B[t_0, \varepsilon] \cup \bigcup_{k=2}^{\infty} B[t_k, \varepsilon] \subset V_1$$

Take now disjoint open sets U_2, V_2 with $\overline{U_2}, \overline{V_2} \subset V_1$ and such that $B[t_2, \varepsilon] \subset U_2$ and

$$B[t_0, \varepsilon] \cup \bigcup_{k=3}^{\infty} B[t_k, \varepsilon] \subset V_2.$$

Following in this way we will get a sequence of open sets (U_n) such that $B[t_n, \varepsilon] \subset U_n$ and the sequence $(\overline{U_n})$ is pairwise disjoint. Applying Theorem 2.2 there is a continuous function $f_n : K \to [0, 1]$ such that $f(t_n) = 1$, $f_n|_{K \setminus U_n} = 0$ and f_n is Lipschitz with constant at most ε^{-1}.
For any $(a_n) \in c_0$, the series $\sum_{n=1}^{\infty} a_n f_n$ is uniformly convergent on K and so define a continuous function f. Note that f is Lipschitz with constant no larger than $2\varepsilon^{-1}\|(a_n)\|_\infty$. Therefore, the mapping

$$(a_n) \to \sum_{n=1}^{\infty} a_n f_n$$

defines an isometry of c_0 into a Lipschitz subspace of $C(K)$. $\square$

The combination of the two previous results of this section gives the following.

Corollary 4.3. *The lower semicontinuous metric d metrizes K if and only if all the Lipschitz subspaces of $C(K)$ are finite dimensional.*

A linear subspace A of a Banach space X is said to be *spaceable* if it contains an infinite-dimensional closed subspace of X. Therefore, the last corollary can be reformulated as $\mathrm{Lip}(K) \cap C(K)$ is spaceable (in $C(K)$) if and only if d does not metrizes K.

The dual point of view of isometric embeddings is useful even in finite dimension.

Proposition 4.4. *A finite dimensional polyhedral space X imbeds isometrically into ℓ_∞^n if and only if $2n$ is not lesser than the number of faces of B_X.*

Proof. Take $K = \{1, \ldots, n\}$ and note that $\ell_\infty^n = C(K)$. The number of extreme points of B_{X*} equals the number of faces of B_X. We only need to define a mapping from K taking at least one point of each antipodal pair from $\mathrm{Ext}(B_{X^*})$ to define an isometric embedding of X into $C(K)$. $\square$

For the remaining results of this section, the hypothesis 'metric compact' stresses the fact that K is metrized by d.

Proposition 4.5 ([9])**.** *If K is an infinite metric compact space, then $C(K)$ contains isometric copies made of Lipschitz functions of any finite dimensional polyhedral space.*

Hint of proof. If X is polyhedral and finite dimensional, its dual X^* is also polyhedral and so $\mathrm{Ext}(B_{X^*}) = \{x_1^*, \ldots, x_N^*\}$ is a finite set. Now use finitely many Urysohn's like disjointly supported functions which are also Lipschitz to build a mapping from K to B_{X^*} that covers $\mathrm{Ext}(B_{X^*})$. $\square$

The fact mentioned at the introduction is now explained in the following result, that implies a relation between the dimension of the Lipschitz copies of the Euclidean spaces and the dimension of K.

Theorem 4.6 ([9])**.** *Let K be a metric compact space and let $n \in \mathbb{N}$. The following are equivalent:*

(i) there is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$;

(ii) $C(K)$ contains isometric Lipschitz copies of all $(n+1)$-dimensional Banach spaces;

(iii) $C(K)$ contains an isometric Lipschitz copy of $(\mathbb{R}^{n+1}, \|\cdot\|_2)$.

Hint of proof. If there is an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$, then with the help of the stereographic projection is possible to build a Lipschitz mapping $\psi : K \to \mathbb{S}^n$, such that $\mathbb{S}^n = \psi(K) \cup (-\psi(K))$. On the other hand, if there a Lipschitz mapping $\psi : K \to \mathbb{S}^n$ such that $\psi(K)$ has nonempty interior relative to $\mathbb{S}^n$, then is possible to find an onto Lipschitz mapping $\phi : K \to \mathbb{I}^n$. $\square$

We will consider the Hilbert cube $C([0,1]^{\mathbb{N}})$ with the metric

$$d((a_n),(b_n)) = \sum_{k=1}^{\infty} 2^{-k}|a_k - b_k|.$$

Corollary 4.7. *The space of continuous functions on the Hilbert cube $C([0,1]^{\mathbb{N}})$ contains Lipschitz copies of all finite dimensional Banach spaces.*

Remark 4.8. *If there is a family of onto Lipschitz mappings $\phi_n : K \to \mathbb{I}^n$ with the Lipschitz constants uniformly bounded, then there is an onto Lipschitz mapping $\Phi : K \to [0,1]^{\mathbb{N}}$. Indeed, for the a standard metric on $\mathbb{I}^n$, the mappings $\eta_n : \mathbb{I}^n \to [0,1]^{\mathbb{N}}$ defined by $\eta_n((a_k)_{k=1}^n) = (a_k)_{k=1}^{\infty}$ taking $a_k = 0$ for $k > n$ are equi-Lipschitz. The family of mappings $(\eta_n \circ \phi_n)$ is equicontinuous, therefore there is a uniformly convergent subsequence whose limit Φ is Lipschitz and onto.*

5 Fragmentability and universality

We say that X is fragmented by metric d if for every nonempty subset $A \subset X$ and every $\varepsilon > 0$ there is $U \subset X$ open such that $A \cap U \neq \emptyset$ and $\mathrm{diam}(A \cap U) < \varepsilon$, where 'diam' is the diameter measured with respect to d. For a metrizable compact space K, fragmentability with respect to a lsc metric d is the same that separability in the d-topology. A compact that is fragmentable by a lsc metric is called *Radon-Nikodym* compact since dual Banach spaces whose weak* compacts are fragmentable by the norm have the Radon-Nikodym property (taking the name of the celebrated result on differentiation of measures), see [3, 5] for instance. A Banach space is said to be Asplund if every of its separable subspaces has separable dual. A celebrated combined result of Namioka, Phelps and Stegall establishes that a Banach space X is Asplund if and only X^* has the Radon-Nikodym property.

Theorem 5.1. *If K is fragmentable by d, then any Lipschitz subspace of $C(K)$ is Asplund.*

Proof. Let X be a Lipschitz subspace of $C(K)$ and let $J : X \to C(K)$ be the isomorphic embedding. By Proposition 3.4, we know that J^* is continuos from K to (X^*, w^*) and Lipschitz. Since fragmentability of compact spaces is preserved by continuous maps that also are Lipschitz for the metrics, we deduce that $J^*(K)$ is a w^*-compact fragmented by the norm Also by Proposition 3.4, we have

$$\delta B_{X^*} \subset \overline{\mathrm{conv}}(J^*(K) \cup (-J^*(K))),$$

that implies the norm fragmentability of B_{X^*} since that property is preserved by finite unions, w^*-closed convex hulls and subsets. Therefore, X^* has the Radon-Nikodym property and so X is Asplund. $\square$

Remark 5.2. *Additional properties can be transferred together wth the fragmentability of K, such as being a descriptive compact. That implies, for instance, that all the Lipschitz subspaces of $C(K)$ have a dual that admits an equivalent locally uniformly rotund dual norm, combine Theorems 2.4, Theorem 2.6 and Theorem 3.1 from [15] with the definitions therein. However, that really makes a difference in case X is not separable.*

Lemma 5.3. *Assume K is not fragmentable by d. Then there exits $\varepsilon > 0$ and two families (U_s) and (V_s) indexed by $\{0,1\}^{<\mathbb{N}}$ satisfying:*

1. *$U_{s\frown 0} \cup U_{s\frown 1} \subset U_s$ and $V_{s\frown 0} \cup V_{s\frown 1} \subset V_s$ for every s;*
2. *$\overline{V_s} \subset U_s$ for every s;*
3. *$d(\overline{V_s}, K \setminus U_s) > \varepsilon$ for every s;*
4. *$U_{s\frown 0} \cap U_{s\frown 1} \neq \emptyset$ for every s.*

Proof. If K is not fragmentable by d there is a closed subset $A \subset K$ and $\varepsilon > 0$ such that every nonempty relatively open set of A has diameter greater than

3ε. The construction of the families will be done by induction on the length of the sequence s, adding one more condition: points $(x_s) \subset A$ with $x_s \in V_s$ and $d(x_{s\frown 0}, x_{s\frown 1}) > 3\varepsilon$. Take two points $x_0, x_1 \in A$ with $d(x_0, x_1) > 3\varepsilon$. Using the lower semicontinuity of the metric, take now two open sets V_0, V_1 with $x_0 \in V_0$ and $x_1 \in V_1$ such that the distance between $\overline{V_0}$ and $\overline{V_1}$ is at least 3ε. The closed sets $B[\overline{V_0}, \varepsilon]$ and $B[\overline{V_1}, \varepsilon]$ are disjoint. Finally take disjoint open sets $U_0 \supset B[\overline{V_0}, \varepsilon]$ and $U_1 \supset B[\overline{V_1}, \varepsilon]$.
Assume everything is built for $|s| \leq n$. For a given s with $|s| = n$ we will construct the objects for $s \frown 0$ and $s \frown 1$. Since $x_s \in A \cap V_s$, we have $A \cap V_s \neq \emptyset$. This relatively open set of A has diameter greater than 3ε. Take points $x_{s\frown 0}, x_{s\frown 1} \in A \cap V_s$ with $d(x_{s\frown 0}, x_{s\frown 1}) > 3\varepsilon$. Take open sets $V_{s\frown 0}, V_{s\frown 1} \subset V_s$ with $x_{s\frown 0} \in V_{s\frown 0}$ and $x_{s\frown 1} \in V_{s\frown 1}$ such that the distance between $\overline{V_{s\frown 0}}$ and $\overline{V_{0s\frown 1}}$ is at least 3ε. The closed sets $B[\overline{V_{s\frown 0}}, \varepsilon]$ and $B[\overline{V_{s\frown 1}}, \varepsilon]$ are disjoint, so they can be separated by open sets $U_{s\frown 0}$ and $U_{s\frown 1}$. Without loss of generality we may assume $U_{s\frown 0}, U_{s\frown 1} \subset U_s$. That completes the induction argument. $\square$

Remark 5.4. *If K is besides metrizable, say by a metric ρ, then we may add to the construction the condition that ρ-diameter of $\overline{U_s}$ goes to 0 with $|s| \to \infty$ (take it smaller that $|s|^{-1}$ for instance).*

The following is the second key result of Lipschitz subspaces outside the metrizable case.

Theorem 5.5 ([16])**.** *Then the following are equivalent:*

(i) K is not fragmentable by d;

(ii) $C(K)$ contains an isometric Lipschitz copy of ℓ_1;

(iii) $C(K)$ contains an isomorphic Lipschitz copy of ℓ_1.

Proof. $(i) \Rightarrow (ii)$ If K is not d-fragmentable, we may produce a Cantor-like closed subset using Lemma 5.3 this way

$$H = \bigcap_{n\in\mathbb{N}} \bigcup_{|s|=n} \overline{V_s}.$$

For every $t \in H$ there is a unique $\sigma(t) \in \{0,1\}^{\mathbb{N}}$ such that $t \in \bigcap_{n\in\mathbb{N}} \overline{V_{\sigma(t)|n}}$. The mapping $\Sigma : H \to \{-1,1\}^{\mathbb{N}}$ defined by taking $\Sigma(t)$ the sequence $\sigma(t)$ after changing the 0's by -1's. It is easy to check that Σ is onto and continuous. Moreover, if $d(t_1, t_2) \leq 2\varepsilon$, then $\Sigma(t_1) = \Sigma(t_2)$. Let p_n the projection on the n'th coordinate of $\{-1,1\}^{\mathbb{N}}$ and consider the function $p_n \circ \Sigma$ and note that it is continuous and ε^{-1}-Lipschitz. By Theorem 2.2, there is a continuous extension f_n of $p_n \circ \Sigma$ to K with the same Lipschitz bound ε^{-1}. The sequence (f_n) is equivalent to the canonical basis of ℓ_1. Indeed, given numbers real numbers (a_n) for $i = 1, \ldots, m$ there is $x \in H$ such that $f_n(x) = \mathrm{sign}(a_n)$ and thus

$$\|\sum_{n=1}^{m} a_n f_n\|_\infty = \sum_{n=1}^{m} |a_n|$$

which means that $E = \overline{span}^{\|\cdot\|_\infty}\{f_n : n \in \mathbb{N}\}$ is isometric to ℓ_1. An easy computation shows that if $f \in E$, then f is $\varepsilon^{-1}\|f\|$-Lipschitz.
$(ii) \Rightarrow (iii)$ It is obvious.
$(iii) \Rightarrow (i)$ If $C(K)$ contains an isometric Lipschitz copy of ℓ_1, then Theorem 5.1 implies that K cannot be fragmentable by d because ℓ_1 is not Asplund. □

Remark 5.6. *It is possible to add an equivalent condition to Theorem 5.5: there is an d-equicontinuous bounded sequence $(f_n) \subset C(K)$ equivalent to the canonical basis of ℓ_1, see [16].*

One longstanding problem in Banach theory was to know if a separable space X having a non separable dual must contain a copy of ℓ_1. A nonseparable reformulation is whether Asplundness equals not containing ℓ_1. The problem was solved negatively by James [7] and Lindenstrauss and Stegall [12], independently. However, we have the following *boutade*.

Corollary 5.7 ([16])**.** *A Banach space X is Asplund if and only if $C(B_{X^*})$ does not contain a Lipschitz copy of ℓ_1.*

Endowing a Hausdorff compact with the discrete metric we retrieve this classic result.

Corollary 5.8. *K is not scattered if and only if $C(K)$ contains an isomorphic copy of ℓ_1.*

Full proof of Theorem 4.2. If K is fragmentable by d then K is sequentially compact [14], therefore the restricted proof above gives the result. Otherwise, if K is not fragmentable by d we may use the sets built in Lemma 5.3 in this way. Let s_n be the sequence starting in 1 and followed by $n-1$ zeroes. The sets U_{s_n} are disjoint. Using Theorem 2.2, there are continuous ε^{-1}-Lipschitz functions $f_n : K \to [0,1]$ such that $f_n|_{V_{s_n}} = 1$ and $f_n|_{K\setminus U_{s_n}} = 0$. Proceeding like in the restricted proof, those functions generate a Lipschitz subspace of $C(K)$ isometric to c_0. □

Now we turn our attention to the "Lipschitz universality" of $C(K)$ spaces. Let Δ be the Cantor space $\{0,1\}^{\mathbb{N}}$ together with the discrete metric.

Proposition 5.9. *The space $C(\Delta)$ is isometric Lipschitz universal for the separable Banach spaces. If K is a metrizable Hausdorff compact together a lsc metric such that $C(K)$ is isomorphic Lipschitz universal for the separable Banach spaces, then K contains a subset equivalent to Δ, that is, homeomorphic and Lipschitz isomorphic.*

Proof. By Mazur's theorem, $C(\Delta)$ is isometric universal in the standard sense and every $f \in C(\Delta)$ is Lipschitz with respect to the discrete norm. On the other hand, if $C(K)$ is isomorphic Lipschitz universal for the separable Banach spaces, then K is not fragmentable by the associated metric d. The proof of Theorem 5.5 together Remark 5.4 provide a set H that is homeomorphic to the Cantor space and its points are uniformly separated, with separation bounded below by 2ε. In order the restriction of d to H be Lipschitz isomorphic to the discrete metric is enough that d be bounded. That is not ensured by the hypotheses, so we may proceed this way. Take any $t_0 \in H$ and consider the closed balls $B[t_0, n]$ for $n \in \mathbb{N}$, that are closed with respect to the topology of H too. Since $H \subset \bigcup_{n=1}^{\infty} B[t_0, n]$, there is one ball with nonempty interior (with respet to H). Now, any nonempty open set of the Cantor space contains an homeomorphic copy of the Cantor space itself, which provides a copy of the Cantor space where d is bounded. $\square$

We do not know if any metrizable K not fragmentable with respect to d is Lipschitz universal for the separable Banach spaces. Actually, we do not know the answer for $K = [0,1]^{\mathbb{N}}$ endowed with the supremum norm (that is, essentially, B_{ℓ_∞} with the weak* topology and the norm metric). A main issue here is that the method to build linear extension operators within spaces of continuous functions seldom preserve Lipchitzness, see [2] or [20] for instance.

6 An ordering for compacta with lsc metrics

We have seen that the problem of identifying Lipschitz subspaces reduces to the study of applications between compact spaces that are Lipschitz with respect to the associated lsc metrics. In this section we will restrict our attention to metrizable compact spaces with associated bounded lsc metrics. Define an order among this class by $K_1 \preceq K_2$ if there exists an onto continuous mapping $\phi : K_2 \to K_1$ that is also Lipschitz for the metrics. The definition implies trivially the following observation.

Proposition 6.1. *If $K_1 \preceq K_2$, any isometric (resp. isomorphic) Lipschitz subspace of $C(K_1)$ is an isometric (resp. isomorphic) Lipschitz subspace of $C(K_2)$.*

In the order $\preceq$ the singleton space plays the role of minimum. On the other hand, Δ is a maximum, however it is not unique. For instance, $\Delta \uplus [0,1]$ (disjoint topological union) is a maximum too. As to intermediate elements in the order $\preceq$, the most interesting example is provided by the Mazur mapping between the unit balls of Lebesgue sequence spaces, namely $\Phi_{q_1,q_2} : B_{\ell_{q_1}} \to B_{\ell_{q_2}}$ defined by

$$\Phi_{q_1,q_2}(\,(x_n)_{n\in\mathbb{N}}\,) := (\operatorname{sign}(x_n)|x_n|^{q_1/q_2})_{n\in\mathbb{N}}$$

which is Lipschitz for $1 \le q_2 \le q_1 < \infty$, see the proof of [5, Theorem 12.50]. The mapping is obviously continuous for the pointwise topologies, that make the balls compact (actually, they are dual unit balls with the weak* topology). Therefore $B_{\ell_{q_2}} \preceq B_{\ell_{q_1}}$ whenever $q_2 \le q_1$.

Proposition 6.2. *Suppose that $C(K)$ contains an isometric Lipschitz copy of some ℓ_p with $p \in (1, +\infty)$, then $C(K)$ contains an isometric Lipschitz copy of $\ell_{p'}$ for every $p' \in [p, +\infty)$.*

Proof. Let $q, q' \in (1, +\infty)$ be the conjugate exponents of p, p' respectively. Let $\Psi : K \to B_{\ell_q}$ witnessing the isometric embedding of ℓ_p as Lipschitz subspace of $C(K)$. Since ℓ_p is smooth, by Corollary 3.7 we have $B_{\ell_q} = \Psi(K) \cup (-\Psi(K))$. Note that the Mazur mapping satisfies $\Phi_{q,q'}(-x) = -\Phi_{q,q'}(x)$. Therefore,

$$B_{\ell'_q} = \Phi_{q,q'}(B_{\ell_q}) = \Phi_{q,q'}(\Psi(K)) \cup \Phi_{q,q'}((-\Psi(K)))$$

$$= (\Phi_{q,q'} \circ \Psi)(K) \cup (-(\Phi_{q,q'} \circ \Psi)(K)),$$

which implies, by Theorem 3.6. that the isometric embedding of $\ell_{p'}$ as a Lipschitz subspace of $C(K)$. $\square$

Now we will show the use of the Szlenk index (see [11] for more information on the use of ordinal indices in Banach space theory) as an obstacle for Lipschitz embeddings For any closed subset $A \subset K$ we define a set derivation

$$\langle A \rangle'_\varepsilon = \{x \in A : \forall U \text{ neighbourhood of } x, \operatorname{diam}(A \cap U) \geq \varepsilon\},$$

where the diameter is computed with respect to d. By iteration, the sets $\langle A \rangle^\gamma_\varepsilon$ are defined for any ordinal γ, taking intersection in the case of limit ordinals. The Szlenk indices of K with respect to d are ordinal numbers defined by

$$Sz(X, \varepsilon) = \inf\{\gamma : \langle X \rangle^\gamma_\varepsilon = \emptyset\}$$

and $Sz(X) = \sup_{\varepsilon>0} Sz(X, \varepsilon)$. If K is fragmentable by d, the Szlenk indices always exist. Otherwise, for some $\varepsilon > 0$ there is an ordinal γ such that $\langle X \rangle^\gamma_\varepsilon = \langle X \rangle^{\gamma+1}_\varepsilon \neq \emptyset$. In that case we put $Sz(X, \varepsilon) = \infty$ and $Sz(X) = \infty$ with the agreement that any ordinal number is less than ∞.

Proposition 6.3 ([17]). *If $K_1 \preceq K_2$, then there is $c > 0$ such that, for all $\varepsilon > 0$, we have*

$$Sz(K_1, \varepsilon) \leq Sz(K_2, c\,\varepsilon).$$

Proof. Let $\phi : K_2 \to K_1$ be the continuous surjection with Lipschitz constant $\lambda > 0$. Take $c = 2\lambda$. It enough to show that

$$\langle \phi(A) \rangle'_{c\,\varepsilon} \subset \phi(\langle A \rangle'_\varepsilon)$$

for every closed subset $A \subset K_2$. Indeed, if $x \in \phi(A) \setminus \phi(\langle A \rangle'_\varepsilon)$, then $\phi^{-1}(x)$ is compact subset of A disjoint with $\langle A \rangle'_\varepsilon$. The set $\phi^{-1}(x)$ can be covered with finitely many open sets $U_1, \ldots, U_n$ such that $\operatorname{diam}(A \cap U) < \varepsilon$. Let $U = \bigcup_{k=1}^n U_k$. Note that $U \cap \langle A \rangle'_\varepsilon = \emptyset$ and for every $y \in A \cap U$ then $d_1(\phi(y), x) < \lambda\,\varepsilon$, implying $\operatorname{diam}(\phi(A \cap U) \leq c\,\varepsilon$. Taking the open set $V = K_1 \setminus \phi(A \setminus U)$ we have $x \in \phi(A) \cap V \subset \phi(A \cap U)$ which implies that $x \notin \langle \phi(A) \rangle'_{c\,\varepsilon}$. $\square$

Corollary 6.4 ([17]). *If X is Gâteaux smooth and embeds as a Lipschitz subspace of $C(K)$, then there is $c > 0$ such that, for all $\varepsilon > 0$, we have*

$$Sz(B_{X^*}, \varepsilon) \leq Sz(K, c\,\varepsilon).$$

It is possible to prove that $Sz(B_{\ell_q}, \varepsilon) \sim \varepsilon^{-q}$ for $q \in [1, +\infty)$, see [17] for the details. Therefore $B_{\ell_{q_2}} \not\preceq B_{\ell_{q_1}}$ if $q_1 < q_2$.

Corollary 6.5 ([18]). *Let $p \in (1, +\infty)$ and let q be its conjugate exponent. Then for every $p' \in [1, p)$, the space $\ell_{p'}$ does not embeds isometrically as a Lipschitz subspace of $C(B_{\ell_q})$.*

Our results with Lipschitz isomorphic embeddings are not so satisfactory distinguishing among the balls B_{ℓ_q}. The necessity of taking the closed convex hull implies some loss of information (the Szlenk index is increased by a factor of the form ε^{-1}). However, some interesting results can be established for infinite Szlenk indices.

Proposition 6.6. *Let K be metrizable compact together a lsc metric d. If the space $C(K)$ is isomorphic Lipschitz universal for the separable reflexive Banach spaces, then K is not fragmentable by d.*

Proof. If K was fragmentable, then $Sz(K)$ would be a countable ordinal and $Sz(B_{X^*}) \leq Sz(K)$ for all X an isomorphic Lipschitz subspace of $C(K)$. Szlenk proved that there are reflexive Banach spaces X with $Sz(B_{X^*})$ an arbitrarily high countable ordinal, leading to a contradiction. □

Acknowledgements

In 2003, I was doing a postdoc at the Hebrew University of Jerusalem under the supervision of Joram Lindenstrauss when I obtained Corollary 5.7. I often enjoy myself remembering the morning as I came to Joram's office and told him "Do you know that Asplundness can be characterized by the lack of copies of ℓ_1?". Some years later, during a Winter School in Czech Republic, I rediscovered Donoghue's surprising result and I tried to publish it. I am still indebted to Bill Johnson who kindly informed me that I was 50 years late in his nonacceptance letter.

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Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, SPAIN
E-mail: matias@um.es