

A covering index for Banach spaces

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March, 2025

Abstract

We introduce a new isomorphic quantity for Banach spaces, the index Θ_X , based on finite convex coverings of the unit ball. This index is closely related to the asymptotic moduli of uniform convexity and uniform smoothness, so that it can be calculated for several classical Banach spaces.

1 Introduction

Let X be a real infinite-dimensional Banach space and let B_X denote its unit ball. Given a subspace $Y \subset X$, evidently we have $B_Y = B_X \cap Y$. We define the *essential inradius* $\varrho(A)$ of a set $A \subset X$ as

$$\varrho(A) = \sup\{r > 0 : \exists x \in A, \exists Y \subset X, \mathcal{D}(X/Y) < \infty, x + rB_Y \subset A\},$$

where \mathcal{D} stands for dimension, so we are measuring radii of finitely codimensional balls. The essential inradius should not be confused with the *asymptotic inradius* that is computed similarly but running on finite-dimensional subspaces of arbitrarily high dimension, see [12]. Let $\mathcal{C}(X)$ denote the family of bounded closed convex subsets of X . Consider the following covering index

$$\Theta_X(n) = \inf \left\{ \max_{1 \leq k \leq n} \varrho(A_k) : (A_k)_{k=1}^n \subset \mathcal{C}(X), \bigcup_{k=1}^n A_k = B_X \right\}.$$

That means, for every convex decomposition of B_X into n sets, there is one of them that contains infinite codimensional balls of radius arbitrarily close to $\Theta(n)$, if not bigger. The purpose of the index Θ_X is to identify a sense of “size” that effectively can shrink as the number of pieces increases, but not “very fast”, so it can eventually be interpreted as a kind of concentration or Ramsey-like property. For instance, covering numbers indices on the standard inradius, the width or the volume (in case of finite dimension) would decrease as $O(n^{-1})$ being n the number of sets. In contrast, the use of the diameter or the asymptotic inradius would make the corresponding covering indices constantly equal to 1, see [12].

*This research has been supported by: Fundación Séneca – ACyT Región de Murcia, project 21955/PI/22; and grant PID2021-122126NB-C32 funded by MCIN/AEI/10.13039/501100011033 and by “ERDF A way of making Europe”.

Evidently, for X an infinite-dimensional Banach space $(\Theta_X(n))_{n=1}^\infty \subset (0, 1]$ is a non-increasing sequence. Moreover, the information provided by Θ_X is of isomorphic nature. Indeed, it is not difficult to prove that if \tilde{X} denotes a renorming of X with a λ -equivalent norm, then

$$\lambda^{-2}\Theta_X(n) \leq \Theta_{\tilde{X}}(n) \leq \lambda^2\Theta_X(n).$$

Another useful observation is that in the definition of $\Theta_X(n)$ we can only consider coverings where all the sets A_k have nonempty interior, [12, Lemma 2.4]. That would allow the use of perturbation arguments to impose some restrictions to the finite codimensional subspaces Y appearing in the definition of $\varrho(A_k)$.

Two examples given in [12] can be reformulated as follows:

- (a) $\Theta_{c_0}(n) = 1$ for all $n \in \mathbb{N}$;
- (b) $\Theta_{\ell_2}(n) = O(n^{-1/2})$;

suggesting a possible connection of Θ_X to some well known isomorphic quantities of Banach spaces, such as the cotype, for instance.

The aim of this paper is establish upper and lower bounds for covering index Θ_X in terms of the geometry of X . The main results can be summarized as that the behavior of $\Theta_X(n)$ depends on the asymptotic moduli of uniform convexity and uniform smoothness for AUC or AUS renormings of X . The magnitude of $\Theta_X(n)$ is completely determined for an ℓ_p sum of finite dimensional spaces.

All the Banach spaces considered are real and separable. We will use the symbol \lesssim instead of Landau's O , in order to use \gtrsim for the reverse estimation, that is, we write $f(t) \gtrsim g(t)$ if there exists $c > 0$ such that $f(t) \geq cg(t)$ for every t in the domain. We will use \simeq in case \lesssim and \gtrsim hold simultaneously. With the exception of these details, our notation is totally standard and we address the reader to generic references such as [8, 5, 3] for any unexplained definition.

2 Upper bounds for $\Theta_X(n)$

Consider a Banach space X with a Schauder finite-dimensional decomposition $(E_n)_{n=1}^\infty$, abbreviated FDD, see [8] for precise definitions. The following notions are relative to a fixed FDD of X for $1 \leq p, q < \infty$:

- (1) we say that X has a upper p estimate if there is $C > 0$ such that

$$\left\| \sum_{i=1}^k x_i \right\| \leq C \left(\sum_{i=1}^k \|x_i\|^p \right)^{1/p}$$

for any $k \in \mathbb{N}$ and choice of $x_i \in E_i$ for $1 \leq i \leq k$;

(2) we say that X has a lower q estimate if there is $c > 0$ such that

$$\left\| \sum_{i=1}^k x_i \right\| \geq c \left(\sum_{i=1}^k \|x_i\|^q \right)^{1/q}$$

for any $k \in \mathbb{N}$ and choice of $x_i \in E_i$ for $1 \leq i \leq k$.

The idea behind the proof of the following result is the Hilbert space example [12, Example 2.1].

Theorem 2.1. *If X has a shrinking FDD satisfying a lower q estimate for some $q \geq 1$, then $\Theta_X(n) \lesssim n^{-1/q}$.*

Proof. Without loss of generality, we will suppose that the FDD indexed by $\mathbb{N} \cup \{0\}$, with E_0 of dimension 1, so we can identify $E_0 = \mathbb{R}$. For every $x \in X$ we will write (x_i) for its decomposition according to (E_i) . Note that under our hypotheses, the FDD induces a bounded linear operator

$$X \longrightarrow \left(\bigoplus_{i=0}^{\infty} E_i \right)_{\ell_q}.$$

We will estimate the covering index Θ_X using a particular decomposition into an even number of pieces $2n$. Now, for $1 \leq j \leq 2n$, consider the set

$$A_j = \left\{ x \in B_X : (-1)^j x_0 \leq 1/2 - c^q n \sum_{m=0}^{\infty} \|x_{2nm+j}\|^q \right\},$$

where $c > 0$ is the constant associated to the lower q estimate. These sets are evidently closed and convex. Moreover, $\bigcup_{j=1}^{2n} A_j = B_X$. Indeed, assume that some $x = (x_i) \in B_X$ belongs to none of the sets. Then we have the reversed inequalities

$$(-1)^j x_0 > 1/2 - c^q n \sum_{n=0}^{\infty} \|x_{2kn+j}\|^q,$$

whose sum over $1 \leq j \leq 2n$ gives

$$0 > n - c^q n \sum_{i=1}^{\infty} \|x_i\|^q.$$

We deduce

$$1 < c^q \sum_{i=1}^{\infty} \|x_i\|^q \leq \|x\|^q$$

that is a contradiction with $x \in B_X$. Now we will estimate the radius of a finite codimensional ball contained in A_j for any j . Let $Y \subset X$ be a finite codimensional subspace. Since the FDD is shrinking, up to a small perturbation we may assume that annihilator of Y is contained in $\bigoplus_{i=0}^N E_i^*$. Let $x \in A_j$ be the center of the ball. We may take $m \in \mathbb{N}$ such that $2mn + j > N$ and the coordinate x_{2mn+j} is

negligible. Therefore, if $x + y \in A_j$, for the estimation of $\|y\|$ only the coordinates 0 and $2mn + j$ are relevant. We have

$$c^q n \|y_{2mn+j}\|^q \leq 1/2 - (-1)^j x_0 \leq 1/2 + |x_0| \leq M$$

where $M > 0$ depends only on the norm of the projection onto E_0 . We get that

$$\|y_{2mn+j}\| \leq \left(\frac{M}{c^q n}\right)^{1/q}.$$

That estimation shows $\varrho(A_j) \lesssim n^{-1/q}$ as wished. \blacksquare

Remark 2.2. *We can extend Theorem 2.1 to Banach spaces having an overspace with a shrinking FDD satisfying a lower q estimate, provided that the FDD also satisfy an upper q estimate. Indeed, instead of reducing y to just one coordinate we can work with the whole pack thanks to the upper estimation.*

We will recall the definition of the *modulus of asymptotic uniform convexity* of a Banach space X , notion coined in [6] but introduced by Milman [9] under a different name, see also [1]. The AUC modulus of X is defined for $\varepsilon > 0$ as

$$\bar{\delta}_X(\varepsilon) = \inf_{\|x\|=1} \sup_{\mathcal{D}(X/Y) < \infty} \inf_{y \in Y, \|y\| \geq \varepsilon} (\|x + y\| - 1).$$

The space is said to be *asymptotically uniformly convex* (AUC) if $\bar{\delta}_X(\varepsilon) > 0$ for every $\varepsilon > 0$. Finally, the space X is said q -AUC for some $q \geq 1$ if $\bar{\delta}_X(\varepsilon) \gtrsim \varepsilon^q$. According to [7] for every separable reflexive AUC space there exists a renorming making it q -AUC.

Corollary 2.3. *Let X be a reflexive Banach space with a FDD. If X admits an equivalent q -AUC norm, then $\Theta_X(n) \lesssim n^{-1/q}$.*

Proof. According to a result of Prus [10] (see also [6, 1]), there exists a blocking of the FDD, that is a further FDD, satisfying a lower q estimate. \blacksquare

3 Lower bounds for $\Theta_X(n)$

The main tool in this section will be the *goal derivation* studied in [11]. For a set $A \in \mathcal{C}(X)$ and $\varepsilon > 0$ we define

$$[A]'_\varepsilon = \{x \in A : \forall U \text{ } w\text{-neighbourhood of } x, \varrho(A \cap U) > \varepsilon\}.$$

It is not evident but true that $[A]'_\varepsilon \in \mathcal{C}(X)$ too. We can define derived sets of superior order by taking $[A]_\varepsilon^n = [[A]_\varepsilon^{n-1}]'_\varepsilon$, for any $n \in \mathbb{N}$ and eventually for ordinal numbers. The *goal Szlenk index*, denoted $Gz(A, \varepsilon)$ is defined as

$$Gz(A, \varepsilon) = \inf\{n \in \mathbb{N} : [A]_\varepsilon^n = \emptyset\}$$

when it is finite. Otherwise, we will simply say that $Gz(A, \varepsilon) \geq \omega$ (here we are not interested in dealing with ordinals). Actually, the original definition of the goal derivation uses “ \geq ” instead of “ $>$ ” as above, but that does not change the behaviour of $Gz(A, \varepsilon)$ as a function of ε .

Proposition 3.1. $Gz(A, \varepsilon) \leq n$ if and only if there is a covering $(A_k)_{k=1}^n$ of B_X by weakly closed sets such that for every $x \in B_X$ there is $1 \leq k \leq n$ such that

$$\sup\{r > 0 : \exists Y \subset X, \mathcal{D}(X/Y) < \infty, x + rB_Y \subset A_k\} \leq \varepsilon.$$

Proof. If $Gz(A, \varepsilon) \leq n$ just take $A_k = [A]_\varepsilon^{k-1}$ for $1 \leq k \leq n$. For the other implication, define the sets

$$B_k = \{x \in B_X : |\{j : x \in A_j\}| > k\}$$

for $0 \leq k \leq n$ (obviously, $B_n = \emptyset$). We claim that $[A]_\varepsilon^k \subset B_k$, that clearly implies the statement. The proof will be by induction. Evidently $B_0 = B_X$. Assume now $k < n$ and $[A]_\varepsilon^k \subset B_k$. Take any $x \in [A]_\varepsilon^k \setminus B_{k+1}$. Then there is a subset $I \subset \{1, \dots, n\}$ with $|I| = k+1$ such that $x \in A_i$ if and only if $i \in I$. That implies

$$V = [A]_\varepsilon^k \setminus \bigcup_{j \notin I} A_j \subset \bigcap_{i \in I} A_i.$$

By construction V is a relatively weakly open neighbourhood of x such that

$$\sup\{r > 0 : \exists Y \subset X, \mathcal{D}(X/Y) < \infty, x + rB_Y \subset V\} < \varepsilon$$

since the hypothesis is fulfilled for some $i \in I$. But the same is true for any other point in V , meaning $\varrho(V) \leq \varepsilon$. That implies $[A]_\varepsilon^{k+1} \subset B_{k+1}$ as wished. ■

Corollary 3.2. If $\varepsilon > \Theta_X(n)$, then $Gz(B_X, \varepsilon) \leq n$.

Since $Gz(B_X, \varepsilon) \gtrsim n^{-1}$, see [11], that gives a first lower bound for Θ_X .

Corollary 3.3. For any Banach space X , we have $\Theta_X(n) \gtrsim n^{-1}$.

In order to give more precise lower bound we need the notion of *modulus of asymptotic uniform smoothness* of a Banach space. As the AUC modulus, the AUS modulus was coined in [6] but introduced by Milman [9] with a different name, see also [1]. The AUS modulus of X is defined for $\varepsilon > 0$ as

$$\bar{\rho}_X(\varepsilon) = \sup_{\|x\|=1} \inf_{\mathcal{D}(X/Y) < \infty} \sup_{y \in Y, \|y\| \leq \varepsilon} (\|x + y\| - 1).$$

The space X is said to be *asymptotically uniformly smooth* if $\lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} \bar{\rho}_X(\varepsilon) = 0$. Finally, the space X is said p -AUS for some $p > 1$ if $\bar{\rho}_X(\varepsilon) \lesssim \varepsilon^p$. According to [7] for every separable AUS space there exists a renorming making it p -AUS. Although we do not have AUS property for $p = 1$ we will consider the notion of “1-AUS” for the next result from [11].

Proposition 3.4. Suppose that X is p -AUS for $p \geq 1$, then $Gz(B_X, \varepsilon) \gtrsim \varepsilon^{-p}$.

The straightforward combination of Corollary 3.2 and the previous proposition gives the following lower bound for Θ_X .

Theorem 3.5. If X is p -AUS renormable for $p \geq 1$, then $\Theta_X(n) \gtrsim n^{-1/p}$.

4 Asymptotic estimations and problems

The combination of upper and lower bounds will allow us to give precise asymptotic estimations of Θ_X . Since we are interested in isomorphic results, the notion of equivalence for sequences fits to our purpose. We say that the sequences $(a_n), (b_n) \subset \mathbb{R}^+$ are equivalent, denoted as $a_n \simeq b_n$, if there exist constants $c, d > 0$ such that $ca_n \leq b_n \leq da_n$ for all $n \in \mathbb{N}$.

Proposition 4.1. *Let X be isomorphic to a subspace of an ℓ_p sum of finite-dimensional normed spaces. Then $\Theta_X(n) \simeq n^{-1/p}$.*

Proof. This is just a combination of Theorem 2.1, Remark 2.2 and Theorem 3.5. ■

In particular, the Θ_X index distinguishes between ℓ_p spaces (in particular, better than the type or the cotype).

Corollary 4.2. $\Theta_{\ell_p}(n) \simeq n^{-1/p}$ for $1 \leq p < +\infty$.

A deep result from [6] says that a reflexive Banach space that admits both p -AUC and p -AUS renormings can be embedded as a subspace of an ℓ_p sum of finite-dimensional normed spaces.

Corollary 4.3. *Let X be a reflexive Banach space and $p > 1$ such that X admits p -AUC and p -AUS renormings, then $\Theta_X(n) \simeq n^{-1/p}$.*

Recall that for a general Banach space $(\Theta_X(n))_{n=1}^{\infty}$ may not converge to 0.

Proposition 4.4. *Let $\varepsilon \in (0, 1]$ be such that $Gz(B_X, \eta) \geq \omega$ for every $\eta \in (0, \varepsilon)$. Then $\lim_n \Theta_X(n) \geq \varepsilon$.*

Proof. Indeed, if $\eta < \varepsilon$, then $\Theta_X(n) \geq \eta$ for all $n \in \mathbb{N}$ by Corollary 3.2. ■

We retrieve the result mentioned in the introduction that $\Theta_{c_0}(n) = 1$ for all $n \in \mathbb{N}$, since the goal set derivation satisfies $[B_{c_0}]'_\eta = B_{c_0}$ for every $\eta \in (0, 1)$ so $Gz(B_{c_0}, \eta)$ is not even defined (we put “ ∞ ” as something above all the ordinals).

The following example shows that the behaviour of Θ_X can be very diverse among reflexive Banach spaces. Consider the dual T^* of the Tsirelson space T , some times also called Tsirelson space, see [8] for a construction. In order to avoid confusion, let us say that T looks like ℓ_1 and T^* takes after c_0 .

Corollary 4.5. *Let T be the Tsirelson space, then $\lim_n \Theta_{T^*}(n) \geq 1/2$.*

Proof. Indeed, if $\varepsilon < 1/2$, then $Gz(B_{T^*}, \varepsilon) \geq \omega$ by [11, Theorem 4.4]. ■

I would like to finish with some general comments and open problems. The index Θ_X is defined in a very simple way, but it can recognize structural features of Banach spaces. Our initial reason for estimating the radius of finite codimensional balls is the fact that it essentially “survive” through uniform homeomorphisms. That is the so called the *Gorelik principle*, see [1, 4], that was used by Godefroy,

Kalton and Lancien [4] to prove that AUS renormabilty of a Banach space is a uniform invariant. However, their proof makes a heavy use of duality so it would be desirable a better understanding in terms of the space itself. Despite this motivation, we feel that our results are “too much linear” to combine well with Gorelik’s.

There is an evident parallelism of Θ_X with the *entropy numbers*. Given a relatively compact subset $K \subset X$, the entropy number $\mathcal{E}_n(K)$ is defined as the infimum of the $\varepsilon > 0$ such that K can be covered with n balls of radius ε . Not just parallelism but a bit of interaction: Let $T : X \rightarrow Z$ be a compact linear operator such that $\lim_n \mathcal{E}_n(T(B_X))/\Theta_X(n) = 0$. It is not difficult to prove that for every $\varepsilon > 0$ there is finite-codimensional subspace $Y \subset X$ such that $\|T|_Y\| < \varepsilon$, meaning that such T can be approximated by finite rank operators.

Now, we will explicitly formulate three problems that we could not solve.

Problem 4.6. *Find the exact value of $\Theta_{\ell_2}(2)$.*

We know that $0.707 \leq \Theta_{\ell_2}(2) \leq 0.931$, see [12] for the upper bound. The lower bound can be obtained with a suitable choice of a 1-codimensional ball (just make a 2-dimensional picture).

Problem 4.7. *Does Θ_X really depends on both the moduli of AUC and AUS?*

Maybe there is a connection to the $\bar{\beta}_X$ modulus of Rolewicz, that takes an intermediate place between the AUC and AUS moduli, see [2, 1].

Problem 4.8. *If $\lim_n \Theta_X(n) > 0$, is c_0 crudely finitely representable in X ?*

The answer is affirmative if $Gz(B_X, \varepsilon) \geq \omega$ for some $\varepsilon > 0$, see [11].

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