ALMOST PRESERVED EXTREME POINTS

RAMÓN J. ALIAGA, LUIS C. GARCÍA-LIROLA, JUAN GUERRERO-VIU,
MATÍAS RAJA, AND ABRAHAM RUEDA ZOCA

ABSTRACT. In this paper we introduce the notion of an almost pre-
served extreme point (APEP) of a set as a weakening of the concept of
preserved extreme points, and we systematically study such points. As
a main result, we prove that a Banach space X has the Radon-Nikodým
property (RNP) if and only if every closed, convex, and bounded subset
of the space has an APEP. Similarly, we prove that X has the RNP if
and only if the unit ball of every equivalent renorming has an APEP. We
further investigate APEPs of the unit ball of classical Banach spaces,
absolute sums, Lipschitz-free spaces, and projective tensor products. In
the latter setting, our work also describes the preserved extreme points
in the unit ball under the assumption that every bounded operator is
compact, thereby partially solving an open problem.

CONTENTS

1

1. Introduction

A longstanding open question in the geometry of Banach spaces is whether the Radon-Nikodým property (RNP) and the Krein-Milman property (KMP) are equivalent. Recall that a Banach space X has the KMP if every closed, bounded, and convex subset of X has an extreme point. On the other hand, one of the equivalent reformulations of the RNP is that every closed, convex and bounded subset of X has a denting point (see e.g. [15, Section VII.6]). Since denting points are always extreme, the implication RNP $\Rightarrow$ KMP is clear. Whether the converse implication holds has motivated a vast literature since the 1980s (see e.g. [9, 10, 12, 24, 27, 34] and references therein).

Observe that there exists a fundamental difference between the concept of denting point (which is a metric notion) and the concept of extreme point (which is a linear notion). This distinction explains the difficulty behind the open question whether the KMP implies the RNP. Halfway between extreme points and denting points, we have an intermediate notion which reveals a rather better interplay with the RNP.

Given a Banach space X and a bounded, closed and convex subset C of X, we say that $x_0 \in C$ is a *preserved extreme point* of C (sometimes called *weak*-extreme point*) if x_0 is an extreme point of $\overline{C}^{w^*}$, where the weak* closure is taken in X^{**}.

Clearly, if $x_0 \in C$ is a preserved extreme point of C then it is an extreme point. Moreover, denting points are preserved extreme points. This can be easily seen from the fact that a point $x_0 \in C$ is a preserved extreme point if, and only if, given any weak neighbourhood W of C such that $x_0 \in W$ there exists a slice S of C with $x_0 \in S \subseteq W$ (see e.g. [19, Chapter 0]).

Coming back to the RNP, since denting points are preserved extreme points, it follows that if a Banach space X has the RNP then every bounded, closed, and convex subset C of X has a preserved extreme point. This time the converse is known to be true. For instance, [35, Theorem 1.1] establishes that if X is a Banach space failing the RNP then, given $\varepsilon > 0$, there exists a closed, convex and bounded subset C of X such that

$$\operatorname{dist}\left(\operatorname{ext}\left(\overline{C}^{w^*}\right), X\right) > \frac{1}{2} - \varepsilon,$$

where $\operatorname{ext}\left(\overline{C}^{w^*}\right)$ stands for the set of all the extreme points of $\overline{C}^{w^*}$.

In order to point out the difference between the notions of denting and preserved extreme points, remark that given a closed, bounded and convex subset C of X and $x_0 \in C$, then:

- x_0 is a denting point in C when the slices of C containing x_0 form a neighbourhood system of x_0 for the norm topology on C.
- x_0 is a preserved extreme point in C when the slices of C containing x_0 form a neighbourhood system of x_0 for the weak topology on C.

To obtain a more geometric description of the RNP, we will consider the following weakening of the concept of preserved extreme point.

Definition 1.1. Let X be a Banach space and let C be a bounded, closed and convex subset of X. We say that a point $x_0 \in C$ is an *almost preserved extreme point (APEP)* if, for every weakly open subset W of C containing x_0 there exists a non-empty slice S of C such that $S \subseteq W$.

The set of APEPs of C will be denoted $\mathrm{ape}\,(C)$.

Note that, unlike preserved extreme points, APEPs only require that every neighbourhood contains a slice, but the slice does not need to contain x_0.

At this point it is clear that if X has the RNP then every closed, convex and bounded subset of X has an APEP. A natural question is whether the converse is also true. We will give an affirmative answer in Section 5.

This paper presents an intensive study of the notion of APEP. We now outline the contents of the paper. In Section 2 we introduce all the necessary notation and preliminary results that we need for the main sections. In Section 3 we obtain the first results about APEP. In Theorem 3.8 we prove that, for a closed, convex and bounded set C of a Banach space X, $x_0 \in C$ is an APEP if and only if x is in the weak* closure (in X^{**}) of extreme points of $\overline{C}^{w^*}$. This characterisation will be exploited throughout the text. In Section 4 we study the APEP in some classical Banach spaces. We characterise the set of APEP of the unit ball of $L_p(\mu)$ spaces for $1 < p < \infty$ (Example 3.3), in $L_1(\mu)$ (Theorem 4.5) and in $C(K)$ spaces (Theorem 4.1). We also study APEP points in the unit ball of ℓ_p-sums of Banach spaces, for $1 \leqslant p \leqslant \infty$, in Subsection 4.3.

In Section 5, we prove that the notion of APEP has the desired interplay with the RNP and we provide new characterisations of that property. First of all, we prove in Theorem 5.3 that a Banach space X has the RNP if and only if every closed, convex, and bounded subset of X has an APEP. Using this result and renorming techniques from [35], we show that if a Banach space X fails the RNP then there exists an equivalent renorming of X whose unit ball fails to have any APEP (Theorem 5.5). As a consequence we get a second characterisation of the RNP using APEP: a Banach space X has the RNP if and only if the unit ball of every equivalent renorming of X contains an APEP (Corollary 5.8).

Next we move in Section 6 to the study of APEPs of the unit ball of Lipschitz-free spaces. We prove that any APEP must be either an elementary molecule or 0. We characterise APEP molecules as the norm limit of denting points, and provide examples of APEPs that are unpreserved extreme points.

In Section 7 we make an intensive study of the APEP in projective tensor products, and more precisely in sets of the form $\overline{\mathrm{co}}(C \otimes D)$ where $C \subseteq X$ and $D \subseteq Y$ are symmetric, bounded, closed, and convex subsets. Theorem 7.1 proves that if z is an APEP of $\overline{\mathrm{co}}(C \otimes D)$ then $z \in C \otimes D$. Conversely, we prove that $x_0 \otimes y_0$ is an APEP of $\overline{\mathrm{co}}(C \otimes D)$ if either x_0 is denting in C and y_0 is APEP in D (Theorem 7.3) or if x_0 and y_0 are APEP in C and D

respectively and $x_0 \otimes y_0$ has a compact neighbourhood system for the weak topology (Theorem 7.7). We close the paper by analysing the APEP and the preserved extreme points under the additional assumption that every operator $T \colon X \longrightarrow Y^*$ is compact. Under this assumption, we prove that if C and D have nonempty interior and $z \neq 0$ is a point in $\overline{\mathrm{co}}(C \otimes D)$, then z is an APEP if and only if $z = x_0 \otimes y_0$ for x_0 and y_0 being APEP in C and D respectively. Similarly, under the same assumptions, we prove that z is a preserved extreme point if and only if $z = x_0 \otimes y_0$ for x_0 and y_0 being preserved extreme points in C and D respectively. This provides a positive solution to [20, Question 3.9], where it is asked whether $x_0 \otimes y_0$ is a preserved extreme point of $B_{X \widehat{\otimes}_\pi Y}$ if both x_0 and y_0 are preserved extreme points in B_X and B_Y, under the assumption that every bounded operator from X to Y^* is compact.

Finally, we collect in Section 8 some remarks and open questions from our work. We also present after Lemma 8.8 a second proof of Theorem 5.3 with a different approach to the one exhibited in Section 5.

2. Notation and preliminary results

We will only consider real Banach spaces. Given a Banach space X, we denote by B_X and S_X its closed unit ball and unit sphere respectively. We also denote by X^* the topological dual of X. Given $E \subset X$, we write $\mathrm{span}(E)$ for the linear span of E. We denote $L(X, Y)$, $K(X, Y)$, and $F(X, Y)$ the spaces of bounded, compact, and finite-rank operators from X to Y, respectively.

Given a subset C of a Banach space X we denote by $\mathrm{co}(C)$ (resp. $\overline{\mathrm{co}}(C)$) the convex hull (respectively the closed convex hull) of C. Given $x^* \in X^*$ and $\alpha > 0$, we denote

$$S(C, x^*, \alpha) = \{x \in C : x^*(x) > \sup x^*(C) - \alpha\}$$

the (open) slice of C produced by x^*. If $X = Y^*$ is a dual Banach space and $x^* \in Y \subseteq Y^{**} = X^*$, the above set will be called a *weak* slice*.

2.1. **Extremal structure and Radon-Nikodým property.** Given a subset C of X, a point $x_0 \in C$ is said to be *an extreme point in C* if it is not the center of any non-degenerate line segment in C; in other words, if $x_0 = \frac{y+z}{2}$ for $y, z \in C$ implies $y = z = x_0$. We denote by $\mathrm{ext}\,(C)$ the set of all the extreme points in C.

Let us point out here several classical results in Banach space theory which shows the importance of extreme points in compact convex sets and that will be used throughout the text without explicit reference.

(1) (Krein-Milman theorem [17, Theorem 3.35]) If C is a weakly compact and convex subset of X then $C = \overline{\mathrm{co}}(\mathrm{ext}\,(C))$. Similarly, if C is a w^*-compact convex subset of X^* then $C = \overline{\mathrm{co}}^{w^*}(\mathrm{ext}\,(C))$.

(2) (Choquet lemma [17, Lemma 3.40]) If C is a weakly compact and convex subset of X, then for every $x \in \text{ext}(C)$ the slices of C containing x form a neighbourhood base of x in the weak topology of C. The same result holds for the weak* topology of X^* replacing slices with weak* slices.

(3) (Milman theorem [17, Theorem 3.41]) If C is a weakly compact and convex subset of X and $B \subseteq C$ is such that $\overline{\text{co}}(B) = C$ then $\text{ext}(C) \subseteq \overline{B}^w$. Similarly, if $C \subseteq X^*$ is w^* compact and convex and $B \subseteq C$ is such that $\overline{\text{co}}^{w^*}(B) = C$ then $\text{ext}(C) \subseteq \overline{B}^{w^*}$.

In the sequel we will present some strenghtenings of the concept of extreme point. Our main reference will be [19, Chapter 0].

Given a bounded set $C \subseteq X$, we say that $x_0 \in C$ is a *preserved extreme point* in C if x_0 is an extreme point of the weak* closure of C in X^{**} (that is, $x_0 \in \text{ext}\left(\overline{C}^{w^*}\right)$). We denote by pre-ext$(C)$ the set of all preserved extreme points of C.

For closed, bounded and convex $C \subseteq X$, it follows from Choquet's lemma that $x_0 \in \text{pre-ext}(C)$ if and only if slices of C containing x_0 form a neighbourhood base of x for the weak topology of C.

If, in the above characterisation, we replace the weak topology with the norm topology, we arrive to the notion of *denting point*. That is, a point $x_0 \in C$ is said to be a *denting point* if, for every $\varepsilon > 0$, there exists a slice S of C with $x_0 \in S$ and $\text{diam}(S) < \varepsilon$, where $\text{diam}(S)$ stands for the diameter of S. We denote by $\text{dent}(C)$ the set of all denting points of C.

The set of denting points plays a very important role in the Banach spaces with the *Radon-Nikodým property*. Let us recall that the Radon-Nikodým property, from now on RNP, was originally defined for Banach spaces by the validity of a vector version of the classic Radon-Nikodým theorem on derivation of measures. Namely, X has the RNP if for any σ-finite measure space (Ω, Σ, μ) and any μ-continuous vector measure $\nu \colon \Sigma \to X$ of bounded variation, there exists a Bochner integrable function $f \colon \Omega \to X$ such that

$$(2.1) \qquad \qquad \nu(A) = \int_A f \, d\mu$$

for every $A \in \Sigma$, see [15] or [11] for details. Moreover, RNP can be localized on closed convex subsets of X. We say that $C \subset X$ has the RNP if for any vector measure $\nu : \Sigma \to X$ that is absolutely continuous with respect to μ, as before, and having average range in C, meaning that $\nu(A)/\mu(A) \in C$ for any $A \in \Sigma$ with $\mu(A) > 0$, there is a Bochner integrable function $f \colon \Omega \to X$ satisfying (2.1). Note that for $C \subset X$ bounded and μ finite, the average range condition implies μ-continuity and bounded variation for ν. Another important observation is that the RNP can be witnessed just by the Lebesgue measure on $\mathbb{R}$ or $[0,1]$.

There are many characterizations of the RNP: differentiation of vector valued functions, integral representation of operators from $L^1(\mu)$ to a Banach space, convergence of vector valued martingales or descriptive topology. However, we are more interested in the geometrical characterizations of the RNP. A closed convex bounded subset $C \subset X$ has the RNP if and only if every nonempty (convex) subset has arbitrarily small slices, in such a case we say that C is (hereditarily) *dentable*. It turns out that if C has the RNP, then it or any of its nonempty subsets have denting points and, even, *strongly exposed points* (a denting point where the arbitrarily small slices can be taken parallel), see [11] for more information.

The equivalence among the different characterizations of the RNP is far from trivial. Despite the geometrical flavor of the main notion and techniques of this paper, we will occasionally appeal to measure theoretical characterizations of the RNP to get our results proven.

2.2. **Lipschitz-free spaces.** Let (M, d) be a complete metric space where a distinguished "base point" $0 \in M$ has been selected. The Lipschitz space $\mathrm{Lip}_0(M)$ is defined as the Banach space of all Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$, endowed with the norm given by the best Lipschitz constant

$$\|f\| = \sup \left\{ \frac{f(x) - f(y)}{d(x, y)} : x \neq y \in M \right\}.$$

For each $x \in M$, the evaluation functional $\delta(x) : f \mapsto f(x)$ belongs to the dual $\mathrm{Lip}_0(M)^*$. The *Lipschitz-free space over M* is defined as the closed space $\mathcal{F}(M) := \overline{\mathrm{span}} \{\delta(x) : x \in M\}$. It is not too hard to see that $\mathcal{F}(M)$ is, in fact, an isometric predual of $\mathrm{Lip}_0(M)$. We refer to [41] for basic facts about Lipschitz and Lipschitz-free spaces.

The most important elements of $\mathcal{F}(M)$ are the so-called *(elementary) molecules*, of the form

$$m_{xy} := \frac{\delta(x) - \delta(y)}{d(x, y)}$$

for $x \neq y \in M$. The set of molecules in $\mathcal{F}(M)$ will be denoted $\mathrm{Mol}\,(M)$. Molecules have norm 1, and it follows easily from the Hahn-Banach separation theorem that $B_{\mathcal{F}(M)} = \overline{\mathrm{co}}(\mathrm{Mol}\,(M))$ (see e.g. [41, Proposition 3.29]). The weak closure of $\mathrm{Mol}\,(M)$ is either $\mathrm{Mol}\,(M)$ or $\mathrm{Mol}\,(M) \cup \{0\}$, depending on whether M bi-Lipschitz embeds into some finite-dimensional Banach space or not (see [21, Proposition 2.9] and [22, Lemma 4.2]). Moreover, norm- and weak convergence agree on $\mathrm{Mol}\,(M)$: a net of molecules $(m_{x_s y_s})$ converges to m_{xy} precisely when $x_s \to x$ and $y_s \to y$ (see e.g. [21, Lemma 2.2] and [39, Lemma 1.2]).

Lipschitz-free spaces provide a convenient toolbox for the construction of Banach spaces with a predetermined extremal structure, because the various types of extremal objects in $B_{\mathcal{F}(M)}$ admit simple metric characterisations when M is complete:

- All extreme points of $B_{\mathcal{F}(M)}$ are molecules [6, Theorem 3.1].

- A molecule m_{xy} is extreme if and only if $d(x,p) + d(p,y) > d(x,y)$ whenever $p \neq x, y$ [5, Theorem 1.1].
- Preserved extreme points and denting points of $B_{\mathcal{F}(M)}$ agree [21, Theorem 2.4].
- A molecule m_{xy} is denting if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x,p) + d(p,y) > d(x,y) + \delta$ whenever $d(x,p) > \varepsilon$ and $d(y,p) > \varepsilon$ [4, Theorem 4.1].

We will also need the following notion introduced in [26]: we say that a function $f \in S_{\mathrm{Lip}_0(M)}$ is *local* if, for every $\varepsilon > 0$, there exist $u \neq v \in M$ such that $f(m_{uv}) > 1 - \varepsilon$ and $d(u,v) < \varepsilon$. This definition is a pointwise version of the notion of local metric space introduced in [23], and it was introduced in order to study Daugavet points in $\mathcal{F}(M)$.

Let us point out our interests in local Lipschitz functions in the following remark fo easy reference.

Remark 2.1. If f is local, then any non-empty slice $S(B_{\mathcal{F}(M)}, f, \varepsilon)$ has diameter 2. Even though this result is well known (and implicitly observed in $(2) \Rightarrow (1)$ in [26, Theorem 3.2]), let us briefly outline the proof for completeness: in the above situation, any such slice will contain molecules m_{uv} for which $d(u,v)$ is arbitrarily small, so our claim follows from [26, Theorem 2.6].

2.3. **Projective tensor products.** The projective tensor product of X and Y, denoted by $X \widehat{\otimes}_\pi Y$, is the completion of the algebraic tensor product $X \otimes Y$ endowed with the norm

$$\|z\|_\pi := \inf \left\{ \sum_{n=1}^{k} \|x_n\| \|y_n\| : z = \sum_{n=1}^{k} x_n \otimes y_n \right\},$$

where the infimum is taken over all such representations of z. The reason for taking the completion is that $X \otimes Y$ endowed with the projective norm is complete if and only if either X or Y is finite dimensional (see [33, p. 43, Exercises 2.4 and 2.5]).

It is well known that $\|x \otimes y\|_\pi = \|x\| \|y\|$ for every $x \in X$, $y \in Y$, and that the closed unit ball of $X \widehat{\otimes}_\pi Y$ is the closed convex hull of the set $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$. Throughout the paper, we will use both facts without any explicit reference.

Observe that every $G \in L(X, Y^*)$ acts on $X \widehat{\otimes}_\pi Y$ via

$$G\left(\sum_{n=1}^{k} x_n \otimes y_n \right) = \sum_{n=1}^{k} G(x_n)(y_n),$$

for $\sum_{n=1}^{k} x_n \otimes y_n \in X \otimes Y$. This action establishes a linear isometry from $L(X, Y^*)$ onto $(X \widehat{\otimes}_\pi Y)^*$ (see e.g. [33, Theorem 2.9]). Throughout this paper we will use the isometric identification $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ without any explicit mention.

Recall that a Banach space X has the *approximation property* (AP) if there exists a net $(S_\alpha)_\alpha$ in $F(X,X)$ such that $S_\alpha(x) \to x$ for all $x \in X$. It is not difficult to show that if either X or Y has the AP then $K(X,Y^*)$ is separating for $X\widehat{\otimes}_\pi Y$ (c.f. e.g. [20, Lemma 2.2]).

Recall that given two Banach spaces X and Y, the *injective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\varepsilon Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\|_\varepsilon := \sup\left\{ \sum_{i=1}^n |x^*(x_i)y^*(y_i)| : x^* \in S_{X^*}, y^* \in S_{Y^*} \right\},$$

where $u = \sum_{i=1}^n x_i \otimes y_i$ (see [33, Chapter 3] for background). Every $u \in X\widehat{\otimes}_\varepsilon Y$ can be viewed as an operator $T_u \colon X^* \to Y$ which is weak*-to-weak continuous. Under this point of view, the norm on the injective tensor product is nothing but the operator norm.

It is known that, given two Banach spaces X and Y, we have $(X\widehat{\otimes}_\varepsilon Y)^* = X^*\widehat{\otimes}_\pi Y^*$ if either X^* or Y^* has the RNP and either X^* or Y^* has the AP [33, Theorem 5.33].

3. First results and examples

In this section we will provide the first general results, examples and characterisations of APEPs in Banach spaces. We begin with a simple fact that will be used in subsequent sections.

Lemma 3.1. *Let X be a Banach space and $C \subseteq X$ be bounded, closed and convex. Then* $\mathrm{ape}\,(C)$ *is weakly closed.*

Proof. Suppose that $x \in C$ belongs to the weak closure of $\mathrm{ape}\,(C)$; we have to prove that $x \in \mathrm{ape}\,(C)$. Let W be a weak neighbourhood of x, and let us see that it contains a non-empty slice of C. By assumption, W contains some $y \in \mathrm{ape}\,(C)$. Thus W is a weak neighborhood of y and, since y is an APEP, is must contain a non-empty slice of C as desired. ∎

This already provides us with examples of situations where APEPs are plentiful. One simple case is the unit ball of an infinite dimensional Banach space X for which $\text{pre-ext}\,(B_X) = S_X$. Since S_X is weakly dense in B_X for every infinite dimensional Banach space X, we have the following consequence of Lemma 3.1.

Proposition 3.2. *Let X be an infinite dimensional Banach space such that* $\text{pre-ext}\,(B_X) = S_X$. *Then* $\mathrm{ape}\,(B_X) = B_X$.

Example 3.3. The above result applies to get that if X is an infinite dimensional $L_p(\mu)$ space, for $1 < p < \infty$, then every point of B_X is APEP.

Moreover this shows that, unlike extreme points, APEPs of B_X do not have to belong to S_X. In particular, in B_X there are APEPs which are not extreme.

This applies in particular for LUR norms. Recall that the norm of a Banach space X is *locally uniformly rotund* (LUR) if for all $x, x_n \in X$ satisfying $\lim_n (2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2) = 0$ one has $\lim_n \|x_n - x\| = 0$ (see [17, Definition 8.16]). It is not difficult to prove that if the norm of a Banach space X is LUR then every point of S_X is strongly exposed (in particular it is a preserved extreme point). Consequently we get the following result.

Corollary 3.4. *If X is an infinite dimensional Banach space whose norm is LUR, then every point of B_X is APEP.*

In particular, every infinite dimensional separable Banach space and every infinite dimensional reflexive space admits an equivalent renorming such that every point of the new unit ball is APEP.

Compare the last statement to Corollary 5.8.

Next, we give another simple lemma that can be understood as a generalisation of Milman's theorem.

Lemma 3.5. *Let X be a Banach space and $C \subseteq X$ be bounded, closed and convex. If $B \subseteq C$ is such that $\overline{\mathrm{co}}(B) = C$, then $\mathrm{ape}(C) \subseteq \overline{B}^w$.*

Proof. Note that every non-empty slice S of C contains a point of B. Indeed, let $S = \{x \in C : f(x) > \alpha\}$ for some $f \in X^*$, $\alpha \in \mathbb{R}$. If $S \cap B = \varnothing$ then $f(x) \leqslant \alpha$ for all $x \in B$ and thus for all $x \in \overline{\mathrm{co}}(B) = C$, so S must be empty.

Now let x be an APEP of C. Then every weak neighbourhood W of x contains a non-empty slice of C and therefore intersects B, so we conclude $x \in \overline{B}^w$. ∎

For some of our results, we will need the following, more precise conclusion in the context of Lemma 3.5.

Lemma 3.6. *Let X be a Banach space, $B \subseteq X$ a bounded set, and $C = \overline{\mathrm{co}}(B)$. Let $x \in \mathrm{ape}(C)$. Then either*

1) *there is $\varepsilon > 0$ and a net $(y_\alpha) \subset B$ such that $y_\alpha \xrightarrow{w} x$ and $\|y_\alpha - x\| \geqslant \varepsilon$ (i.e. $x \in \overline{B \setminus B(x, \varepsilon)}^w$), or*
2) *for every weakly open set W containing x and $\varepsilon > 0$ there exists a slice S of C such that $S \subseteq W$ and $\mathrm{diam}(S) < \varepsilon$.*

Proof. Assume that 2) does not hold. Then we can find a weakly open set W_0 containing x and $\varepsilon_0 > 0$ so that every slice S of C contained in W_0 satisfies that $\mathrm{diam}(S) \geqslant \varepsilon_0$.

Now, let W be a weakly open set containing x. Since x is APEP, we can find a slice $S = S(C, f, \alpha)$ with $S \subset W \cap W_0$. Take $0 < \delta < 1$. By [19, Lemma 5.2.14] we have

$$\varepsilon_0 \leqslant \mathrm{diam}(S(C, f, \delta\alpha)) \leqslant 2\,\mathrm{diam}(S(C, f, \alpha) \cap B) + 4\delta.$$

If we choose δ small enough to guarantee that $\frac{\varepsilon_0 - 4\delta}{2} > \frac{\varepsilon_0}{4}$, then it follows that there are $y, z \in S(C, f, \alpha) \cap B$ with $\|y - z\| \geqslant \frac{\varepsilon_0}{4}$. Then either $\|y - x\| \geqslant \frac{\varepsilon_0}{8}$ or $\|z - x\| \geqslant \frac{\varepsilon_0}{8}$.

In any case, we have proved that there is $y_W \in W \cap B$ with $\|y_W - x\| \geqslant \frac{\varepsilon_0}{8}$. It is clear that (y_W) is a net in B that converges weakly to x.   ∎

The following example shows that condition 2) in Lemma 3.6 can really fail to hold.

Example 3.7. In [2], a Banach space X isomorphic to $C([0,1])$ is constructed such that every non-empty relatively weakly open subset of B_X has diameter 2 and such that every point of S_X is a preserved extreme point of B_X. In particular, every point of B_X is APEP (Proposition 3.2).

We now provide a characterisation of APEPs of a given closed, bounded and convex set in terms of the extremal structure of its weak* closure.

Theorem 3.8. *Let X be a Banach space, $C \subseteq X$ be closed convex and bounded and $x \in C$. Let $D := \overline{C}^{w^*}$ be its weak* closure in X^{**}. The following are equivalent:*

(1) $x \in \mathrm{ape}\,(C)$.
(2) *For every weak* open subset W of D containing x there exists a weak* slice S of D such that $S \subseteq W$.*
(3) $x \in \overline{\mathrm{ext}\,(D)}^{w^*}$.

Proof. (1)⇒(2). Let W be a weak* open subset of D such that $x \in W$. Choose a weak* open subset V of D such that $x \in V \subseteq \overline{V}^{w^*} \subseteq W$. Consider $U := V \cap C$, which is a weakly open set of C with $x \in U$. Since x is APEP we can find a non-empty slice $S = S(C, f, \alpha)$ such that $S \subseteq U$. We claim that

$$S(D, f, \alpha) \subseteq \overline{V}^{w^*}.$$

Indeed, given any $z^{**} \in S(D, f, \alpha) \subseteq D$ we can find a net $z_s \xrightarrow{w^*} z^{**}$ such that $z_s \in C$ holds for every s. Since $f(z_s) \to f(z^{**}) > 1 - \alpha$ we can find an index s_0 such that $f(z_s) > 1 - \alpha$ holds for every $s \succcurlyeq s_0$. Since $z_s \in C$ we infer that $z_s \in S(C, f, \alpha) \subseteq U \subseteq V$ holds for every $s \succcurlyeq s_0$. Since $z_s \xrightarrow{w^*} z^{**}$ we infer that $z^{**} \in \overline{V}^{w^*}$, as desired. Since $\overline{V}^{w^*} \subseteq W$ we get that $S(D, f, \alpha) \subseteq W$, as required.

(2)⇒(3). Select a relatively weak* open subset W of D containing x and let us find an extreme point e^{**} of D such that $e^{**} \in W$. By (2) we can find a weak* slice S of D such that $S \subseteq W$. Now Krein-Milman theorem ensures that $D = \overline{\mathrm{co}}^{w^*}(\mathrm{ext}\,(D))$ (observe that D is weak* compact since C is bounded). Since S is a weak* slice of D we get that $\emptyset \neq S \cap \mathrm{ext}\,(D) \subseteq W \cap \mathrm{ext}\,(D)$. We have proved that every weak* neighbourhood of x contains an extreme point of D, so the implication is proved.

(3)⇒(1). Let W be a weakly open subset of C containing x, and let us prove that there exists a slice S of C with $S \subseteq W$. In order to do so define $\tilde{W}$ as the weak* open subset of D defined by W, that is, such that $\tilde{W} \cap C = W$. By (3) there exists an extreme point e^{**} of D such that

$e^{**} \in \tilde{W}$. Since e^{**} is an extreme point of D, Choquet lemma implies that there exists a slice $S = S(D, f, \alpha)$ such that $e^{**} \in S \subseteq \tilde{W}$. It is now clear that $S(C, f, \alpha) \subseteq \tilde{W} \cap C = W$, and the proof is finished. ∎

With Theorem 3.8 we can now provide an example of an APEP which is not an extreme point. The following example shows that this phenomenon may happen even in finite-dimensional Banach spaces.

Example 3.9. Let $C \subseteq \mathbb{R}^3$ be a compact set whose set of extreme points is not closed (c.f. e.g. [17, Exercise 3.86]), and set $x_0 \in \overline{\text{ext}(C)} \setminus \text{ext}(C) = \overline{\text{pre-ext}(C)} \setminus \text{pre-ext}(C)$ (since in the finite dimensional framework clearly every extreme point is preserved). Then x_0 is an APEP of C which is not an extreme point of C.

An example of an extreme point which is not APEP will be obtained in Example 6.6. Moreover, an example of an extreme point which is APEP but fails to be a preserved extreme point will be exhibited in Example 6.7.

Remark 3.10. It is a well-known result that if X is a 2-dimensional Banach space and $C \subseteq X$ is closed, convex, and bounded then the set $\text{ext}(C)$ is closed (c.f. e.g. [17, Exercise 3.86]). Consequently, in such situation every extreme point of C is APEP.

Let us provide another example, which shows that if a closed, convex, and bounded set satisfies that every point is APEP then this does not necessarily imply that every point is an extreme point, even if the set of extreme points is norm-dense.

Example 3.11. In [30] a compact convex set $K \subseteq \ell_2$ is constructed with the property that the set of all extreme points is dense in K. In such compact, all points are APEP but there are (densely many) points which are not extreme points.

4. APEP IN CLASSICAL BANACH SPACES

In this section, we pursue to characterise the APEPs of the unit ball of classical Banach spaces (namely $C(K)$ spaces and $L_p(\mu)$ for $1 \leqslant p \leqslant \infty$). We will also study APEPs in absolute sums of Banach spaces.

It is immediate that if $X = \ell_p^n$ for $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ then the set of (preserved) extreme points of B_X is closed. Consequently, the APEPs of B_X coincide with the extreme points. On the other hand, if $1 < p < \infty$ and $X = L_p(\mu)$ is infinite-dimensional, Example 3.3 shows that all points of B_X are APEP. The rest of the section will focus on the remaining cases.

4.1. $C(K)$ **spaces.** Let K be a compact Hausdorff topological space. In this subsection, we aim to provide a description of those points of $B_{C(K)}$ which are APEP. The main result is the following.

Theorem 4.1. *Let K be a compact Hausdorff topological space. Let $f \in B_{C(K)}$. The following are equivalent:*

(1) *f is an APEP of $B_{C(K)}$.*
(2) *$|f(t)| = 1$ holds for every $t \in K$.*
(3) *f is an extreme point of $B_{C(K)}$.*
(4) *f is a preserved extreme point of $B_{C(K)}$.*

Proof. (2)$\Rightarrow$(3) is straightforward, whereas (3)$\Rightarrow$(4) is well known (c.f. e.g. [29, p. 295]). Moreover, (4)$\Rightarrow$(1) is general. It remains to show (1)$\Rightarrow$(2). To this end, it will suffice to prove that for any $t_0 \in K$ and any non-empty slice S of $B_{C(K)}$ there exists some function $\varphi \in S$ with $|\varphi(t_0)| = 1$. Indeed, it follows that the weakly open subset

$$W := \{g \in B_{C(K)} : |\delta_{t_0}(g)| < 1\}$$

cannot contain any non-empty slice of $B_{C(K)}$ and therefore cannot contain any APEP either.

In order to do so, fix $t_0 \in K$ and a slice S of $B_{C(K)}$. We may assume that $S = S(B_{C(K)}, \mu, \alpha)$ where $\mu \in C(K)^* = M(K)$ is a regular Borel measure with $\|\mu\| = 1$. Observe that we can decompose $\mu = \lambda \delta_{t_0} + \nu$, where $\lambda \in \mathbb{R}$ and $\nu \in M(K)$ is such that $\nu(\{t_0\}) = 0$. Now select $h \in B_{C(K)}$ such that $\mu(h) > 1 - \frac{\alpha}{8}$. Since ν is a regular measure and $\nu(\{t_0\}) = 0$, there exists an open subset $U \subseteq K$ with $t_0 \in U$ such that $|\nu|(U) \leqslant \frac{\alpha}{8}$. Consider a Urysohn function $g \in S_{C(K)}$ such that $g(t) = 0$ if $t \notin U$, $0 \leqslant g \leqslant 1$ and $g(t) = 1$ at $\overline{V}$ for some open set V such that $t_0 \in V \subseteq \overline{V} \subseteq U$. Now take $j \in S_{C(K)}$ another Urysohn function such that $j(t) = 0$ if $t \notin V$ and $j(t_0) = 1$.

We consider $\mathrm{sign}(\lambda) = |\lambda|/\lambda$ if $\lambda \neq 0$ and $\mathrm{sign}(0) = 1$. Now, define

$$\varphi := (1 - g)h + \mathrm{sign}(\lambda)j.$$

It is clear that $\varphi(t_0) = \mathrm{sign}(\lambda) \in \{-1, 1\}$, so we only have to prove that $\varphi \in S$. Let us start by proving that $\|\varphi\|_\infty \leqslant 1$. Select any $t \in K$. Now we have two possibilities:

(1) If $t \notin V$ then $j(t) = 0$, so $|\varphi(t)| = |1 - g(t)||h(t)| \leqslant |h(t)| \leqslant 1$.
(2) If $t \in V$ then $g(t) = 1$ and thus $|\varphi(t)| = |j(t)| \leqslant 1$.

In any case we get $|\varphi(t)| \leqslant 1$. It remains to estimate $\mu(\varphi)$. Observe that

$$\mu(\varphi) = \mu((1 - g)h) + \mu(\mathrm{sign}(\lambda)j).$$

On the one hand, since $\delta_{t_0}(h(1-g)) = 0$ we get that $\mu((1-g)h) = \nu((1-g)h)$. Now

$$\nu((1-g)h) = \int_K (1-g)h\ d\nu = \int_{K\setminus U} (1-g)hd\nu + \int_U (1-g)h\ d\nu$$

$$\geqslant \int_{K\setminus U} h\ d\nu - \|(1-g)h\|_\infty |\nu|(U)$$

$$\geqslant \int_K h\ d\nu - \int_U h\ d\nu - |\nu|(U)$$

$$\geqslant \nu(h) - 2|\nu|(U) \geqslant \nu(h) - \frac{\alpha}{4}.$$

On the other hand,

$$\mu(\text{sign}(\lambda)j) = \text{sign}(\lambda)\lambda j(t_0) + \nu(\text{sign}(\lambda)j)$$

$$= |\lambda| + \text{sign}(\lambda)\int_K j\ d\nu$$

$$= |\lambda| + \text{sign}(\lambda)\int_V j\ d\nu$$

$$\geqslant |\lambda| - |\nu|(V) \geqslant |\lambda| - |\nu|(U)$$

$$\geqslant |\lambda| - \frac{\alpha}{8}.$$

Putting everything together we infer

$$\mu(\varphi) \geqslant \nu(h) - \frac{\alpha}{4} + |\lambda| - \frac{\alpha}{8} = |\lambda| + \nu(h) - \frac{3\alpha}{8}.$$

Taking into account that $|\lambda| \geqslant \lambda\delta_{t_0}(h)$ we clearly get that

$$\mu(\varphi) \geqslant (\lambda\delta_{t_0} + \nu)(h) - \frac{3\alpha}{8} = \mu(h) - \frac{3\alpha}{8} > 1 - \frac{\alpha}{8} - \frac{3\alpha}{8} > 1 - \alpha.$$

This implies that $\varphi \in S$ and the proof is finished. ∎

Let us show an immediate consequence of Theorem 4.1, for describing APEP in the unit ball of L_∞ spaces.

Corollary 4.2. *Let (Ω, Σ, μ) be a measure space and let I be an arbitrary set. Let $x = (x_i)_{i\in I} \in B_{\ell_\infty(I)}$ and $f \in B_{L_\infty(\mu)}$. Then,*

a) *$f \in \text{ape}\left(B_{L_\infty(\mu)}\right)$ if and only if $|f(\omega)| = 1$ for a.e. $\omega \in \Omega$.*

b) *$x \in \text{ape}\left(B_{\ell_\infty(I)}\right)$ if and only if $|x_i| = 1$ for all $i \in I$.*

Proof. a) We have $L_\infty(\mu) = C(K_\mu)$ isometrically, where K_μ is the maximal ideal space of $L_\infty(\mu)$ (see e.g. [43, Theorem 9.6]). The result follows from Theorem 4.1 and the well-known characterization of extreme points of $B_{L_\infty(\mu)}$.

b) is a particular case of a) taking μ as the counting measure on I, but can also be justified directly: we have $\ell_\infty(I) = C(\beta I)$ isometrically, where βI is the Stone-Čech compactification of I, and the result follows again

from Theorem 4.1 and the well-known characterization of extreme points of $B_{\ell_\infty(I)}$. ∎

4.2. $L_1(\mu)$ spaces. In this section we aim to characterise when $f \in B_{L_1(\mu)}$ is APEP, for a given measure space (Ω, Σ, μ). We limit our analysis to localisable measure spaces, which are precisely those for which $L_1(\mu)^*$ is isometrically isomorphic to $L_\infty(\mu)$ [18, Theorem 243G]. This is no loss of generality, as it is known that every $L_1(\mu)$ space is isometrically isomorphic to an ℓ_1-sum of spaces of the form $L_1(\mu_i)$ where μ_i is a finite, hence localisable, measure (c.f. e.g. [14, P. 501]). We will deal with APEPs in ℓ_1-sums of Banach spaces in Section 4.3.

Before we proceed, let us introduce a bit of notation. Recall that a measurable set $A \subset \Omega$ is called an *atom* for μ if $\mu(A) > 0$ and if $\mu(B) = 0$ for every measurable subset $B \subset A$ such that $\mu(B) < \mu(A)$. As a consequence of [25, Theorem 2.1] we can decompose $L_1(\mu)$ as

$$(4.1) \qquad\qquad L_1(\mu) = L_1(\nu) \oplus_1 \ell_1(I),$$

where ν is the continuous part of μ, and I is the set of all atoms for μ (up to a measure 0 set).

With the above description in mind, we will first analyse the APEPs of the unit ball of $L_1(\mu)$ in the case that μ either fails to have any atom or in the case that μ is purely atomic, and then we will complete the information with the stability results of the APEPs in ℓ_1-sums of spaces of the next section (see Proposition 4.7). In order to do so, let us start with the atomless case.

Proposition 4.3. *Let (Ω, Σ, μ) be a localisable measure space such that μ is atomless. Then* ape$\big(B_{L_1(\mu)}\big)$ *is empty.*

Proof. In this proof we will denote $L_1 = L_1(\mu)$ and $L_\infty = L_\infty(\mu) = L_1(\mu)^*$. Let $f \in B_{L_1}$ and let us prove that f is not APEP. In order to do so, let us begin with the case $f \neq 0$. Since $f = f^+ - f^-$, we may assume without loss of generality that $f^+ \neq 0$. Since $\int_\Omega f^+ d\mu \neq 0$ and μ is atomless we can find a subset $A \subseteq \Omega$ and $\alpha > 0$ such that

$$\alpha < \int_A f^+ \, d\mu = \int_A f \, d\mu < \frac{1}{2}.$$

Define $g := \chi_A \in S_{L_\infty}$ and set

$$W := \left\{ \varphi \in B_{L_1} : \alpha < g(\varphi) = \int_A \varphi \, d\mu < \frac{1}{2} \right\}.$$

Observe that $f \in W$. Indeed,

$$g(f) = \int_\Omega f \chi_A \, d\mu = \int_A f \, d\mu = \int_A f^+ \, d\mu \in \left(\alpha, \frac{1}{2} \right).$$

Let us now prove that the relatively weakly open set W cannot contain any slice of B_{L_1}.

Indeed, take a slice $S = S(B_{L_1}, h, \alpha)$ for $\alpha > 0$ and $h \in S_{L_\infty}$. By the very definition of essential supremum, there exists $\xi \in \{-1, 1\}$ and $B \subseteq \Omega$ with $\mu(B) > 0$ and such that

$$B = \{t \in \Omega : \xi h(t) > 1 - \alpha\}.$$

Now we have two different possibilities:

a) If $\mu(A \cap B) \neq 0$ then define the function $\varphi := \xi \frac{\chi_{A \cap B}}{\mu(A \cap B)}$. On the one hand we have

$$h(\varphi) = \int_\Omega h\varphi \, d\mu = \frac{1}{\mu(A \cap B)} \int_{A \cap B} \xi h(t) \, d\mu(t) > 1 - \alpha$$

since $\xi h(t) = |h(t)|$ on $A \cap B$. This implies $\varphi \in S$. On the other hand,

$$g(\varphi) = \int_\Omega g\varphi \, d\mu = \frac{\xi}{\mu(A \cap B)} \int_{A \cap B} g \, d\mu = \xi$$

since $g = 1$ on $A \cap B$. In particular $g(\varphi)$ is either 1 or -1, so $\varphi \notin W$.

b) If $\mu(A \cap B) = 0$ then define $\varphi := \xi \frac{\chi_B}{\mu(B)}$. As before, $h(\varphi) > 1 - \alpha$ (i.e. $\varphi \in S$), but clearly $g(\varphi) = \int_A \xi \frac{\chi_B}{\mu(B)} \, d\mu = 0$ since $\mu(A \cap B) = 0$. This proves that $\varphi \notin W$, as desired.

To finish the proof, it remains to be proved that 0 is also not an APEP of B_{L_1}. In order to do so, define

$$W := \left\{\varphi \in B_{L_1} : \left|\int_\Omega \varphi \, d\mu\right| < \frac{1}{2}\right\}.$$

It is immediate that W is a relatively weakly open set containing 0. However, it does not contain any slice of B_{L_1}. Indeed, given any slice S of B_{L_1}, by the proof of the above case we can find $\xi \in \{-1, 1\}$ and $C \subseteq \Omega$ with $\mu(C) > 0$ such that $\varphi := \xi \frac{\chi_C}{\mu(C)} \in S$. However,

$$\int_\Omega \varphi \, d\mu = \frac{\xi}{\mu(C)} \int_\Omega \chi_C \, d\mu = \xi \notin \left(-\frac{1}{2}, \frac{1}{2}\right).$$

∎

Now we move to the purely atomic case, obtaining the following result.

Proposition 4.4. *Let I be a non-empty set and consider $X = \ell_1(I)$. Then the APEPs of $B_{\ell_1(I)}$ are the standard basis vectors $\pm e_i$, $i \in I$.*

Proof. Since $B_{\ell_1(I)} = \overline{\text{co}}(\{\pm e_i : i \in I\})$ we infer from Lemma 3.5 that if x is an APEP of $B_{\ell_1(I)}$ then $x \in \overline{\{\pm e_i : i \in I\}}^w$. Evaluating against functionals of the form $e_i^* \in \ell_\infty(I)$ and the constant function 1_I yields $e_i^*(x) \in \{-1, 0, 1\}$ and $1_I(x) \in \{-1, 1\}$, and we deduce that $x \in \{\pm e_i : i \in I\}$. In the opposite direction, it is clear that every element of the form $\pm e_i$ is a denting point and therefore an APEP. ∎

Now we can give a description of the APEPs in an $L_1(\mu)$ space.

Theorem 4.5. *Let (Ω, Σ, μ) be a localisable measure space and let $f \in B_{L_1(\mu)}$. The following are equivalent:*

(1) *f is an APEP of $B_{L_1(\mu)}$.*
(2) *f is a denting point of $B_{L_1(\mu)}$.*
(3) *$f = \pm \frac{\chi_A}{\mu(A)}$, where A is an atom of μ.*

Proof. It is well known that $(3) \Rightarrow (2) \Rightarrow (1)$. We prove $(1) \Rightarrow (3)$. According to the decomposition in (4.1) we see $f = (g, h) \in L_1(\nu) \oplus_1 \ell_1(I)$.

Proposition 4.6 below together with the preceding paragraph yield that f is APEP if, and only if, either $g = 0$ and $h \in \mathrm{ape}\left(B_{\ell_1(I)}\right)$ or $h = 0$ and $g \in \mathrm{ape}\left(B_{L_1(\nu)}\right)$. However the latter is impossible due to Proposition 4.3. Consequently, f is APEP if, and only if, $f = (0, h)$, where h is APEP in $B_{\ell_1(I)}$. But now Proposition 4.4 implies that the above holds true if, and only if, $h = \pm e_i$ for some $i \in I$. Now, taking into account the identification of $\ell_1(I)$ with the purely atomic measures in (4.1), the result follows. ∎

4.3. **Absolute sums of Banach spaces.** Now, we focus on studying APEPs of the unit ball in ℓ_p-sums of Banach spaces. We start with the case $p = 1$. For finite sums, we have an easy characterisation.

Proposition 4.6. *Let Y, Z be Banach spaces. Then*

$$\mathrm{ape}\left(B_{Y \oplus_1 Z}\right) = (\mathrm{ape}\left(B_Y\right) \times \{0\}) \cup (\{0\} \times \mathrm{ape}\left(B_Z\right)).$$

Proof. Let $X = Y \oplus_1 Z$, then $X^{**} = Y^{**} \oplus_1 Z^{**}$ and therefore

$$\mathrm{ext}\left(B_{X^{**}}\right) = (\mathrm{ext}\left(B_{Y^{**}}\right) \times \{0\}) \cup (\{0\} \times \mathrm{ext}\left(B_{Z^{**}}\right)).$$

Clearly, $Y^{**} \times \{0\} \subseteq X^{**}$ is weak*-weak*-homeomorphic to Y^{**} (and similarly for $\{0\} \times Z^{**}$), so

$$\overline{\mathrm{ext}\left(B_{X^{**}}\right)}^{w^*} = (\overline{\mathrm{ext}\left(B_{Y^{**}}\right)}^{w^*} \times \{0\}) \cup (\{0\} \times \overline{\mathrm{ext}\left(B_{Z^{**}}\right)}^{w^*}).$$

The result now follows immediately from Theorem 3.8. ∎

The argument of Proposition 4.6 extends seamlessly (or by induction) to finite ℓ_1-sums. For infinite sums, however, the bidual does not admit such a simple expression so a different argument is needed. In that case, we are able to characterise all non-zero APEPs.

Proposition 4.7. *Let X be the ℓ_1-sum of a family $\{X_i : i \in I\}$ of Banach spaces.*

a) *If $x = (x_i) \neq 0$ is an APEP of B_X, then there exists an index $j \in I$ such that $x_j \in \mathrm{ape}\left(B_{X_j}\right)$ and $x_i = 0$ for all $i \neq j$.*
b) *If $x_j \in \mathrm{ape}\left(B_{X_j}\right)$, then the element $x = (u_i)$ defined as $u_j = x_j$ and $u_i = 0$ for $i \neq j$ is an APEP of B_X.*

Proof. a) Let $x = (x_i)$ be an APEP of B_X, and suppose that there are two indices $j_1 \neq j_2 \in I$ such that x_{j_1}, x_{j_2} are non-zero. For $k = 1, 2$ let $\varphi_k \in S_{X_{j_k}}$

be such that $\varphi_k(x_{j_k}) = \|x_{j_k}\|$, and consider the set

$$W = \{z = (z_i) \in B_X \,:\, \varphi_1(z_{j_1}) > 0 \text{ and } \varphi_2(z_{j_2}) > 0\}\,.$$

Then W is a relatively weakly open neighbourhood of x. Since x is APEP, W must contain a non-empty slice of the form $S = S(B_X, f, \alpha)$ for some $f \in S_{X^*}$ and $\alpha > 0$. Identify X^* with the ℓ_∞-sum of the spaces $\{X_i^* : i \in I\}$ and write $f = (f_i)$. Suppose that $\|f_j\| > 1 - \alpha$ for some $j \in I$, choose $y \in B_{X_j}$ such that $f_j(y) > 1 - \alpha$ and let $z = (z_i)$ be such that $z_j = y$ and $z_i = 0$ for $i \neq j$. Then $f(z) = f_j(y) > 1 - \alpha$ so $z \in S \subseteq W$, but this is not possible as either $z_{j_1} = 0$ or $z_{j_2} = 0$. This shows that $\|f_j\| \leqslant 1 - \alpha$ for all $j \in I$, contradicting $\|f\| = 1$. Hence, there must exist $j \in I$ such that $x_i = 0$ for all $i \neq j$.

Let us see that x_j is an APEP of B_{X_j} if $x_j \neq 0$. Let $W \subseteq B_{X_j}$ be a relatively weakly open neighbourhood of x_j, and let $\varphi \in X_j^*$ such that $\varphi(x_j) = \|x_j\| > 0$. Define

$$W' = \{z = (z_i) \in B_X : z_j \in W,\ \varphi(z_j) > 0\}\,,$$

which is clearly a relatively weakly open subset of B_X and $x \in W'$. Since x is APEP, there is a non-empty slice $S = S(B_X, f, \alpha)$, for some $f = (f_i) \in S_{X^*}$ and $\alpha > 0$, such that $S \subseteq W'$. We claim that $\|f_i\| \leqslant 1 - \alpha$ for all $i \neq j$. If not, there is some $k \neq j$ such that $\|f_k\| > 1 - \alpha$, and so we can find $y \in B_{X_k}$ such that $f_k(y) > 1 - \alpha$. Hence, the point $z = (z_i) \in B_X$ such that $z_i = 0$ for all $i \neq k$ and $z_k = y$ satisfies that $f(z) = f_k(y) > 1 - \alpha$. Thus, $z \in S \subseteq W'$ but $\varphi(z_j) = 0$, obtaining a contradiction. Therefore, $\|f_i\| \leqslant 1 - \alpha$, for all $i \neq j$, and since $X^* = (\bigoplus X_i^*)_\infty$, it follows that $\|f_j\| = 1$. Now, define the slice $S_j = \{y \in B_{X_j} : f_j(y) > 1 - \alpha\}$ which is clearly non-empty. If $y \in S_j$, then the point $z = (z_i) \in B_X$ such that $z_i = 0$ for $i \neq j$ and $z_j = y$ satisfies that $z \in S \subseteq W'$. Thus, $z_j = y \in W$, concluding that $S_j \subseteq W$ and x_j is then an APEP.

b) Let $x = (x_i)$ be a point in B_X such that there is $j \in I$ with $x_i = 0$ for $i \neq j$ and $x_j \in \operatorname{ape}\left(B_{X_j}\right)$. Let us show that $x \in \operatorname{ape}\left(B_X\right)$. Let W be a relatively weakly open neighbourhood of x in B_X. We may assume that W is a basic weakly open set of the form

$$W = \{y \in B_X : |f^1(y - x)| < \varepsilon, \ldots, |f^n(y - x)| < \varepsilon\}$$

for some functionals $f^1 = (f_i^1), \ldots, f^n = (f_i^n)$ in the unit ball of $X^* = (\bigoplus X_i^*)_\infty$ and some $\varepsilon > 0$. Now, consider

$$W_j = \left\{z \in B_{X_j} : |f_j^1(z - x_j)| < \frac{\varepsilon}{2}, \ldots, |f_j^n(z - x_j)| < \frac{\varepsilon}{2}\right\},$$

which is a relatively weakly open neighbourhood of x_j in B_{X_j}. Since x_j is APEP and $x_j \in W_j$, we can find a slice $S_j = \{z \in B_{X_j} : g(z) > 1 - \alpha\}$ for some $g \in S_{X_j^*}$ and $\alpha > 0$, such that $S_j \subseteq W_j$. We can always assume that $\alpha \leqslant \frac{\varepsilon}{2}$ since S_j is non-empty for all $\alpha > 0$. Finally, consider the slice $S = \{y = (y_i) \in B_X : g(y_j) > 1 - \alpha\}$. It is clear that if $z \in S_j \neq \emptyset$, then the point $y = (y_i) \in B_X$ such that $y_i = 0$ for $i \neq j$ and $y_j = z$, satisfies that

$y \in S$, so $S \neq \emptyset$. Furthermore, pick $y = (y_i) \in S$, and denote by $\hat{y} = (\hat{y}_i)$ the element of X such that $\hat{y}_i = y_i$ for $i \neq j$ and $\hat{y}_j = 0$. It is clear that $\|y_j\| \geqslant g(y_j) > 1 - \alpha \geqslant 1 - \frac{\varepsilon}{2}$, so $\|\hat{y}\| = \|y\| - \|y_j\| < \frac{\varepsilon}{2}$. Therefore, we have

$$|f^p(y-x)| \leqslant |f_j^p(y_j - x_j)| + |f^p(\hat{y})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad 1 \leqslant p \leqslant n,$$

since $y_j \in S_j \subseteq W_j$. This proves that $y \in W$, from which we conclude that $S \subseteq W$. Thus x is an APEP of B_X. ∎

The only case that remains unclear is whether it is possible to have $0 \in \mathrm{ape}\,(B_X)$ when $0 \notin \mathrm{ape}\,(B_{X_i})$ for all $i \in I$.

Next, we consider ℓ_p-sums of Banach spaces for $1 < p < \infty$. It is easier to use arguments based on Theorem 3.8 in this case as, given a family $\{X_i : i \in I\}$ of Banach spaces, one has $(\bigoplus_i X_i)_p^{**} = (\bigoplus_i X_i^{**})_p$ and
(4.2)
$$\mathrm{ext}\,\big(B_{(\bigoplus X_i)_p}\big) = \left\{ (x_i) \in S_{(\bigoplus X_i)_p} \ : \ \forall i, x_i = 0 \text{ or } \frac{x_i}{\|x_i\|} \in \mathrm{ext}\,(B_{X_i}) \right\}.$$

Using these descriptions, we can begin with the following necessary condition for APEPs in the unit sphere.

Lemma 4.8. *Let X and Y be two Banach spaces and let $(x_0, y_0) \in S_{X \oplus_p Y}$ be an APEP of $B_{X \oplus_p Y}$. Then either $x_0 = 0$ or $\frac{x_0}{\|x_0\|}$ is an APEP of B_X.*

Proof. Assume that $x_0 \neq 0$ and let us prove that $\frac{x_0}{\|x_0\|}$ is an APEP. Since (x_0, y_0) is an APEP of $B_{X \oplus_p Y}$ we can find by virtue of Theorem 3.8 a net (e_s, f_s) of extreme points of $B_{(X \oplus_p Y)^{**}} = B_{X^{**} \oplus_p Y^{**}}$ such that $(e_s, f_s) \xrightarrow{w^*} (x_0, y_0)$. This implies that both $(e_s) \xrightarrow{w^*} x_0$ and $(f_s) \xrightarrow{w^*} y_0$. Since both (e_s) and (f_s) are bounded nets we can assume, up to taking subnets, that $\|e_s\| \to \lambda$ and $\|f_s\| \to \mu$. The w^*-lower semicontinuity of the norm of X^{**} and Y^{**} implies $\|x_0\| \leqslant \lambda$ and $\|y_0\| \leqslant \mu$. Note that

$$1 = \|x_0\|^p + \|y_0\|^p \leqslant \lambda^p + \mu^p = \lim_s \|e_s\|^p + \|f_s\|^p = \lim_s \|(e_s, f_s)\|^p \leqslant 1$$

so indeed $\|x_0\| = \lambda$ and $\|y_0\| = \mu$.

Now, up to taking a further subnet, since $\|e_s\| \to \lambda = \|x_0\| > 0$ we can assume that $e_s \neq 0$ holds for every s. Since (e_s, f_s) is an extreme point of $B_{X^{**} \oplus_p Y^{**}}$ and $e_s \neq 0$ we infer $\frac{e_s}{\|e_s\|} \in \mathrm{ext}\,(B_{X^{**}})$ for every s. Since $e_s \xrightarrow{w^*} x_0$ and $\|e_s\| \to \|x_0\|$ we get that

$$\frac{e_s}{\|e_s\|} \xrightarrow{w^*} \frac{x_0}{\|x_0\|},$$

obtaining that $\frac{x_0}{\|x_0\|}$ is an APEP of B_X by Theorem 3.8, as it is the weak* limit of a net of extreme points of $B_{X^{**}}$. ∎

Now we are able to characterise APEPs of norm 1 for $1 < p < \infty$.

Proposition 4.9. *Let X be the ℓ_p-sum of a family $\{X_i : i \in I\}$ of Banach spaces, where $1 < p < \infty$, and let $(x_i) \in S_X$. The following assertions are equivalent:*

(1) *(x_i) is an APEP of B_X.*
(2) *For every $i \in I$, either $x_i = 0$ or $\frac{x_i}{\|x_i\|}$ is an APEP of B_{X_i}.*

Proof. (1)$\Rightarrow$(2): Observe that given $i \in I$ we have

$$X = X_i \oplus_p Y,$$

where Y is ℓ_p-sum of the family $\{X_j : i \in I \setminus \{i\}\}$ and the above identification is an isometric isomorphism. Now the result is a direct application of Lemma 4.8.

(2)$\Rightarrow$(1): Let U be a weak* neighbourhood of (x_i) in X^{**}. We will show that U intersects $\mathrm{ext}\,(B_{X^{**}})$ and this will be enough by Theorem 3.8. Since the weak* topology of X^{**} is the product topology of the weak* topologies in X_i^{**}, we may assume that $U = \prod_{i \in I} U_i$ where U_i is a weak* neighbourhood of x_i (or X_i^{**}) for each $i \in I$. If $x_i \neq 0$ then, by assumption and Theorem 3.8, there exists $e_i \in \mathrm{ext}\,(B_{X_i^{**}})$ such that $\|x_i\|\,e_i \in U_i$. Let $y = (y_i) \in X^{**}$ be defined by

$$y_i = \begin{cases} \|x_i\|\,e_i & \text{, if } x_i \neq 0 \\ 0 & \text{, if } x_i = 0 \end{cases}.$$

Note that $y \in U$ and

$$\|y\| = \left(\sum_{i \in I} \|y_i\|^p\right)^{1/p} = \left(\sum_{i \in I} \|x_i\|^p\right)^{1/p} = \|(x_i)\| = 1$$

since $(x_i) \in S_X$, so $y \in \mathrm{ext}\,(B_{X^{**}})$ by (4.2). This ends the proof. ∎

Of course, Proposition 4.9 does not cover all APEPs of the unit ball of ℓ_p-sums as it is possible to have APEPs of norm strictly less than 1. For instance, the space ℓ_p can be expressed as the ℓ_p-sum of countably many copies of $\mathbb{R}$, and $\mathrm{ape}\,(B_{\ell_p}) = B_{\ell_p}$ for $1 < p < \infty$ by Proposition 3.2.

The analysis of APEPs in an infinite ℓ_∞-sum of Banach spaces becomes troublesome because we do not have a simple description of its dual, let alone its bidual. So, instead, we conclude with the case of c_0-sums of Banach spaces. In this case, we can characterise APEPs of the unit ball completely.

Proposition 4.10. *Let X be the c_0-sum of a family $\{X_i : i \in I\}$ of Banach spaces. Then $x = (x_i)$ is an APEP of B_X if and only if x_i is an APEP of B_{X_i} for each $i \in I$.*

Proof. The space X^{**} can be identified with the ℓ_∞-sum of the spaces X_i^{**} and, therefore, the extreme points of $B_{X^{**}}$ are precisely the elements of the form (x_i^{**}) where $x_i^{**} \in \mathrm{ext}\,(B_{X_i^{**}})$ for all $i \in I$. Since the weak* topology of X^{**} is the product topology of the weak* topologies in X_i^{**}, the result is now clear from Theorem 3.8. ∎

Example 4.11. As consequence of Proposition 4.10, the unit ball of $c_0(I)$ does not have any APEP.

5. A CHARACTERISATION OF THE RNP IN TERMS OF APEP

In this section we aim to prove a strong connection between the notion of APEP and the RNP. On the one hand, we aim to prove that a Banach space X has the RNP if, and only if, every bounded, closed and convex subset of X has an APEP. On the other hand, we will prove that a Banach space X has the RNP if, and only if, the unit ball of every equivalent renorming of X has an APEP.

Let us start with the first of our objectives. In order to do so, we need a bit of notation. We say that a bounded subset $A \subset X^*$ is *relatively convexly resolvable* if for every w^*-compact $K \subset X^*$ with $A \cap K = \emptyset$, there is a decreasing transfinite sequence of convex and w^*-compact sets $(D_\alpha)_{\alpha \leqslant \kappa}$ such that

$$A \subset \bigcup_{\alpha \text{ is odd}} (D_\alpha \setminus D_{\alpha+1}),$$

and

$$K \cap \bigcup_{\alpha \text{ is odd}} (D_\alpha \setminus D_{\alpha+1}) = \emptyset.$$

We say that a subset of a Banach space $A \subset X$ is relatively convexly resolvable if it is so as considered in X^{**}. Those notions are an elaboration on the definition of resolvable sets, see [28, Appendix A.5] for instance, of topological nature. The label "relative" refers to the fact that the family of sets depends on the disjoint w^*-compact.

Recall that the average range of a vector measure $\nu \colon \Sigma \to X$ with respect to a positive measure μ on the same measurable space (Ω, Σ) is the set

$$\left\{ \frac{\nu(A)}{\mu(A)} : A \in \Sigma, \mu(A) \neq 0 \right\} \subset X.$$

Proposition 5.1. *Let $C \subset X$ be a bounded closed convex set. Assume it is relatively convexly resolvable. Then C has the RNP.*

The following proof we will based on the theory of liftings, see [38]. Given a measure space (Ω, Σ, μ) denote by $\mathcal{L}^\infty(\mu)$ the set of real-valued bounded measurable functions equipped with the essential supremum seminorm and denote by $L^\infty(\mu)$ the Banach space obtained by identifying the functions that agree almost everywhere. A lifting on a measure space (Ω, Σ, μ) is a linear and multiplicative operator $\rho \colon L^\infty(\mu) \to \mathcal{L}^\infty(\mu)$ that acts as a right inverse to the canonical quotient map from $\mathcal{L}^\infty(\mu)$ to $L^\infty(\mu)$. In other words, a lifting determines a representative of any coset from $\mathcal{L}^\infty(\mu)$ in an algebraically coherent way. The existence of a lifting requires some technical assumptions on the measure space that are fulfilled by Lebesgue measure, which is enough for our characterization of the RNP.

Proof. The following argument is essentially developed in [31]. Firstly, recall an old result of Tortrat, see [36], saying that X is universally measurable in (X^{**}, w^*), that is, that X is μ-measurable for every finite Radon measure μ on X^{**}. This implies that C is universally measurable in (X^{**}, w^*) as well. Let (Ω, Σ, μ) a probability space and let $\nu \colon \Sigma \to X$ be a μ-continuous vector measure with average range in C. We claim that there exists a w^*-Borel measurable density $f \colon \Omega \to X^{**}$, that is, $\langle \nu(D), x^* \rangle = \int_D \langle f, x^* \rangle \, d\mu$ for every $D \in \Sigma$ and $x^* \in X^*$.

Indeed, for any $x^* \in X^*$, the signed measure $\langle \nu, x^* \rangle$ is μ-continuous, so it has a Radon-Nikodým derivative $f_{x^*} \in L^1(\mu)$. Let ρ be a lifting of $L^\infty(\mu)$. It is easy to check that the map $x^* \to \rho(f_{x^*})(\omega)$ is linear for every $\omega \in \Omega$ and bounded by $\|x^*\|$, so there is $x_\omega^{**} \in B_{X^{**}}$ such that $x_\omega^{**}(x^*) = \rho(f_{x^*})(\omega)$. Clearly, the map defined by $f(\omega) = x_\omega^{**}$ is w^*-scalarly measurable, so it is w^*-Baire measurable by [16, Theorem 2.3]. Recall that the measure image, also called pushforward measure, is defined as $\mu \circ f^{-1}(A) = \mu(f^{-1}(A))$. We claim that f also is w^*-Borel measurable and $\mu \circ f^{-1}$ is w^*-Radon. Indeed, if ρ_K is the abstract lifting considered in [8, §2] for the compact space $K = B_{X^{**}}$, then

$$h \circ \rho_K(f)(\omega) = \rho(h \circ f)(\omega)$$

for every $\omega \in \Omega$ and every $h \in C(K)$. From the definition of f we get that $\rho_K(f) = f$ just taking as continuous functions h the elements $x^* \in X^*$. The desired properties follow from [8, Theorem 2.1].

Now note that we may assume, without loss of generality, that f takes values in $\overline{C}^{w^*}$. We will prove that, actually, f has almost all of its values in C. Indeed, take any w^*-compact subset $K \subset \overline{C}^{w^*} \setminus C$. By the hypothesis, there is a decreasing transfinite sequence of convex and w^*-compact sets $(D_\alpha)_{\alpha \leqslant \kappa}$ such that

$$C \subset \bigcup_{\alpha \text{ is odd}} (D_\alpha \setminus D_{\alpha+1}).$$

In particular, $\overline{C}^{w^*} \subset D_1$ and

$$K \subset \bigcup_{\alpha \text{ is even}} (D_\alpha \setminus D_{\alpha+1}).$$

Assume $(\mu \circ f^{-1})(K) > 0$. Then there would be a smallest α such that

$$(\mu \circ f^{-1})(K \cap D_\alpha) = (\mu \circ f^{-1})(K), \text{and}$$

$$(\mu \circ f^{-1})(K \cap D_{\alpha+1}) < (\mu \circ f^{-1})(K).$$

That implies $(\mu \circ f^{-1})(K \cap (D_\alpha \setminus D_{\alpha+1})) > 0$. Take $S \subset K \cap (D_\alpha \setminus D_{\alpha+1})$ a w^*-compact with $(\mu \circ f^{-1})(S) > 0$ and such that S supports $(\mu \circ f^{-1})|_S$ (Radon measures have always support). By the Hahn-Banach theorem there is a w^*-closed halfspace H such that $H \cap D_{\alpha+1} = \emptyset$ and $H \cap S \neq \emptyset$. Since S is a measure support, $(\mu \circ f^{-1})(H \cap S) > 0$. That would imply together

$(\mu \circ f^{-1})(\overline{C}^{w^*} \cap H) > 0$ and $H \cap C = \emptyset$, meaning that the average range of ν lies outside of C, which is a contradiction.

Now we know that f takes almost all of its values in C. Note that $\mu \circ f^{-1}$ is w^*-Radon and therefore w-Radon in X. A classic result attributed by Talagrand to Phillips and Grothendieck [37, 16] says that $\mu \circ f^{-1}$ is the restriction of a Radon measure on $(X, \|\cdot\|)$. That implies that the range of f is essentially separable. Since f is scalarly measurable, we deduce that f is Bochner measurable, and that concludes the proof that C has the RNP. ∎

Proposition 5.2. *Let $C \subset X$ be a bounded closed convex set. Assume that for any w^*-compact $K \subset X^{**}$ such that $C \cap K = \emptyset$ and any nonempty convex subset $B \subset C$ there exists a w^*-open halfspace H in X^{**} such that $B \cap H \neq \emptyset$ and*

$$\overline{B}^{w^*} \cap H \cap K = \emptyset.$$

Then C is relatively convexly resolvable.

Proof. Let K be a w^*-compact set in X^{**} such that $C \cap K = \emptyset$. We will build a sequence (D_α) satisfying the definition and such that $D_\alpha = \overline{B_\alpha}^{w^*}$ for α an odd ordinal. Take $D_1 = \overline{C}^{w^*}$. By assumption, there is a w^*-open halfspace H such that $\overline{C}^{w^*} \cap H \cap K = \emptyset$. Then take $D_2 = D_1 \setminus H$ and

$$D_3 = \overline{C \setminus H}^{w^*}.$$

Assume that α is odd and $D_\alpha = \overline{B_\alpha}^{w^*}$ is nonempty. Find a w^*-open halfspace H as in the hypothesis so $D_\alpha \cap H \cap K = \emptyset$. Now take $D_{\alpha+1} = D_\alpha \setminus H$ and $B_{\alpha+2} = B_\alpha \setminus H$, thus $D_{\alpha+2} = \overline{B_{\alpha+2}}^{w^*}$. For limit ordinals α, just take the intersection $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$ and $B_{\alpha+1} = D_\alpha \cap C$ (recall that limit ordinals are even). Continue this process while the sets are nonempty. In that case, C will be exhausted and therefore

$$C \subset \bigcup_{\alpha \text{ is odd}} (D_\alpha \setminus D_{\alpha+1}),$$

and that last set does not intersect K. ∎

Now we obtain the first main result in this section.

Theorem 5.3. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has RNP.*
(2) *Every closed, convex and bounded subset C of X has an APEP.*

Proof. (1)⇒(2): Every closed, convex and bounded subset C of X has a denting point since X has RNP, and denting points are APEP.

(2)⇒(1): Let C be a bounded, closed and convex subset of X and let us prove that C satisfies the hypothesis of Proposition 5.2. When this is proved, an application of Propositions 5.2 and 5.1 will yield the result.

In order to prove that C satisfies the hypothesis of Proposition 5.2, let $K \subseteq X^{**}$ be a w^*-compact subset such that $C \cap K = \emptyset$, and take any non-empty convex subset $B \subseteq C$. We can assume with no loss of generality that B is closed. Take $x_0 \in \text{ape}\,(B)$, and note that $x_0 \notin K$. Since K is w^*-closed then we can find a w^*-open set W in X^{**} such that $x_0 \in W$ and $\overline{W}^{w^*} \cap K = \emptyset$. Since x_0 is APEP and $W \cap B$ is a weakly open subset of B containing x_0 we can find a non-empty slice $S = \{x \in X : x^*(x) > \sup x(B) - \alpha\} \subseteq W \cap B$. Consider the w^*-open halfspace

$$H := \{x^{**} \in X^{**} : x^{**}(x^*) > \sup x^*(B) - \alpha\}.$$

Clearly $H \cap B = S$ is non-empty. We claim that

$$\overline{B}^{w^*} \cap H \cap K = \emptyset.$$

This will follow from the fact that $\overline{B}^{w^*} \cap H \subseteq \overline{W}^{w^*}$. Let us prove that. In order to do so, take $x^{**} \in \overline{B}^{w^*} \cap H$. By the w^*-density of B we can find a net (x_s) in B that w^*-converges to x^{**}. Since H is w^*-open there exists s_0 such that $x_s \in H$ holds for every $s \succeq s_0$. Now, for $s \succeq s_0$ we get that $x_s \in B \cap H = S \subseteq W \cap B \subseteq W$. Therefore $x^{**} \in \overline{W}^{w^*}$, as desired. This concludes the proof. ∎

Remark 5.4. We provide an alternative proof of Theorem 5.3, with different techniques, at the end of Section 8.

Once we have accomplished one of the main aims of the section, we want to take advantage of Theorem 5.3 to prove that, in any Banach space X that fails the RNP, there exists an equivalent renorming whose unit ball has no APEP. That is what we do in the following theorem.

Theorem 5.5. *Let X be a Banach space failing the RNP. Then there exists an equivalent norm $|\cdot|$ on X such that $\text{ape}\,\big(B_{(X,|\cdot|)}\big) = \emptyset$.*

For the proof we need a number of auxiliar lemmata whose aim is to construct, from a bounded, closed and convex subset C of a Banach space X with no APEP, another subset D of X which is bounded, closed, convex and symmetric with non-empty interior (and henceforth D will be the unit ball of some equivalent renorming of the space). Let us begin with the following lemma which asserts that if C does not have any APEP then the closure of the extreme points of $\overline{C}^{w^*}$ is far from X.

Lemma 5.6. *Let X be a Banach space and let $C \subseteq X$ a closed, convex and bounded subset such that $\text{ape}\,(C) = \emptyset$. Then*

$$\text{dist}\left(\overline{\text{ext}\left(\overline{C}^{w^*}\right)}^{w^*}, X\right) > 0.$$

Proof. Assume towards a contradiction that, for every $n \in \mathbb{N}$, there exist elements $c_n \in \overline{\text{ext}\left(\overline{C}^{w^*}\right)}^{w^*}$ and $x_n \in X$ such that $\|c_n - x_n\| < \frac{1}{n}$. Let c

be a weak* cluster point of (c_n), so $c \in \overline{\mathrm{ext}\left(\overline{C}^{w^*}\right)}^{w^*}$. Since $\overline{C}^{w^*} \cap X = C$ and $\mathrm{ape}\,(C) = \emptyset$ we infer by Theorem 3.8 that $c \notin X$. Since X is closed we get by Hahn-Banach theorem a functional $f \in X^*$ such that $f(c) = \alpha > 0$ and $f(x) = 0$ holds for every $x \in X$. Find $n \in \mathbb{N}$ such that $\alpha > \frac{2}{n}\|f\|$; since c is a weak* cluster point of (c_n), we can choose it so that $f(c_n) > \frac{\alpha}{2}$. Consequently, since $f(x_n) = 0$ we get

$$\frac{\alpha}{2} < f(c_n) = f(c_n - x_n) \leqslant \|f\|\|c_n - x_n\| \leqslant \frac{\|f\|}{n} < \frac{\alpha}{2},$$

a contradiction. So the result follows. ∎

Now let us obtain a result which allows us to construct symmetric convex bounded subsets without APEP points.

Lemma 5.7. *Let X be a Banach space and let $C \subseteq X$ be a bounded closed and convex subset of X such that $\mathrm{ape}\,(C) = \emptyset$. Then*

$$\mathrm{ape}\,(\overline{\mathrm{co}}(C \cup -C)) = \emptyset.$$

Proof. Set $K := \overline{\mathrm{co}}(C \cup -C)$ and $D := \overline{C}^{w^*} \subseteq X^{**}$. We claim that

$$(5.1) \quad \overline{K}^{w^*} = \mathrm{co}(D \cup -D) = \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1], x \in D, y \in -D\}.$$

The inclusion $\supseteq$ is clear because D and $-D$ are contained in $\overline{K}^{w^*}$ since K is weak* closed and clearly contains C and $-C$, so it must contain their weak* closures. Since $\overline{K}^{w^*}$ is convex then the inclusion $\mathrm{co}(D \cup -D) \subseteq \overline{K}^{w^*}$ follows. For the reverse inclusion observe that $\mathrm{co}(D \cup -D)$ is w^* compact since D is w^* compact and by the description given in (5.1). Moreover, it is clear that $\mathrm{co}(C \cup -C) \subseteq \mathrm{co}(D \cup -D)$ trivally. Consequently $\overline{K}^{w^*} = \overline{\mathrm{co}(C \cup -C)}^{w^*} \subseteq \mathrm{co}(D \cup -D)$.

With the above description it is clear that

$$\mathrm{ext}\left(\overline{K}^{w^*}\right) \subseteq \mathrm{ext}\,(D) \cup \mathrm{ext}\,(-D).$$

Consequently, taking closures we infer that

$$\overline{\mathrm{ext}\left(\overline{K}^{w^*}\right)}^{w^*} \subseteq \overline{\mathrm{ext}\,(D)}^{w^*} \cup -\overline{\mathrm{ext}\,(D)}^{w^*}.$$

Since $\mathrm{ape}\,(C) = \emptyset$, Theorem 3.8 implies that $\overline{\mathrm{ext}\,(D)}^{w^*} \cap X = \emptyset$, so we get that $\overline{\mathrm{ext}\left(\overline{K}^{w^*}\right)}^{w^*} \cap X = \emptyset$ and, by Theorem 3.8 again, $\mathrm{ape}\,(K) = \emptyset$, as desired. ∎

Now we are ready to provide the pending proof.

Proof of Theorem 5.5. Since X fails the RNP, by Theorem 5.3 there exists a subset $C \subseteq X$ such that $\mathrm{ape}\,(C) = \emptyset$. By Lemma 5.7 we can assume

that C is closed, convex, bounded and symmetric. Call $D := \overline{C}^{w^*}$ and, by Lemma 5.6, we get that

$$\alpha := \text{dist}\left(\overline{\text{ext}(D)}^{w^*}, X\right) > 0.$$

Let $0 < \varepsilon < \alpha$ and consider $B := C + \varepsilon B_X$, which is a closed, bounded, convex and symmetric subset of X with non-empty interior (thus B is the unit ball of some equivalent norm on X). Let us prove that $\text{ape}(B) = \emptyset$. In order to do so, observe that

$$\overline{B}^{w^*} = D + \varepsilon B_{X^{**}}.$$

Observe also that $\text{ext}\left(\overline{B}^{w^*}\right) \subseteq \text{ext}(D) + \varepsilon \text{ext}(B_{X^{**}})$. Taking this into account, we infer that $\text{dist}(\overline{B}^{w^*}, X) \geqslant \alpha - \varepsilon$. Indeed, given any $c + \varepsilon x^{**} \in \text{ext}\left(\overline{B}^{w^*}\right)$ we get that $c \in \text{ext}(D)$ and $x^{**} \in \text{ext}(B_{X^{**}})$, and given any $x \in X$ we get

$$\|c + \varepsilon x^{**} - x\| \geqslant \|c - x\| - \varepsilon \geqslant \alpha - \varepsilon$$

since $\|c - x\| \geqslant \text{dist}\left(\overline{\text{ext}(D)}^{w^*}, X\right) = \alpha$. This implies that $\text{ape}(B) = \emptyset$ by Theorem 3.8. Consequently, the equivalent norm whose unit ball is B does the trick. ∎

As a consequence of Theorem 5.5 we are now able to obtain the following characterisation of the RNP.

Corollary 5.8. *Let X be a Banach space. The following assertions are equivalent:*

(1) *X has the RNP.*
(2) *The unit ball of any equivalent renorming of X has an APEP.*

6. Lipschitz-free spaces

In this section, we will study APEPs of the unit ball of Lipschitz-free spaces and relate them to extremal structure. We refer the reader to Section 2.2 for notation and basic facts. Throughout the section, (M, d) will denote a complete metric space with base point $0 \in M$.

We begin by vastly reducing our list of suspects:

Proposition 6.1. *Every APEP of $B_{\mathcal{F}(M)}$ is either a molecule or 0.*

Proof. Since $B_{\mathcal{F}(M)} = \overline{\text{co}}(\text{Mol}(M))$, Lemma 3.5 implies that any APEP of $B_{\mathcal{F}(M)}$ must belong to $\overline{\text{Mol}(M)}^{w}$. However, $\overline{\text{Mol}(M)}^{w} \subseteq \text{Mol}(M) \cup \{0\}$ by [21, Proposition 2.9]. ∎

By [22, Lemma 4.2], $\text{Mol}(M)$ is weakly closed if and only if M bi-Lipschitz embeds into $\mathbb{R}^n$ for some n. If that is the case, the argument above shows that any APEP of $B_{\mathcal{F}(M)}$ must be a molecule. If M does not bi-Lipschitz

embed into Euclidean space, then 0 can be or fail to be an APEP of $B_{\mathcal{F}(M)}$, as witnessed by the following examples.

Example 6.2. Let X be any infinite-dimensional uniformly convex Banach space and $M := S_X$. Then M is uniformly concave and therefore every molecule in $\mathcal{F}(M)$ is a preserved extreme point of $B_{\mathcal{F}(M)}$ [41, Theorem 3.39]. Clearly, M does not bi-Lipschitz embed into Euclidean space, as it is not even locally compact. Thus 0 is an APEP of $B_{\mathcal{F}(M)}$ by Theorem 3.8 and [22, Lemma 4.2].

Example 6.3. Let $M := \{0\} \cup \{e_n : n \in \mathbb{N}\} \subset \ell_1$. Clearly, M does not bi-Lipschitz embed into Euclidean space as it is not totally bounded. However, $\mathcal{F}(M)$ is linearly isometric to ℓ_1 (see e.g. [41, Example 3.10]) and therefore 0 is not an APEP of $B_{\mathcal{F}(M)}$ by Proposition 4.4.

We do not know of a precise metric condition characterising when 0 is an APEP of $B_{\mathcal{F}(M)}$. But we are able to characterise those molecules that are APEPs as follows.

Theorem 6.4. *Let* $m_{xy} \in \mathcal{F}(M)$ *be a molecule. Then the following are equivalent:*

(1) $m_{xy} \in \overline{\mathrm{ape}}\left(B_{\mathcal{F}(M)}\right)$,

(2) $m_{xy} \in \overline{\mathrm{dent}}\left(B_{\mathcal{F}(M)}\right)$,

(3) *there exist* $m_{x_n y_n} \in \mathrm{dent}\left(B_{\mathcal{F}(M)}\right)$ *such that* $x_n \to x$ *and* $y_n \to y$.

Proof. The equivalence (2)$\Leftrightarrow$(3) follows from the fact that all denting points of $B_{\mathcal{F}(M)}$ are molecules and that norm convergence of molecules translates to convergence of the underlying pair of points. On the other hand, (2) is equivalent to $m_{xy} \in \overline{\mathrm{dent}\left(B_{\mathcal{F}(M)}\right)}^w$ by [21, Lemma 2.2], and this clearly implies (1) by Lemma 3.1. So it only remains to be proved that (1) implies $m_{xy} \in \overline{\mathrm{dent}\left(B_{\mathcal{F}(M)}\right)}^w$.

Let m_{xy} be an APEP of $B_{\mathcal{F}(M)}$ and let W be a weak neighbourhood of m_{xy}. We will show that W contains a denting point of $B_{\mathcal{F}(M)}$ and this will finish the proof. Apply Lemma 3.6 to m_{xy}, with $B = \mathrm{Mol}\,(M)$. Since norm and weak convergence of molecules agree, the situation in option 1) of Lemma 3.6 is impossible, so we deduce that W contains slices of arbitrarily small diameter. It follows from Remark 2.1 that W contains a slice $S = S(B_{\mathcal{F}(M)}, f, \alpha)$ where f is not local. By [40, Proposition 2.7] S, and thus W, contains a denting point of $B_{\mathcal{F}(M)}$. This ends the proof. ■

We will now use Theorem 6.4 to provide further examples of the interplay between APEPs and extremality. In Section 3, we showed that APEPs are not necessarily extreme points. This can also happen to molecules in Lipschitz-free spaces.

Example 6.5. Let $M = \{0, 1\} \times [0, 1] \cup \left\{(\frac{1}{2}, 0)\right\} \subset \mathbb{R}^2$. Then each $m_n := m_{(0, \frac{1}{n}), (1, \frac{1}{n})}$ is an extreme point of $B_{\mathcal{F}(M)}$; in fact, it is preserved because M

is compact [4, Theorem 4.2]. Moreover, m_n converges to $m := m_{(0,0),(1,0)}$, so m is APEP by Theorem 6.4. But m is not extreme because the segment between $(0,0)$ and $(1,0)$ is not empty.

Observe that, up to this point, we have not obtained any example of a (non-preserved) extreme point which fails to be an APEP. The next example shows that this is indeed possible.

Example 6.6. Let M be a uniformly discrete metric space such that there exists an extreme point $m_{xy} \in \mathrm{ext}\left(B_{\mathcal{F}(M)}\right)$ that is not preserved (such examples exist, e.g. [4, Example 4.3]). We claim that m_{xy} is not an APEP. Indeed, by Theorem 6.4, m_{xy} can only be an APEP if there exist preserved extreme points $m_{x_n y_n}$ such that $x_n \to x$ and $y_n \to y$ but, since M is topologically discrete, this implies that $x_n = x$, $y_n = y$ for n large enough, and thus m_{xy} is preserved.

It is also possible to construct a metric space M such that $B_{\mathcal{F}(M)}$ has a non-preserved extreme point that is the norm limit of preserved extreme points, hence APEP by Theorem 6.4:

Example 6.7. In $\mathbb{R}^2$ with the ℓ_1 metric, consider the set N consisting of the points $p = (0,0)$, $q = (1,0)$, and $p_k = (0, \frac{1}{k})$, $q_k = (1, \frac{1}{k})$ for $k \in \mathbb{N}$. Let $M = N \cup \{x_n : n \in \mathbb{N}\}$, endowed with the aforementioned metric for N and

$$d(x_n, p) = d(x_n, q) = \tfrac{1}{2} + \tfrac{1}{n}$$
$$d(x_n, p_k) = d(x_n, q_k) = \tfrac{1}{2} + \tfrac{1}{k} + \tfrac{1}{n}$$
$$d(x_n, x_m) = 1 + \tfrac{1}{n} + \tfrac{1}{m}$$

for $n \neq m \in \mathbb{N}$. Note that the quantity

$$d(p_k, x) + d(q_k, x) - d(p_k, q_k)$$

is at least $\frac{1}{k} - \frac{1}{k+1}$ for all $k \in \mathbb{N}$ and $x \in M \setminus \{p_k, q_k\}$, therefore every molecule $m_{p_k q_k}$ is a preserved extreme point of $B_{\mathcal{F}(M)}$. Since $p_k \to p$, $q_k \to q$, the molecule m_{pq} is an APEP by Theorem 6.4. Similarly, the quantity

$$d(p, x) + d(q, x) - d(p, q)$$

is strictly positive for $x \in M \setminus \{p, q\}$, however its value for $x = x_n$ is $\frac{2}{n}$, which can be made arbitrarily small while $d(p, x), d(q, x) > \frac{1}{2}$. Thus m_{pq} is an unpreserved extreme point of $B_{\mathcal{F}(M)}$.

Observe that Theorem 6.4 reveals that, for a molecule m_{xy} in $\mathcal{F}(M)$, if m_{xy} is an APEP then every weakly open subset W of $B_{\mathcal{F}(M)}$ containing m_{xy} contains slices of arbitrarily small diameter. At this point, it could be wondered whether, in the particular case of Lispchitz-free spaces, this fenomenon occurs because, in fact, m_{xy} is contained in non-empty weakly open subsets of arbitrarily small diameter. In the following example we show that this does not hold and that it is possible for a molecule m_{xy} to be an APEP and have the property that every non-empty weakly open subset W of $B_{\mathcal{F}(M)}$ containing m_{xy} satisfies $\mathrm{diam}\left(W\right) = 2$.

Before exhibiting the example, let us recall that a point x in the unit sphere of a Banach space X is said to be a Δ-*point* (respectively *super Δ-point*) if every slice (resp. relatively weakly open subset) of the unit ball of X containing x contains points at distance $2 - \varepsilon$ from x for every $\varepsilon > 0$.

Example 6.8. Consider the metric space M_V constructed by Veeorg in [39, Section 3] as follows. In $\mathbb{R}^2$, consider the points $p = (0,0)$ and $q = (1,0)$ and, for $n \in \mathbb{N}$, the set

$$S_n = \left\{ (2^{-n}k, 2^{-n}) : k = 0, 1, \ldots, 2^n \right\}.$$

Then M_V is the set $\{p, q\} \cup \bigcup_{n=1}^{\infty} S_n$ endowed with the metric

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| & , \text{ if } y_1 = y_2 \\ |y_1 - y_2| + \min\{x_1 + x_2, 2 - (x_1 + x_2)\} & , \text{ if } y_1 \neq y_2 \end{cases}.$$

Now let $p_n = (0, \frac{1}{2^n+1})$, $q_n = (1, \frac{1}{2^n+1})$ and set $M = M_V \cup \{p_n, q_n : n \in \mathbb{N}\}$ with the metric defined by the same formula. See Figure 1.
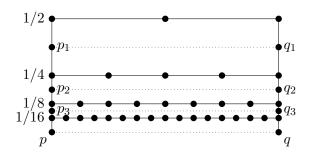

FIGURE 1. The metric space M from Example 6.8.

Similarly to Example 6.7, the molecule m_{pq} is an unpreserved extreme point of $B_{\mathcal{F}(M)}$ that is an APEP. Indeed, any $u \in M \setminus \{p, q\}$ has the form $u = (x, y)$ for some $y > 0$ and therefore

$$d(p, u) + d(q, u) - d(p, q) = (x + y) + (1 - x + y) - 1 = 2y$$

is always positive, but can be made arbitrarily small while keeping e.g. $x = \frac{1}{2}$ to make sure that $d(p, u)$ and $d(q, u)$ remain large, so m_{pq} is an unpreserved extreme point. Moreover, each molecule $m_{p_n q_n}$ is a denting point of $B_{\mathcal{F}(M)}$ as, given any $u = (x, y) \in M \setminus \{p_n, q_n\}$, a similar computation yields

$$d(p_n, u) + d(q_n, u) - d(p_n, q_n) = 2 \left| y - \frac{1}{2^n + 1} \right|$$

which has a positive lower bound in M. Thus m_{pq} is an APEP of $B_{\mathcal{F}(M)}$ by Theorem 6.4 as $p_n \to p$, $q_n \to q$.

In this example, m_{pq} is even a Δ-point. This follows from [1, Theorem 6.7] as p and q are *discretely connectable* in M, i.e. their distance can be approximated by discrete paths in M with arbitrarily small jumps (passing through the sets S_n).

Remark 6.9. Recently, E. Basset, Y. Perreau, A. Procházka and T. Veeorg announced the result that, in any Lipschitz-free space, every molecule that is a Δ-point is a super Δ-point. This (still unpublished) result would imply that the APEP m_{pq} from Example 6.8 is a super-Δ point. In particular, APEPs can fail to be points of (weak- to norm-)continuity even in Banach spaces with the RNP.

An even stronger notion is that of *Daugavet point*, i.e. a point $x \in S_X$ such that every slice S of B_X contains points at distance $2 - \varepsilon$ from x, regardless of whether $x \in S$ or not. However, for Lipschitz-free spaces, APEPs of $B_{\mathcal{F}(M)}$ can never be Daugavet points, as Daugavet points are at distance 2 from any denting point by [26, Proposition 3.1].

7. TENSOR PRODUCTS

In this section, we aim to study APEPs in projective tensor product spaces. Following the spirit of previous works dealing with the extremal structure in projective tensor products like [20, 42], we will focus on studying the APEP of sets of the form $\overline{\text{co}}(C \otimes D)$ in $X \widehat{\otimes}_\pi Y$ for bounded, closed and convex subsets $C \subseteq X$ and $D \subseteq Y$. We refer the reader to Subsection 2.3 for necessary notation and background on tensor product theory.

Let us start with the search of necessary conditions for APEPs in $\overline{\text{co}}(C \otimes D)$. It is natural that they would have to be elementary tensors. This is precisely the statement of the next result under appropriate assumptions on the space $K(X, Y^*)$.

Theorem 7.1. *Let X and Y be Banach spaces such that $K(X, Y^*)$ is separating for $X \widehat{\otimes}_\pi Y$. Let $C \subseteq X$ and $D \subseteq Y$ be bounded, closed and convex subsets. If z is an APEP of $\overline{\text{co}}(C \otimes D) \subseteq X \widehat{\otimes}_\pi Y$, then $z = x \otimes y$ for some $x \in C$ and $y \in D$.*

This result should be compared with [20, Theorem 1.1], where a similar statement is proved for preserved extreme points.

Proof. Let z be an APEP of $\overline{\text{co}}(C \otimes D)$. An application of Lemma 3.5 yields that
$$z \in \overline{C \otimes D}^w = \overline{C}^w \otimes \overline{D}^w,$$
thanks to [20, Theorem 2.3]. Finally, since C, D are weakly closed, it follows that $z = x \otimes y$ for some $x \in C$ and $y \in D$. ∎

Remark 7.2. In view of [20, Theorem 1.1] it is natural to suspect that, in the above theorem, if $z = x \otimes y \neq 0$ is an APEP then both x and y should be APEPs in C and D respectively. However, we will show in Example 7.8 that such result does not hold.

In order to establish sufficient conditions for APEPs of a set of the form $\overline{\text{co}}(C \otimes D)$, the above result says that we can reduce to analysing the elementary tensors. Now in our first sufficient condition we center our attention

on [20, Proposition 3.2], where it is proved that if x_0 is a strongly exposed point of C and y_0 is a w-strongly exposed point then $x_0 \otimes y_0$ is a w-strongly exposed point. However, since in the notion of APEP we do not need to localise the point $x_0 \otimes y_0$, it seems that the assumptions that x_0 is denting and y_0 is APEP should be enough to get that $x_0 \otimes y_0$ is APEP. This is precisely the content of the following result.

Theorem 7.3. *Let X, Y be Banach spaces. Let $C \subseteq X$ and $D \subseteq Y$ be bounded, closed and convex subsets. Let x_0 be a denting point of C and y_0 be an APEP of D. Then, $x_0 \otimes y_0$ is APEP of $\overline{co}(C \otimes D)$.*

Proof. We can assume that C, D are both different from $\{0\}$, since otherwise $\overline{co}(C \otimes D) = \{0\}$. Hence, without loss of generality, assume that $\sup_{z \in C} \|z\| = \sup_{w \in D} \|w\| = 1$. Let U be a relatively weakly open neighbourhood of $x_0 \otimes y_0$ in $\overline{co}(C \otimes D)$. We may assume that $U = \bigcap_{i=1}^{n} S(\overline{co}(C \otimes D), T_i, \alpha_i)$ for some $T_1, \ldots, T_n \in S_{L(X,Y^*)}$ and some $\alpha_1, \ldots, \alpha_n > 0$. Since $x_0 \otimes y_0 \in U$, we have $T_i(x_0)(y_0) > \sup_{z \in \overline{co}(C \otimes D)} T_i(z) - \alpha_i$ for every $1 \leqslant i \leqslant n$. Thus, we may find $\varepsilon_0 > 0$ so that $T_i(x_0)(y_0) > \sup_{z \in \overline{co}(C \otimes D)} T_i(z) - \alpha_i + \varepsilon_0$ for every $1 \leqslant i \leqslant n$. Since $x_0 \in \text{dent}\,(C)$, there are some $\delta' > 0$ and $x^* \in X^*$ with $\sup_{x \in C} x^*(x) = 1$ such that $\text{diam}\,(S(C, x^*, \delta')) < \frac{\varepsilon_0}{4}$. Moreover, notice that y_0 belongs to the set

$$W = \bigcap_{i=1}^{n} \left\{ y \in D : T_i(x_0)(y) > \sup_{z \in \overline{co}(C \otimes D)} T_i(z) - \alpha_i + \varepsilon_0 \right\}$$

which is a relatively weakly open subset of D. Since y_0 is an APEP of D, there are some δ'' and $y^* \in Y^*$ with $\sup_{y \in D} y^*(y) = 1$ such that $S(D, y^*, \delta'') \subseteq W$. Finally, taking $\delta = \min\left\{ \delta', \delta'', \frac{\varepsilon_0}{4} \right\}$ and considering the non-empty slice $S = S(\overline{co}(C \otimes D), x^* \otimes y^*, \eta^2)$ where $0 < \eta < \min\left\{ \frac{\delta}{4}, \frac{1}{2} \right\}$, we conclude that $S \subseteq U$, exactly as in the proof of [20, Proposition 3.2]. Thus, $x_0 \otimes y_0$ is an APEP of $\overline{co}(C \otimes D)$ as desired.   ∎

Now we continue looking for sufficient conditions for a point $x_0 \otimes y_0$ to be an APEP. Let us consider the following definition from [20].

Definition 7.4. Let X and Y be Banach spaces, and let $A \subseteq X \widehat{\otimes}_\pi Y$. We say that $u \in A$ *has a compact neighbourhood system for the weak topology in A* if, given any weakly open subset U containing u, there are slices $S(A, T_i, \alpha_i)$ given by compact operators $T_i \in K(X, Y^*)$ such that

$$u \in \bigcap_{i=1}^{n} S(A, T_i, \alpha_i) \subseteq U.$$

Remark 7.5. It is immediate that the above definition is an equivalent reformulation to the definition given in [20, Definition 3.3]. It is also clear that u has a compact neighbourhood system for the weak topology in $A \subset X \widehat{\otimes}_\pi Y$ if and only if u is a point of cointinuity of the identity map

$$id \colon (A, \sigma(X \widehat{\otimes}_\pi Y, K(X, Y^*))) \to (A, w).$$

It is now time for some examples of this situation.

Example 7.6.

(1) Given two Banach spaces X, Y such that $L(X, Y^*) = K(X, Y^*)$, it is clear that every $u \in A$ has a compact neighbourhood system for the weak topology in A for every $A \subseteq X \widehat{\otimes}_\pi Y$.

(2) Let $C \subseteq X$ and $D \subseteq Y$ be two closed, absolutely convex and bounded subsets and let $x_0 \in \text{dent}\,(C)\,, y_0 \in \text{dent}\,(D)$. Then $x_0 \otimes y_0$ has a compact neighbourhood system for the weak topology in $\overline{\text{co}}(C \otimes D)$. Indeed, it is proved in [42, Theorem 1] that given $\varepsilon > 0$ there exist $x_0^* \in X^*$ and $y_0^* \in Y^*$ such that $x_0 \otimes y_0 \in S(\overline{\text{co}}(C \otimes D), x_0^* \otimes y_0^*, \alpha)$ and that $\text{diam}\,(S(\overline{\text{co}}(C \otimes D), x_0^* \otimes y_0^*, \alpha)) < \varepsilon$. The result follows since the operator

$$x_0^* \otimes y_0^*: \quad \begin{array}{ccc} X & \longrightarrow & Y^* \\ x & \longmapsto & x_0^*(x)y_0^* \end{array}$$

is clearly compact.

(3) In [20, Example 3.8] an equivalent norm $|\cdot|$ on ℓ_2 is given such that e_1 is a w-strongly exposed point of $B_{(\ell_2, |\cdot|)}$ but $e_1 \otimes e_1$ is not a w-strongly exposed point of $B_{(\ell_2, |\cdot|)\widehat{\otimes}_\pi(\ell_2, |\cdot|)}$. According to [20, Theorem 1.3], the point $e_1 \otimes e_1$ fails to have a compact neighbourhood system for the weak topology in $B_{(\ell_2, |\cdot|)\widehat{\otimes}_\pi(\ell_2, |\cdot|)}$.

The following result establishes that APEPs remain stable under tensor products, provided that a suitable compact neighbourhood system exists.

Theorem 7.7. *Let X and Y be two Banach spaces, and let $C \subseteq X$, $D \subseteq Y$ be bounded, closed and convex subsets. Let $x_0 \in \text{ape}\,(C)$ and $y_0 \in \text{ape}\,(D)$. Assume that $x_0 \otimes y_0$ has a compact neighbourhood system for the weak topology in $\overline{\text{co}}(C \otimes D) \subseteq X \widehat{\otimes}_\pi Y$. Then $x_0 \otimes y_0$ is an APEP of $\overline{\text{co}}(C \otimes D)$.*

This result should be compared with [20, Theorem 1.3], where an analogous statement is proved for w-strongly exposed point. Moreover, we employ here many ideas from the proof of that result.

Proof. We can assume that C, D are both different from $\{0\}$, since otherwise $\overline{\text{co}}(C \otimes D) = \{0\}$. Hence, without loss of generality, assume that $\sup_{z \in C} \|z\| = \sup_{w \in D} \|w\| = 1$. Let U be a weak neighbourhood of $x_0 \otimes y_0$ in $\overline{\text{co}}(C \otimes D)$. By the assumption, we can assume that $U = \bigcap_{i=1}^{n} S(\overline{\text{co}}(C \otimes D), T_i, \alpha_i)$ for certain compact operators $T_1, \ldots, T_n \colon X \to Y^*$. Furthermore, we can assume $\sup_{u \in \overline{\text{co}}(C \otimes D)} T_i(u) = 1$ for every i. Let $\eta > 0$ small enough so that $T_i(x_0 \otimes y_0) > 1 - \alpha_i + \eta$ holds for every $1 \leqslant i \leqslant n$. Moreover, observe that $x_0 \in \bigcap_{i=1}^{n} \{z \in C : T_i(z)(y_0) > 1 - \alpha_i + \eta\}$, which is a relatively weakly open subset of C. Since x_0 is an APEP of C there exists a non-empty slice

$S(C, x_0^*, \delta')$ such that $\sup_{z \in C} x_0^*(z) = 1$ and that

$$S(C, x_0^*, \delta') \subseteq \bigcap_{i=1}^{n} \{z \in C : T_i(z)(y_0) > 1 - \alpha_i + \eta\}.$$

Now, for every $1 \leqslant i \leqslant n$, the set $T_i(S(C, x_0^*, \delta'))$ is a relatively compact subset of Y^*. Using the compactness condition on all the $T_i's$ we can find a finite set $x_1, \ldots, x_m \in S(C, x_0^*, \delta')$ so that the balls $B(T_i(x_j), \frac{\eta}{2}), 1 \leqslant j \leqslant m$, cover $T_i(S(C, x_0^*, \delta'))$ for every $1 \leqslant i \leqslant n$. Observe that $T_i(x_j)(y_0) > 1 - \alpha_i + \eta$ holds for every $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Consequently,

$$y_0 \in \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}.$$

Since y_0 is an APEP of D we can find a slice $S(D, y_0^*, \delta'')$ such that $\sup_{w \in D} y_0^*(w) = 1$ and that

$$S(D, y_0^*, \delta'') \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}.$$

We claim now that

$$S(C, x_0^*, \delta') \otimes S(D, y_0^*, \delta'') \subset \bigcap_{i=1}^{n} S\left(\overline{\text{co}}(C \otimes D), T_i, \alpha_i - \frac{\eta}{2}\right).$$

Indeed, let $x \in S(C, x_0^*, \delta')$ and $y \in S(D, y_0^*, \delta'')$. We have, for every $i \in \{1, \ldots, n\}$, an index $j_i \in \{1, \ldots, m\}$ such that $\|T_i(x) - T_i(x_{j_i})\| < \frac{\eta}{2}$. On the other hand, since $S(D, y_0^*, \delta'') \subseteq \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} \{y \in D : T_i(x_j)(y) > 1 - \alpha_i + \eta\}$ we have that, for every $1 \leqslant i \leqslant n$, $T_i(x_{j_i})(y) > 1 - \alpha_i + \eta$. Consequently

$$T_i(x)(y) \geqslant T_i(x_{j_i})(y) - \|T_i(x_{j_i}) - T_i(x)\| > 1 - \alpha_i + \eta - \frac{\eta}{2} = 1 - \alpha_i + \frac{\eta}{2}.$$

Take $\delta := \min\{\delta', \delta'', \frac{\eta}{8 \max_{1 \leqslant i \leqslant n} \|T_i\|}, \frac{1}{2}\}$ and consider $S := S(\overline{\text{co}}(C \otimes D), x_0^* \otimes y_0^*, \delta^2)$ which is non-empty since $\sup_{z \in C} x_0^*(z) = 1 = \sup_{w \in D} y_0^*(w)$. Moreover,

$$S \subseteq \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta)) + 4\delta B_{X \widehat{\otimes}_\pi Y}$$

by virtue of [20, Lemma 2.1]. Now, given $1 \leqslant i \leqslant n$, since $1 - \delta \geqslant \max\{1 - \delta', 1 - \delta''\}$ we conclude that every element $x \otimes y$ of $S(C \otimes D, x_0^* \otimes y_0^*, \delta)$ satisfies $x_0^*(x) > 1 - \delta'$ and $y_0^*(y) > 1 - \delta''$, so $T_i(x)(y) > 1 - \alpha_i + \frac{\eta}{2}$. Since T_i is a linear continuous functional on $X \widehat{\otimes}_\pi Y$ we conclude that $T_i(z) \geqslant 1 - \alpha_i + \frac{\eta}{2}$ holds for every $1 \leqslant i \leqslant n$ and every $z \in \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta))$. Henceforth, given $z \in S$ we can find $u \in \text{co}(S(C \otimes D, x_0^* \otimes y_0^*, \delta))$ and $v \in B_{X \widehat{\otimes}_\pi Y}$ so that $z = u + 4\delta v$. Now, given $1 \leqslant i \leqslant n$ we get

$$T_i(z) = T_i(u) + 4\delta T_i(v) \geqslant 1 - \alpha_i + \frac{\eta}{2} - 4\delta \|T_i\| > 1 - \alpha_i,$$

from where we conclude that $z \in \bigcap_{i=1}^{n} S(\overline{\mathrm{co}}(C \otimes D), T_i, \alpha_i) = U$. This proves that $S \subseteq U$.

Summarising, we have proved that every relatively weakly open subset of $\overline{\mathrm{co}}(C \otimes D)$ containing $x_0 \otimes y_0$ actually contains a non-empty slice $S(\overline{\mathrm{co}}(C \otimes D), x_0^* \otimes y_0^*, \alpha)$. From here we conclude that $x_0 \otimes y_0$ is an APEP, as requested. ∎

Let us now observe that we do not need both elements x_0 and y_0 to be APEPs in Theorem 7.7.

Example 7.8. Let $X = C([0,1])$ and let $Y = \ell_p$ for $2 < p < \infty$. Clearly both X and Y have the AP and, moreover,

$$L(X, Y^*) = L(C(K), \ell_{p'}) = K(C(K), \ell_{p'}) = K(X, Y^*)$$

by [3, Exercise 6.10] since $1 < p' < 2$. Let $x = \frac{1}{2}f$ where $f(t) = 1$ for every $t \in [0,1]$, and take $y \in S_Y$. It is clear that x is not an APEP of B_X by Theorem 4.1. However $x \otimes y = f \otimes \left(\frac{1}{2}y\right)$ is an APEP by Theorem 7.7 since f is an APEP of B_X (Theorem 4.1) and $\frac{1}{2}y$ is an APEP of B_Y by Example 3.3.

It is clear that in above example is based on the absence of uniqueness in the representation of an elementary tensor in a projective tensor product. In order to deal with this difficulty, and taking into account the hypotheses in Theorems 7.1 and 7.7, we will end the section by studying APEPs under the assumption that $K(X, Y^*) = L(X, Y^*)$, in other words, that every $T \in L(X, Y^*)$ is compact. If we additionally require that either X or Y has the AP then we get

$$(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*) = K(X, Y^*) = X^* \widehat{\otimes}_\varepsilon Y^*.$$

Note that, for instance, this is the case when $X = \ell_p$ and $Y = \ell_{q'}$ with $1 \leqslant q < p < \infty$ and $1/q + 1/q' = 1$, thanks to Pitt's theorem (see e.g. Proposition 4.49 in [17]); a version for Lorentz and Orlicz sequence spaces holds too [7]. Recall also that for a reflexive space X and a Banach space Y, one of them with the compact approximation property, the condition $K(X, Y) = L(X, Y)$ is equivalent to the fact that every operator from X to Y attains its norm. This is shown in [13], extending previous results of Holub and Mujica in the reflexive case.

In the next result we study APEPs of the unit ball of a projective tensor product $X \widehat{\otimes}_\pi Y$ under the assumption that $L(X, Y^*) = K(X, Y^*)$. The main technique will be the one used in [32, Theorem 2.1]. As a byproduct, we obtain a description of all preserved extreme points in this case.

Theorem 7.9. *Let X, Y be Banach spaces such that X^* or Y^* has the AP and $K(X, Y^*) = L(X, Y^*)$. Let $z \in B_{X \widehat{\otimes}_\pi Y}$ with $z \neq 0$. Then:*

   a) *$z \in \mathrm{ape}\left(B_{X \widehat{\otimes}_\pi Y}\right)$ if, and only if, $z = x \otimes y$ for some $x \in \mathrm{ape}(B_X)$ and $y \in \mathrm{ape}(B_Y)$.*

    b) $z \in \text{pre-ext}\left(B_{X\widehat{\otimes}_\pi Y}\right)$ if, and only if, $z = x \otimes y$ for some $x \in$ $\text{pre-ext}\,(B_X)$ and $y \in \text{pre-ext}\,(B_Y)$.

For the proof, we will make use of the following lemma, whose statement should be compared with [20, Theorem 2.3].

Lemma 7.10. *Let X, Y be Banach spaces. If $C \subseteq X^*$ and $D \subseteq Y^*$ are bounded subsets, then*

$$\overline{C}^{w^*} \otimes \overline{D}^{w^*} = \overline{C \otimes D}^{w^*},$$

considering $X^* \otimes Y^* \subseteq (X\widehat{\otimes}_\varepsilon Y)^*$.

Proof. If $C = \{0\}$ or $D = \{0\}$ then the result is trivial, so assume $C, D \neq \{0\}$. First of all, since C and D are bounded we may assume that

$$\sup_{x^* \in C} \|x^*\| = 1 = \sup_{y^* \in D} \|y^*\|.$$

We show first that $\overline{C}^{w^*} \otimes \overline{D}^{w^*} \subseteq \overline{C \otimes D}^{w^*}$. For each $x^* \in X^*$, define the operator $T_{x^*}\colon X\widehat{\otimes}_\varepsilon Y \to Y$ given by $T_{x^*}(x \otimes y) = x^*(x)y$ for all $x \in X$ and $y \in Y$, which extends by linearity and continuity to the whole $X\widehat{\otimes}_\varepsilon Y$. Hence, the adjoint $T_{x^*}^*\colon Y^* \to (X\widehat{\otimes}_\varepsilon Y)^*$ is given by

$$T_{x^*}^*(y^*) = x^* \otimes y^* \in (X\widehat{\otimes}_\varepsilon Y)^*, \quad \forall y^* \in Y^*,$$

and it is w^*-w^*-continuous. Therefore,

$$\{x^*\} \otimes \overline{D}^{w^*} = T_{x^*}^*\left(\overline{D}^{w^*}\right) \subseteq \overline{T_{x^*}^*(D)}^{w^*} = \overline{\{x^*\} \otimes D}^{w^*} \subseteq \overline{C \otimes D}^{w^*},$$

and this holds for every $x^* \in C$. So, $C \otimes \overline{D}^{w^*} \subseteq \overline{C \otimes D}^{w^*}$ and also $\overline{C}^{w^*} \otimes D \subseteq \overline{C \otimes D}^{w^*}$, by a symmetric argument. Hence,

$$\overline{C}^{w^*} \otimes \overline{D}^{w^*} \subseteq \overline{\overline{C}^{w^*} \otimes D}^{w^*} \subseteq \overline{\overline{C \otimes D}^{w^*}}^{w^*} = \overline{C \otimes D}^{w^*}.$$

It remains to prove that $\overline{C}^{w^*} \otimes \overline{D}^{w^*} \supseteq \overline{C \otimes D}^{w^*}$. Take $z \in \overline{C \otimes D}^{w^*}$ and pick a net $(x_\alpha^* \otimes y_\alpha^*)_\alpha \subseteq C \otimes D$ converging weak* to z. Since $\overline{C}^{w^*}$ and $\overline{D}^{w^*}$ are weak*-compact, we may assume (by taking subnets if necessary) that $(x_\alpha^*)_\alpha$ and $(y_\alpha^*)_\alpha$ converge weak* to some $x^* \in \overline{C}^{w^*}$ and $y^* \in \overline{D}^{w^*}$ respectively. Let us show that $(x_\alpha^* \otimes y_\alpha^*)_\alpha$ converges weak* to $x^* \otimes y^*$ in $(X\widehat{\otimes}_\varepsilon Y)^*$. It is clear that $((x_\alpha^* \otimes y_\alpha^*)(x \otimes y))_\alpha = (x_\alpha^*(x)y_\alpha^*(y))_\alpha$ converges to $x^*(x)y^*(y) = (x^* \otimes y^*)(x \otimes y)$ for all $x \in X$ and $y \in Y$. By linearity, we also have $(x_\alpha^* \otimes y_\alpha^*)(v) \to (x^* \otimes y^*)(v)$ for $v \in X \otimes Y$. Finally, take $u \in X\widehat{\otimes}_\varepsilon Y$ and $\varepsilon > 0$. On the one hand, pick $v \in X \otimes Y$ such that $\|u - v\| \leqslant \frac{\varepsilon}{4}$. Hence, for every α, we have

$$|(x_\alpha^* \otimes y_\alpha^* - x^* \otimes y^*)(u - v)| \leqslant (\|x_\alpha^*\| \|y_\alpha^*\| + \|x^*\| \|y^*\|)\frac{\varepsilon}{4} \leqslant \frac{\varepsilon}{2}.$$

On the other hand, pick β such that

$$|(x_\alpha^* \otimes y_\alpha^*)(v) - (x^* \otimes y^*)(v)| < \frac{\varepsilon}{2}, \quad \forall \alpha \succcurlyeq \beta.$$

Then,
$$|(x_\alpha^* \otimes y_\alpha^* - x^* \otimes y^*)(u)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall \alpha \succcurlyeq \beta,$$
so $(x_\alpha^* \otimes y_\alpha^*)_\alpha$ converges weak* to $x^* \otimes y^*$. By uniqueness of the limit, $z = x^* \otimes y^* \in \overline{C}^{w^*} \otimes \overline{D}^{w^*}$. ∎

Now we can provide the pending proof.

Proof of Theorem 7.9. a) Thanks to Theorem 7.7 we know that if x is an APEP of B_X and y is an APEP of B_Y, then $x \otimes y$ is APEP of $B_{X \widehat{\otimes}_\pi Y}$. Conversely, assume that z is APEP of $B_{X \widehat{\otimes}_\pi Y}$. By Theorem 7.1, $z \in B_X \otimes B_Y$. It remains to be shown that $z = x \otimes y$ for some $x \in \text{ape}\,(B_X)$ and $y \in \text{ape}\,(B_Y)$. By virtue of Theorem 3.8, that is equivalent to $x \in \overline{\text{ext}\,(B_{X^{**}})}^{w^*}$ and $y \in \overline{\text{ext}\,(B_{Y^{**}})}^{w^*}$. Furthermore, since $z \in B_{X \widehat{\otimes}_\pi Y}$ is APEP we have $z \in \overline{\text{ext}\,\left(B_{(X \widehat{\otimes}_\pi Y)^{**}}\right)}^{w^*} \cap B_{X \widehat{\otimes}_\pi Y}$. Therefore, let us show that

$$(7.1) \qquad \overline{\text{ext}\,(B_{X^{**}})}^{w^*} \otimes \overline{\text{ext}\,(B_{Y^{**}})}^{w^*} = \overline{\text{ext}\,\left(B_{(X \widehat{\otimes}_\pi Y)^{**}}\right)}^{w^*}$$

Indeed,

$$\overline{\text{ext}\,(B_{X^{**}})}^{w^*} \otimes \overline{\text{ext}\,(B_{Y^{**}})}^{w^*} = \overline{\text{ext}\,(B_{X^{**}}) \otimes \text{ext}\,(B_{Y^{**}})}^{w^*},$$

using Lemma 7.10. Finally, $\text{ext}\,(B_{X^{**}}) \otimes \text{ext}\,(B_{Y^{**}}) = \text{ext}\,\left(B_{(X^* \widehat{\otimes}_\varepsilon Y^*)^*}\right)$ by [32, Theorem 1.1] and $X^* \widehat{\otimes}_\varepsilon Y^* = K(X, Y^*) = L(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$, thanks to [33, Corollary 4.13]. Hence,

$$\overline{\text{ext}\,\left(B_{(X^* \widehat{\otimes}_\varepsilon Y^*)^*}\right)}^{w^*} = \overline{\text{ext}\,\left(B_{(X \widehat{\otimes}_\pi Y)^{**}}\right)}^{w^*}.$$

This proves (7.1).

Finally, since z is an APEP of $B_{X \widehat{\otimes}_\pi Y}$, by (7.1) we have $z = x^{**} \otimes y^{**}$ for some $x^{**} \in \overline{\text{ext}\,(B_{X^{**}})}^{w^*} \subset B_{X^{**}}$ and $y^{**} \in \overline{\text{ext}\,(B_{Y^{**}})}^{w^*} \subset B_{Y^{**}}$. Moreover, thanks to Theorem 7.1, we also have $z \in B_X \otimes B_Y$. It follows easily that $x^{**} \in B_X$ and $y^{**} \in B_Y$, because $z \neq 0$. Thus, x^{**} is an APEP of B_X and y^{**} is an APEP of B_Y, which concludes the proof of a).

b) The proof follows immediately by the description of the extreme points of $B_{(X \widehat{\otimes}_\pi Y)^{**}}$ given before. ∎

Now the following remark is pertinent.

Remark 7.11.
  (1) In [20, Question 3.9] it is asked whether $x \otimes y$ is a preserved extreme point of $B_{X \widehat{\otimes}_\pi Y}$ when $x \in B_X$ and $y \in B_Y$ are preserved extreme points of B_X and B_Y respectively. Theorem 7.9 gives an affirmative answer under the assumption that $L(X, Y^*) = K(X, Y^*)$ and that either X^* or Y^* has the AP.

(2) In connection with the above question, in [20, Example 3.8], an equivalent norm $|\cdot|$ on ℓ_2 and a point $x_0 \in B_{(X,|\cdot|)}$ are given such that x_0 is w-strongly exposed point (in particular, it is a preserved extreme point) by a certain functional $f \in S_{X^*}$ such that $x_0 \otimes x_0$ is not a w-strongly exposed point.

It is a natural question whether $x_0 \otimes x_0$ is a preserved extreme point, and indeed it seems to be the first example to check in order to look for a negative answer to [20, Question 3.9] above.

Let us point out, however, that $x_0 \otimes x_0$ is an extreme point. Indeed, it is not difficult to prove that $x_0 \otimes x_0$ is an exposed point (by $f \otimes f$). Moreover, it can be proved that $x_0 \otimes x_0 \in \overline{\text{dent}}\left(B_{(X,|\cdot|)\widehat{\otimes}_\pi(X,|\cdot|)}\right)$, so in particular it is an APEP. If $x_0 \otimes x_0$ is not a preserved extreme point, then this furnishes another example of an extreme, almost preserved extreme point that is not preserved extreme.

We finish by extending Theorem 7.9 from the case of the unit ball of a projective tensor product to the case of $\overline{\text{co}}(C \otimes D)$ when C and D have non-empty interior.

Theorem 7.12. *Let X and Y be two Banach spaces such that $L(X,Y^*) = K(X,Y^*)$ and such that either X or Y has the AP. Let $C \subseteq X$ and $D \subseteq Y$ be two bounded, closed, convex and symmetric subsets with non-empty interior. Given $z \in \overline{\text{co}}(C \otimes D)$ with $z \neq 0$ we have:*

a) *$z \in \text{ape}\left(\overline{\text{co}}(C \otimes D)\right)$ if, and only if, $z = x \otimes y$ for some $x \in \text{ape}(C)$ and $y \in \text{ape}(D)$.*

b) *$z \in \text{pre-ext}\left(\overline{\text{co}}(C \otimes D)\right)$ if, and only if, $z = x \otimes y$ for some $x \in \text{pre-ext}(C)$ and $y \in \text{pre-ext}(D)$.*

Proof. Let us begin by observing that 0 is an interior point of C (the same holds for D). Indeed, take any interior point $u \in C$ and $\delta > 0$ such that $u + \delta B_X^0 \subseteq C$, where B_X^0 stands for the open unit ball. By symmetry, we also have $-u + \delta B_X^0 \subset C$. It follows that $\delta B_X^0 \subset C$ since C is convex.

Now, by the properties of C, we have that C is the unit ball of some equivalent norm on X. We denote by $\tilde{X}$ such an equivalent renorming for which $B_{\tilde{X}} = C$. Similarly, we define $\tilde{Y}$ to be an equivalent renorming of Y such that $B_{\tilde{Y}} = D$. Observe that by the assumptions either $\tilde{X}$ or $\tilde{Y}$ has the AP, and that $L(\tilde{X}, \tilde{Y}^*) = K(\tilde{X}, \tilde{Y}^*)$. Finally, note that

$$B_{\tilde{X}\widehat{\otimes}_\pi\tilde{Y}} = \overline{\text{co}}(B_{\tilde{X}} \otimes B_{\tilde{Y}}) = \overline{\text{co}}(C \otimes D).$$

Now the conclusion follows by Theorem 7.9 since $X\widehat{\otimes}_\pi Y$ and $\tilde{X}\widehat{\otimes}_\pi\tilde{Y}$ are isomorphic Banach spaces, so the weak* topologies of their corresponding biduals are the same. ∎

Remark 7.13. The above result should be compared with [32, Theorem 2.2], which generalises [32, Theorem 1.1] from the unit ball of the dual of a space of operators to certain weak* compact neighbourhoods of 0 in such space.

8. Concluding remarks and open questions

In this section we collect some comments and open questions which are derived from our work.

From our study of APEP in the unit ball of ℓ_1-sums of spaces, the following remains open.

Question 8.1. Let $\{X_i : i \in I\}$ an arbitrary infinite family of Banach spaces and let X be its ℓ_1-sum. If $0 \in B_X$ is APEP in B_X, must there exist i such that $0 \in X_i$ is APEP in B_{X_i}?

For the case of ℓ_p-sums, in Proposition 4.9 we have characterised when the norm-one elements are APEP of the unit ball. However, we were unable to describe those APEP whose norm is strictly smaller than 1.

Question 8.2. Let X be the ℓ_p-sum of a family $\{X_i : i \in I\}$ of Banach spaces, where $1 < p < \infty$, and let $(x_i) \in B_X$ with $\|(x_i)\| < 1$. When is (x_i) an APEP of B_X?

Even though we do not have a complete characterisation, let us now present a sufficient condition concerning the above question.

Proposition 8.3. *Let X be the ℓ_p-sum of a family $\{X_i : i \in I\}$ of Banach spaces, where $1 < p < \infty$, and let $(x_i) \in B_X$ with $\|(x_i)\| < 1$. Set $J := \{i \in I : \|x_i\| \neq 0\}$. Assume that there exists $i_0 \in I \setminus J$ such that 0 is APEP in $B_{X_{i_0}}$ and that $\frac{x_j}{\|x_j\|}$ is APEP in B_{X_j} for every $j \in J$. Then (x_i) is APEP in B_X.*

Proof. The proof will be quite similar to that of Proposition 4.9. Let U be a weak* neighbourhood of (x_i) in X^{**}. We will show that U intersects $\mathrm{ext}\,(B_{X^{**}})$ and this will be enough by Theorem 3.8. Since the weak* topology of X^{**} is the product topology of the weak* topologies in X_i^{**}, we may assume that $U = \prod_{i \in I} U_i$ where U_i is a weak* neighbourhood of x_i (or X_i^{**}) for each $i \in I$. Given $j \in J$, since $x_j \neq 0$ then, by assumption and Theorem 3.8, there exists $e_j \in \mathrm{ext}\,(B_{X_i^{**}})$ such that $\|x_j\|\, e_j \in U_j$. Moreover, since 0 is an APEP of $B_{X_{i_0}}$ we may select $e_{i_0} \in \mathrm{ext}\left(B_{X_{i_0}^{**}}\right)$ such that $(1 - \|(x_i)\|^p)^{\frac{1}{p}}\, e_{i_0} \in U_{i_0}$. Let $y = (y_i) \in X^{**}$ be defined by

$$y_i = \begin{cases} \|x_i\|\, e_i & \text{, if } i \in J \\ (1 - \|(x_i)\|^p)^{\frac{1}{p}}\, e_{i_0} & \text{, if } i = i_0 \\ 0 & \text{, otherwise.} \end{cases}$$

Note that $y \in U$ and

$$\|y\| = \left(\sum_{i \in I} \|y_i\|^p \right)^{1/p} = \left(\sum_{j \in J} \|x_j\|^p + \left((1 - \|(x_i)\|^p)^{\frac{1}{p}} \right)^p \right)^{1/p}$$

$$= \left(\sum_{j \in J} \|x_j\|^p + 1 - \sum_{j \in J} \|x_j\|^p \right)^{\frac{1}{p}} = 1$$

since $(x_i) \in S_X$, so $y \in \text{ext}\,(B_{X^{**}})$ by (4.2). This ends the proof. ∎

Another question coming from Subsection 4.3 is the following.

Question 8.4. Let $\{X_i : i \in I\}$ be an arbitrary infinite family of Banach spaces and let X be its ℓ_∞-sum. Can necessary or sufficient conditions for a point $(x_i) \in B_X$ to be APEP be given?

Regarding Lipschitz-free spaces, in Section 6 we proved that every APEP of $B_{\mathcal{F}(M)}$ must be either a molecule or 0, and we characterised those molecules that are APEP. We showed that 0 can be an APEP, but only if M does not bi-Lipschitz embed in $\mathbb{R}^n$, and that this necessary condition is not sufficient (Examples 6.2 and 6.3). The following remains open.

Question 8.5. For which metric spaces M is 0 an APEP of $B_{\mathcal{F}(M)}$?

Finally, let us collect some open questions from our results in Section 7. First of all, concerning Theorem 7.1 we wonder the following.

Question 8.6. In the statement of Theorem 7.1, can we infer that if $z \neq 0$ then x and y are APEP in C and D respectively?

At a first glance it seems that from Example 7.8 the answer to the above question should be no. Observe, however, that the actual problem there is the existence of an appropriate representation of the element z as an elementary tensor of APEP, which certainly does exist in the above mentioned Example 7.8.

Our last question has to do with the possibility of removing the assumption of the existence of a compact neighbourhood system for the weak topology in Theorem 7.7.

Question 8.7. Let X and Y be Banach spaces, and let $C \subseteq X$, $D \subseteq Y$ be symmetric bounded closed convex subsets. If $x_0 \in C$ and $y_0 \in D$ are APEPs, is $x_0 \otimes y_0$ an APEP of $\overline{\text{co}}(C \otimes D)$? Can we obtain this at least when $C = B_X$ and $D = B_Y$?

We conclude this section by presenting an alternative proof of Theorem 5.3. This proof, inspired by a construction due to W. Schachermayer, A. Sersouri, and E. Werner [35], offers a more geometric perspective, in contrast to the measure-theoretic approach developed in Section 5.

We need the following elementary result, whose proof is included for completeness.

Lemma 8.8. *Let C be a convex set and let $y \in X$ such that $y \notin \overline{C}$. Then*

$$\operatorname{dist}(y, C) = \operatorname{dist}(y, \overline{C}^{w^*}),$$

where the weak-star closure above is taken in X^{**}.

Proof. Take any $0 < r < \operatorname{dist}(y, C)$. Then $\overline{C}$ and $B(x, r)$ can be separated by a hyperplane determined by some $x^* \in X^*$. Then the same hyperplane separates $\overline{B(x, r)}^{w^*}$ and $\overline{C}^{w^*}$ in X^*, thus $\operatorname{dist}(y, \overline{C}^{w^*}) \geqslant r$. ∎

Following [35], for a measurable subset $A \subset [0, 1]$ with positive measure, we denote

$$\mathcal{F}_A = \{f \in L_1[0, 1] : f = f \cdot \chi_A, f \geqslant 0, \|f\|_1 = 1\}.$$

Second proof of Theorem 5.3. First, note that we may assume that X is separable. Indeed, if X fails the RNP then there is a separable $Y \subset X$ failing the RNP. Now, for a bounded closed convex set $C \subset Y$ and a point $x_0 \in C$, we have that $x_0 \in \operatorname{ape}(C)$ as a subset of Y if and only if $x_0 \in \operatorname{ape}(C)$ as a subset of X, since the weak topology in Y coincides with the weak topology inherited from X.

Now, let X be a separable Banach space failing the RNP. We will show that there is a bounded closed convex set $C \subset X$ with $\operatorname{dist}\left(X, \overline{\operatorname{ext}\left(\overline{C}^{w^*}\right)}^{w^*}\right) > 0$ (and so, by Theorem 3.8, $\operatorname{ape}(C) = \emptyset$).

Since X fails the RNP, there is a non-representable operator $T \colon L_1[0, 1] \to X$. Let $(y_n)_{n=1}^{\infty}$ be a dense sequence in X. The proof of Theorem 1.1 in [35] shows that there exists $\gamma > 0$ and subsets $D_1^n, \ldots, D_{N(n)}^n \subset [0, 1]$ such that, if we denote $E := \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} D_i^n$ and $E_i^n = E \cap D_i^n$, we have:

a) $\operatorname{dist}(y_n, T(\mathcal{F}_{D_i^n})) > \gamma$ for all $i \in \{1, \ldots, N(n)\}$ and $n \in \mathbb{N}$.

b) For each $n \in \mathbb{N}$, every extreme point y^{**} of $\overline{T(\mathcal{F}_E)}^{w^*}$ belongs to $\overline{T(\mathcal{F}_{E_i^n})}^{w^*}$ for some $i \in \{1, \ldots, N(n)\}$.

We claim that $C = \overline{T(\mathcal{F}_E)}$ satisfies the desired property. First, note that C is bounded, closed and convex (since $\mathcal{F}_E$ is a convex set). Now, let $y^{**} \in \overline{\operatorname{ext}\left(\overline{C}^{w^*}\right)}^{w^*}$ and take a net $(y_s^{**})_s \subset \operatorname{ext}\left(\overline{C}^{w^*}\right)$ with $y_s^{**} \xrightarrow{w^*} y^{**}$. Fix $n \in \mathbb{N}$. By property b) above, there is $i_n \in \{1, \ldots, N(n)\}$ such that the net $(y_s^{**})_s$ is frequently in $\overline{T(\mathcal{F}_{E_{i_n}^n})}^{w^*}$. Thus, $y^{**} \in \overline{T(\mathcal{F}_{E_{i_n}^n})}^{w^*}$. By Lemma 8.8, we get

$$\operatorname{dist}(y_n, y^{**}) \geqslant \operatorname{dist}(y_n, \overline{T(\mathcal{F}_{E_{i_n}^n})}^{w^*}) = \operatorname{dist}(y_n, T(\mathcal{F}_{E_{i_n}^n}))$$
$$\geqslant \operatorname{dist}(y_n, T(\mathcal{F}_{D_{i_n}^n})) > \gamma$$

and so $\operatorname{dist}(X, y^{**}) \geqslant \gamma$. This proves the desired claim. ∎

References

[1] T. A. Abrahamsen, R. J. Aliaga, V. Lima, A. Martiny, Y. Perreau, A. Procházka and T. Veeorg, *Delta-points and their implications for the geometry of Banach spaces*, J. London Math. Soc. **109** (2024), e12913.

[2] T. A. Abrahamsen, P. Hájek, O. Nygaard, J. Talponen and S. Troyanski, *Diameter 2 properties and convexity*, Studia Math. **232**, 3 (2016), 227–242.

[3] F. Albiac and N. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics 233, Springer-Verlag, New York, 2006.

[4] R. J. Aliaga and A. J. Guirao, *On the preserved extremal structure of Lipschitz-free spaces*, Studia Math. **245** (2019), 1–14.

[5] R. J. Aliaga and E. Pernecká, *Supports and extreme points in Lipschitz-free spaces*, Rev. Mat. Iberoam. **36** (2020), 2073–2089.

[6] R. J. Aliaga, E. Pernecká and R. J. Smith, *A solution to the extreme point problem and other applications of Choquet theory to Lipschitz-free spaces*, arXiv preprint (2024), arXiv:2412.04312.

[7] E. A. Ausekle and È. F. Oya, *Pitt's theorem for Lorentz and Orlicz sequence spaces*. Mat. Zametki **61**, 18–25 (1997)

[8] A. Bellow, *Lifting compact spaces*, Lect. Notes in Math. **794** Springer-Verlag, 233–253 (1980).

[9] J. Bourgain, *La Proprieté de Radon-Nikodym*, Publications Mathématiques de L'Université Pierre et Marie Curie, vol.36, 1979.

[10] J. Bourgain and M. Talagrand, *Dans un espace réticulé, la propriété de Radon-Nikodym et celle de Krein-Milman sont équivalentes*, Proc. Am. Math. Soc. **81** (1981), 93–96.

[11] R.D. Bourgin, *Geometric Aspects of Convex Sets with Radon-Nikodým Property*, Lect. Notes in Math. 993, Springer Verlag, 1980.

[12] V. Caselles, *A short proof of the equivalence of KMP and RNP in Banach lattices and preduals of von Neumann algebras*, Proc. Am. Math. Soc. **102** (1988), 973–974.

[13] S. Dantas, M. Jung, G. Martínez-Cervantes, *On the existence of non-norm-attaining operators.* J. Inst. Math. Jussieu **22** (2023), no. 3, 1023–1035.

[14] A. Defant, K.Floret, *Tensor norms and operator ideals*, North-Holland Publishing Co., Amsterdam, (1993).

[15] J. Diestel and J. J. Uhl, *Vector measures*, American Matematical Society Providence, Rhode Island (1977).

[16] G. A. Edgar, *Measurability in a Banach space I*, Indiana Univ. Math. J. **26** (1977), 63–667.

[17] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, CMS Books in Mathematics. Springer-Velag New York 2001.

[18] D. H. Fremlin. *Measure theory. Vol. 2. Broad oundations.* Colchester: Torres Fremlin, corrected second printing of the 2001 original edition, 2003.

[19] L. C. García Lirola, *Convexity, optimization and geometry of the ball in Banach spaces*, PhD thesis, Universidad de Murcia, 2017. Available at *DigitUM* with reference http://hdl.handle.net/10201/56573.

[20] L. C. García-Lirola, G. Grelier, G. Martínez-Cervantes and A. Rueda Zoca, *Extremal structure of projective tensor products*, Results Math. **78** (2023), article 196.

[21] L. García-Lirola, C. Petitjean, A. Procházka and A. Rueda Zoca, *Extremal structure and duality of Lipschitz free spaces*, Mediterr. J. Math. (2018) 15: 69.

[22] L. García-Lirola and A. Rueda Zoca, *Unconditional almost squareness and applications to spaces of Lipschitz functions*, J. Math. Anal. Appl. **451** (2017), 117–131.

[23] Y. Ivakhno, V. Kadets and D. Werner, *The Daugavet property for spaces of Lipschitz functions*, Math. Scand. **101**, 2 (2007), 261–279.

[24] R. C. James, *Some interesting Banach spaces*, Rocky Mt. J. Math. **23** (3) (1993), 911–937.

[25] R. A. Johnson, *Atomic and nonatomic measures*, Proc. Am. Math. Soc. **25** (1970), 650–655.

[26] M. Jung and A. Rueda Zoca, *Daugavet points and Δ-points in Lipschitz-free spaces*, Studia Math. **265** (2022), 37–55.

[27] G. López-Pérez and R. Medina, *The equivalence between CPCP and strong regularity under Krein-Milman property*, J. Funct. Anal. **286** (2024), article 110273.

[28] J. Lukes, J. Malý, I. Netuka, and J. Spurný, *Integral Representation Theory: Applications to Convexity, Banach Spaces and Potential Theory*, de Gruyter Studies in Mathematics 35, 2010.

[29] R. R. Phelps, *Extreme points of polar convex sets*, Proc. Amer. Math. Soc. **12** (1961), 291–296.

[30] E. T. Poulsen. *A simplex with dense extreme points.* Ann. Inst. Fourier. (Grenoble) **11** (1961), 83–87.

[31] M. Raja, *First Borel class sets in Banach spaces and the asymptotic-norming property*, Israel J. Math. **138** (2003), 253–270.

[32] W. M. Ruess and C. Stegall, *Extreme points in duals of operator spaces*, Math. Ann. **261** (1982), 535–546.

[33] R. A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag, London, (2002).

[34] W. Schachermayer, *The Radon-Nikodym property and the Krein-Milman property are equivalent for strongly regular sets*, Trans. Am. Math. Soc. **303** (2) (1987) 673–687.

[35] W Schachermayer, A. Sersouri and E. Werner, *Moduli of nondentability and the Radon-Nikodým property in Banach spaces*, Isr. J. Math. **65**, 3 (1989), 225–257.

[36] L. Schwartz, *Certaines propriétés des mesures sur les espaces de Banach*, Séminaire Maurey Schwartz 1975–1976, Exposé No. 23.

[37] M. Talagrand, *Pettis integral and measure theory*, Memoirs Amer. Math. Soc., **307**, AMS, Providence, 1984.

[38] A. Ionescu Tulcea, C. Ionescu Tulcea, *Topics in the Theory of Lifting*, Ergebnisse der Mathematik und ihrer Grenzgebiete **48**, Springer-Verlag, 1969.

[39] T. Veeorg, *Characterizations of Daugavet points and Delta-points in Lipschitz-free spaces*, Studia Math. **268** (2023), 213–233.

[40] T. Veeorg, *Daugavet- and Delta-points in spaces of Lipschitz functions*, RACSAM **119** (2025), article 71.

[41] N. Weaver, *Lipschitz algebras*, 2nd ed., World Scientific, Singapore, 2018.

[42] D. Werner, *Denting points in tensor products of Banach spaces*, Proc. Amer. Math. Soc. **101**, 1 (1987), 122–126.

[43] K. Zhu, *An Introduction to Operator Algebras* (1st ed.), CRC Press, (1993).

(R. J. Aliaga) Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera S/N, 46022 Valencia, Spain

Email address: raalva@upv.es

URL: https://raalva.wordpress.com

(L.C. García-Lirola) Departamento de Matemáticas, Universidad de Zaragoza, 50009, Zaragoza, Spain

Email address: luiscarlos@unizar.es

URL: https://personal.unizar.es/luiscarlos/

(J. Guerrero-Viu) Departamento de Matemáticas, Universidad de Zaragoza, 50009, Zaragoza, Spain

Email address: j.guerrero@unizar.es

(M. Raja) Departamento de Matemáticas, Universidad de Murcia, 30100, Murcia, Spain

Email address: matias@um.es

(A. Rueda Zoca) Universidad de Granada, Facultad de Ciencias. Departamento de Análisis Matemático, 18071-Granada (Spain)

Email address: abrahamrueda@ugr.es

URL: https://arzenglish.wordpress.com