Riemann integration in $\mathbb{R}^d$

degree in mathematics – second year

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1 Rectangles and partitions

In all that follows we will assume that the dimension of the space is a fixed number $d \in \mathbb{N}$. The case $d = 1$ is the one dimensional Riemann integral that has been studied previously in first year, but the characterizations of Riemann integrability in terms of continuity point are not likely studied in that setting. A rectangle in $\mathbb{R}^d$ will always be a *compact rectangle* $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ unless we specify other kind of rectangle (open). The d-dimensional volume of the rectangle is the nonnegative number

$$\operatorname{vol}(R) = (b_1 - a_1)(b_2 - a_2) \dots (b_d - a_d)$$

The rectangle is non degenerate if $\operatorname{vol}(R) > 0$. Clearly, the topological interior of R is the set

$$(a_1, b_1) \times \cdots \times (a_d, b_d)$$

Two rectangles R_1 and R_2 are said not overlapping if they meet only on their borders.

Any non degenerate rectangle R can be tiled with smaller non degenerate rectangles $\{R_i\}_{i=1}^n$ which are pairwise not overlapping. To see that, just consider the rectangles of the form $I_1 \times \cdots \times I_d$ where each I_k is an interval coming from a finite partition of $[a_k, b_k]$. Then arrange all these rectangles into a sequence $\{R_i\}_{i=1}^n$. The tiling $\{R_i\}_{i=1}^n$ of R obtained in this way is called a *grill* of R. It is not difficult to see that $\operatorname{vol}(R) = \sum_{i=1}^n \operatorname{vol}(R_i)$ in this case, but something more general is true. Given a rectangle R, a collection $\pi = \{R_i\}_{i=1}^n$ is said a partition of R if they are not overlapping and $\bigcup_{i=1}^n R_i = R$.

Proposition 1.1. *If $\{R_i\}_{i=1}^n$ is a partition of a rectangle R, then*

$$\operatorname{vol}(R) = \sum_{i=1}^n \operatorname{vol}(R_i).$$

Proof. Assume that $R = [a_1, b_1] \times \cdots \times [a_d, b_d]$ is non degenerate, since in other case the result is trivial. As well, we may assume that $\{R_i\}_{i=1}^n$ contain no degenerate rectangle, since after removing them we still have $\bigcup_{i=1}^n R_i = R$ (the union of the interiors are dense in R). Fix a coordinate $1 \le k \le d$. The k-projection of R_i is a subinterval of $[a_k, b_k]$. Consider the one-dimensional partition of $[a_k, b_k]$ generated for all the endpoints of such intervals for $1 \le i \le n$, and then consider the grill $\{R'_j\}_{j=1}^m$ obtained form those intervals by cartesian products. For each $1 \le j \le m$ there is exactly one $1 \le i \le n$ such that $R'_j \subset R_i$ since both have nonempty interior. Consider the sets $A_i = \{j : R'_j \subset R_i\}$ for $1 \le i \le n$ which are disjoint and $\bigcup_{i=1}^n A_i = \{1, \ldots, m\}$. Observe that $\{R'_j\}_{j \in A_i}$ is a partition of R_i. Now

$$\sum_{i=1}^n \mathrm{vol}(R_i) = \sum_{i=1}^n \sum_{j \in A_i} \mathrm{vol}(R'_j) = \sum_{j=1}^m \mathrm{vol}(R'_j) = \mathrm{vol}(R)$$

■

With similar arguments it is possible to prove the following

Proposition 1.2. *If $\{R_i\}_{i=1}^n$ is collection of non overlapping rectangles and $\{R'_j\}_{i=1}^m$ is another collection of rectangles such that $\bigcup_{i=1}^n R_i \subset \bigcup_{j=1}^m R'_j$, then $\sum_{i=1}^n \mathrm{vol}(R_i) \le \sum_{j=1}^m \mathrm{vol}(R'_j)$.*

A partition $\pi' = \{R'_j\}_{j=1}^m$ is finer than $\pi = \{R_i\}_{i=1}^n$ if for every $j : 1 \ldots m$ there is $i : 1 \ldots n$ such that $R'_j \subset R_i$. Observe that in this case we have

$$R_i = \bigcup \{R'_j : R'_j \subset R_i\}.$$

Given two partitions $\pi = \{R_i\}_{i=1}^n$ and $\pi' = \{R'_j\}_{j=1}^m$ is always possible to find a third partition which is finer. Just take the rectangles $R_i \cap R'_j$ having nonempty interior.

2 Integrals on compact rectangles

Given a bounded function $f : R \to \mathbb{R}$ defined on a rectangle and partition $\pi = \{R_i\}_{i=1}^n$ of R, we consider the numbers

$$L(f, \pi) = \sum_{i=1}^n \inf\{f, R_i\} \mathrm{vol}(R_i)$$

$$U(f,\pi) = \sum_{i=1}^{n} \sup\{f, R_i\} \mathrm{vol}(R_i)$$

named lower and upper sums respectively. Observe that for $\pi_1 \le \pi_2$ partitions of R we always have

$$L(f,\pi_1) \le L(f,\pi_2) \le U(f,\pi_2) \le U(f,\pi_1)$$

The Darboux lower and upper integrals of f (on R) are defined this way

$$\underline{\int} f = \sup\{L(f,\pi) : \pi \text{ partition of } R\}$$

$$\overline{\int} f = \inf\{U(f,\pi) : \pi \text{ partition of } R\}.$$

Definition 2.1. *A bounded function $f : R \to \mathbb{R}$ is said Riemann integrable (on R) if $\underline{\int} f = \overline{\int} f$. In that case, its integral (in Riemann sense) is that common value $\int f = \int_R f := \underline{\int} f = \overline{\int} f$.*

Recall that the oscillation of a function $f : R \to \mathbb{R}$ on a set $A \subset R$ is the number

$$\mathrm{osc}(f, A) = \sup\{|f(x) - f(y)| : x, y \in A\}$$

In order to establish the properties of integrable functions the following criterion will be very useful.

Proposition 2.2. *A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if for every $\varepsilon > 0$ there is a partition $\pi = \{R_i\}_{i=1}^n$ of R such that*

$$\sum_{i=1}^{n} \mathrm{osc}(f, R_i) \mathrm{vol}(R_i) < \varepsilon$$

Hint of Proof. Just notice that $\mathrm{osc}(f, R_i) = \sup\{f, R_i\} - \inf\{f, R_i\}$. ■

The first application provides us with an important class of integrable functions.

Corollary 2.3. *If $f : R \to \mathbb{R}$ is continuous, then it is Riemann integrable.*

Proof. Since R is compact, then f is uniformly continuous. Given $\varepsilon > 0$, take a partition $\pi = \{R_i\}_{i=1}^n$ made of rectangles small enough to guarantee that $\mathrm{osc}(f, R_i) < \varepsilon/\mathrm{vol}(R)$. ■

Proposition 2.4. *Let $\mathfrak{R}(R)$ denote the set of functions which are Riemann integrable on R. Then*

1. *$\mathfrak{R}(R)$ is a vector space and $\int_R(\alpha f+\beta g)=\alpha\int_R f+\beta\int_R g$ whenever $f,g\in\mathfrak{R}(R)$ and $\alpha,\beta\in\mathbb{R}$.*
2. *$\mathfrak{R}(R)$ is stable by products (so it is an algebra).*
3. *If $f,g\in\mathfrak{R}(R)$ and $f\leq g$, then $\int_R f\leq\int_R g$.*
4. *If $f\in\mathfrak{R}(R)$, then $f^+,f^-,|f|\in\mathfrak{R}(R)$ and $|\int_R f|\leq\int_R|f|$.*
5. *If $f\in\mathfrak{R}(R)$ and $S\subset R$ a rectangle, then $f|_S\in\mathfrak{R}(S)$.*
6. *If $f\in\mathfrak{R}(R)$ and $\{R_i\}_{i=1}^n$ is a partition of R, then $\int_R f=\sum_{i=1}^n\int_{R_i}f$.*

Hint of Proof. Observe that

$$\underline{\int_R} f+\underline{\int_R} g\leq\underline{\int_R}(f+g)\leq\overline{\int_R}(f+g)\leq\overline{\int_R} f+\overline{\int_R} g$$

and

$$\overline{\int_R}\alpha f=\alpha\overline{\int_R} f,\quad \underline{\int_R}\alpha f=\alpha\underline{\int_R} f$$

for $\alpha>0$, while if $\alpha<0$ then

$$\overline{\int_R}\alpha f=\alpha\underline{\int_R} f,\quad \underline{\int_R}\alpha f=\alpha\overline{\int_R} f.$$

Integrability of products can be reduced to integrability of squares of positive functions. In such a case, we have

$$\operatorname{osc}(f^2,A)\leq 2\sup\{f,A\}\operatorname{osc}(f,A)$$

which is suitable for that purpose. ∎

3 Integrability and continuity points

The goal of this section is to give a characterization of Riemann integrability by means of the set of continuity points of the function. Let us begin with a simple but useful observation.

Proposition 3.1. *If $f \in \mathfrak{R}(R)$, $f \geq 0$ and $\int_R f = 0$, then $f(x) = 0$ whenever $x \in R$ is a point of continuity of f.*

A bounded set $A \subset \mathbb{R}^d$ is said of null content if for every $\varepsilon > 0$ there is a family of rectangles $\{R_i\}_{i=1}^n$ such that $A \subset \bigcup_{i=1}^n R_i$ and $\sum_{i=1}^n \mathrm{vol}(R_i) < \varepsilon$. Notice that being of content null is stable by subsets, finite unions and closures.

A set $A \subset \mathbb{R}^d$ is said of null measure if for every $\varepsilon > 0$ there is a family of rectangles $\{R_i\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty R_i$ and $\sum_{i=1}^\infty \mathrm{vol}(R_i) < \varepsilon$. Measure null sets are stable by subsets and countable unions. Of course, content null sets are measure null, but the converse is not true: just consider $A = [0, 1] \cap \mathbb{Q}$. As the countable union of its singletons it is of null measure. On the other hand, $\overline{A} = [0, 1]$ so this set cannot be of null content. Notice that a compact set of null measure is of null content since there is no restriction in considering the cover made of open rectangles (slightly larger ones).

Given a function $f : R \to \mathbb{R}$, we may define its oscillation at some $x \in R$ as

$$\mathrm{osc}(f, x) = \inf\{\mathrm{osc}(f, U) : U \text{ neighborhood of } x\}.$$

Observe that f is continuous at x if and only if $\mathrm{osc}(f, x) = 0$. Moreover, for every $\delta > 0$ the set $\{x \in R : \mathrm{osc}(f, x) < \delta\}$ is open (relatively to R). The following is the celebrated Riemann-Lebesgue characterization of the Riemann integrability.

Theorem 3.2. *Let $f : R \to \mathbb{R}$ be a bounded function defined on a non degenerate compact rectangle $R \subset \mathbb{R}^d$. The following statements are equivalent:*

i) f is Riemann integrable on R;

ii) $\{x \in R : \mathrm{osc}(f, x) \geq \delta\}$ is of null content for every $\delta > 0$;

iii) the set of discontinuity points of f is of null measure.

Proof. Note that the equivalence between *ii*) and *iii*) is consequence of this set equality

$$\{x \in R : \mathrm{osc}(f, x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in R : \mathrm{osc}(f, x) \geq 1/n\}$$

bearing in mind that the first are the discontinuity points of f and the second is represented as a union of compact subsets of R.

Suppose that f is Riemann integrable. For $\varepsilon, \delta > 0$, take a partition $\{R_i\}_{i=1}^n$ of R into rectangles such that

$$\sum_{i=1}^{n} \operatorname{osc}(f, R_i)\operatorname{vol}(R_i) < \delta\varepsilon$$

Consider the open set $O = \bigcup_{i=1}^n R_i^\circ$. If $y \in O \cap \{x \in R : \operatorname{osc}(f,x) > \delta\}$, then $\operatorname{osc}(f, R_i) > \delta$ if $y \in R_i$. Take $N = \{i : 1 \le i \le n, \operatorname{osc}(f, R_i) > \delta\}$ and observe that

$$\delta \sum_{i\in N} \operatorname{vol}(R_i) < \sum_{i\in N} \operatorname{osc}(f, R_i)\operatorname{vol}(R_i) < \delta\varepsilon$$

following that $O \cap \{x \in R : \operatorname{osc}(f,x) > \delta\}$ is covered by $\{R_i\}_{i\in N}$. Since $R \setminus O = \bigcup_{i=1}^n \partial R_i$ is of null content and $\varepsilon > 0$ arbitrary, we deduce that $\{x \in R : \operatorname{osc}(f,x) > \delta\}$ is of null content.
Supose now that statement *ii*) holds. Given $\varepsilon > 0$, set $M = \operatorname{osc}(f, R)$ and take a cover $\{S_j\}_{j=1}^m$ by open rectangles of the set $\{x : \operatorname{osc}(f,x) \ge \varepsilon/\operatorname{vol}(R)\}$ such that $\sum_{j=1}^m \operatorname{vol}(S_j) < \varepsilon/M$. If $O = \bigcup_{j=1}^m S_j$, then $R \setminus O$ is compact. Every $x \in R \setminus O$ has an open neighborhood U_x such that $\operatorname{osc}(f, U_x) < \varepsilon/\operatorname{vol}(R)$. Let $\xi > 0$ be the Lebesgue number of the covering $\{U_x\}_{x\in R\setminus O}$. Note that $R \setminus O$ is a finite union of non overlaping rectangles, that can be decomposed into smaller nonoverlaping rectangles of diameter less than ξ. That family of rectangles can be extended to a partition $\{R_i\}_{i=1}^n$ of R adding rectangles filling $R \cap O$. With all these ingredients we have

$$\sum_{i=1}^{n} \operatorname{osc}(f, R_i)\operatorname{vol}(R_i) = \sum_{R_i^\circ \subset O} \operatorname{osc}(f, R_i)\operatorname{vol}(R_i) + \sum_{R_i \subset R\setminus O} \operatorname{osc}(f, R_i)\operatorname{vol}(R_i)$$

$$\le M \sum_{R_i^\circ \subset O} \operatorname{vol}(R_i) + \frac{\varepsilon}{\operatorname{vol}(R)} \sum_{R_i \subset R\setminus O} \operatorname{vol}(R_i) \le M\frac{\varepsilon}{M} + \frac{\varepsilon}{\operatorname{vol}(R)}\operatorname{vol}(R) = 2\varepsilon.$$

That proves the Riemann integrability of f. ■

Corollary 3.3. *If $f \in \mathfrak{R}(R)$, $f \ge 0$ and $\int_R f = 0$, then $\{x \in R : f(x) \neq 0\}$ is of null measure.*

4 Integration on domains

Let $D \subset \mathbb{R}^d$ a bounded subset and $f : D \to \mathbb{R}$ a bounded function. We say that f is Riemann integrable on D if given a compact rectangle $R \supset D$, the function $\tilde{f} : R \to \mathbb{R}$ defined as $\tilde{f}(x) = f(x)$ if $x \in D$ and $f(x) = 0$ if $x \in R \setminus D$ is Riemann integrable on R. In such a case, we take

$$\int_D f := \int_R \tilde{f}.$$

It is not difficult to check that the definition is independent of the chosen rectangle R, and taking $\mathfrak{R}(D)$. Properties of function integrables on rectangles extend naturally to $\mathfrak{R}(D)$. In a similar fashion, for $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, that is, if the set $\{x \in \mathbb{R}^d : f(x) \neq 0\}$ is bounded, we may define $\int f$ in terms of integration on rectangles.

The integrability of $f : D \to \mathbb{R}$ depends on the continuity points of the extended function $\tilde{f}$ which in turn depends both on the values of f and the "distribution" of D into $\mathbb{R}^d$. It seems to be a good idea to investigate the sets of $\mathbb{R}^d$ where the continuous functions, at least, are integrable.

Definition 4.1. *A bounded subset $A \subset \mathbb{R}^d$ is said Jordan measurable (or Jordan domain) if its indicator function χ_A is Riemann integrable. In such a case, the number $c(A) = \int \chi_A$ is called the Jordan content of A.*

Observe that null content sets are those Jordan measurable sets having content zero. For a bounded set $A \subset \mathbb{R}^d$ we may define the inner content $c_*(A) = \underline{\int} \chi_A$ and the outer content as $c^*(A) = \overline{\int} \chi_A$. We have that a bounded set$A$ is measurable Jordan if and only if $c_*(A) = c^*(A)$, whose interpretation is related to the Greek's exhaustion method for areas and volumes.

Proposition 4.2. *A bounded subset $A \subset \mathbb{R}^d$ is Jordan measurable if and only if its boundary ∂A is of null content.*

Proof. The discontinuities of χ_A happen exactly at the points of ∂A. ∎

We have defined the Jordan content from the Riemann integral. The other way around is posible as shows the following result. The details of the proof are left to the reader.

Proposition 4.3. *Let $R \subset \mathbb{R}^d$ be a rectangle.*

1. *If $f : R \to [0, +\infty)$ a bounded function and consider $F = \{(x, t) : x \in R, 0 \leq t \leq f(x)\}$. Then*

$$\underline{\int_R} f = c_*(F), \qquad \overline{\int_R} f = c^*(F)$$

where the Jordan content is taken in $\mathbb{R}^{d+1}$. In particular, f is Riemann integrable if and only if F is Jordan measurable, and then $\int_R f = c(F)$.

2. *Bounded sets defined by subgraphs and epigraphs of Riemann integrable functions are Jordan measurable.*

3. *A bounded function $f : R \to \mathbb{R}$ is Riemann integrable if and only if its graph $\{(x, f(x)) : x \in R\}$ is of null content in $\mathbb{R}^{d+1}$.*

We have the mean value property of the integral.

Proposition 4.4. *If D is a Jordan set and $f \in \mathfrak{R}(D)$ then*

$$\inf\{f, D\} \leq \frac{1}{c(D)} \int_D f \leq \sup\{f, D\}.$$

Proof. Just compare f with $\lambda\chi_D$ with $\lambda \in \{\inf\{f, D\}, \sup\{f, D\}\}$ and integrate. ■

The characterization Theorem 3.2 is extended with no trouble.

Proposition 4.5. *A bounded function $f : D \to \mathbb{R}$ is Riemann integrable on a Jordan domain D if and only if the set of its points of discontinuity is of null measure (equivalently, the set of points where the oscillation is bigger than δ is of null content for every $\delta > 0$).*

Note that Jordan sets are stable by finite unions, finite intersections and differences. We say that two Jordan sets A and B do not overlap if $A \cap B \subset \partial A \cup \partial B$. The problem of measuring sets in $\mathbb{R}^d$ is solved in the frame of Jordan sets.

Proposition 4.6. *If $A_{i\,i=1}^{\,n} \subset \mathbb{R}^d$ is a non overlapping finite family of Jordan sets, then its union is Jordan as well and $c(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n c(A_i)$.*

A Jordan partition of a Jordan set D is a non overlapping finite family $\{D_i\}_{i=1}^n$ of Jordan sets such that $D = \bigcup_{i=1}^n D_i$. Jordan partitions provide a good frame for Riemann sums, which provide a more explicit way for the computation of integrals.

Theorem 4.7. *Let $f \in \mathfrak{R}(D)$ where D is a Jordan domain. For every $\varepsilon > 0$ there is $\delta > 0$ such that if $\{D_i\}_{i=1}^n$ is a Jordan partition of D into sets of diameter less than δ, then*

$$|\sum_{i=1}^n f(t_i)c(D_i) - \int_D f| < \varepsilon$$

for any choice of points $t_i \in D_i$.

Proof. Without loss of generality we may assume that D is compact. Indeed, take $\overline{D}$ and extend f to $\overline{D} \setminus D$ as zero. Fix $\varepsilon > 0$. Let $M = \operatorname{osc}(f, D)$. The set $x \in D : \operatorname{osc}(f, x) \geq \varepsilon/c(D)$ is covered by finitely many open rectangles such that its union is an open set O with $c(O) < \varepsilon/2M$. For any point $x \in D \setminus O$, take $U_x \ni x$ such that $\operatorname{osc}(f, U_x) < \varepsilon/2c(D)$ and consider the Lebesgue number ξ of the open cover $\{O\} \cup \{U_x\}_{x \in D \setminus O}$. If $\{D_i\}_{i=1}^n$ is a Jordan partition such any set D_i has diameter less than ξ, take N to be the set of such indices i for which $D_i \subset O$. We have

$$|\sum_{i=1}^n f(t_i)c(D_i) - \int_D f| \leq \sum_{i=1}^n \int_{D_i} |f(t_i) - f|$$

$$= \sum_{i \in N} \int_{D_i} |f(t_i) - f| + \sum_{i \notin N} \int_{D_i} |f(t_i) - f|$$

$$\leq \sum_{i \in N} Mc(D_i) + \sum_{i \notin N} \frac{\varepsilon}{2c(D)} c(D_i) \leq Mc(O) + \frac{\varepsilon}{2c(D)} c(D) \leq \varepsilon$$

whenever the points $t_i \in D_i$ are chosen. ∎

In fact, the thesis in the previous statement implies the Riemann integrability suitably reformulated. Indeed, if the Riemann sums

$$\sum_{i=1}^n f(t_i)c(D_i)$$

have a common limit when the Jordan partition $\{D_i\}_{i=1}^n$ is either refined or the maximum diameter of its sets goes to zero, then the function f must be integrable on D.

The convergence of Riemann sum can be applied to prove the change of variables formula in a very important particular case.

Theorem 4.8. *Let* $E \subset [0, +\infty) \times [0, 2\pi]$ *a Jordan domain mapped on the Jordan domain* $D \subset \mathbb{R}^2$ *by the map* $(\theta, r) \to (r \cos\theta, r \sin\theta)$. *Then for any* $f \in \mathfrak{R}(D)$ *we have*

$$\iint_D f(x, y)\, dxdy = \iint_E f(r\cos\theta, r\sin\theta)\, r\, drd\theta$$

Proof. Without loss of generality we may assume that E is a rectangle, since the extension of f to be zero on the complement do not change the value of the integrals. Set $\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$. Take a partition on E with nodes $\{(r_i,\theta_j)\}_{i=1,j=1}^{n,m}$. The rectangles are mapped on sectors $D_{i,j}$ having area

$$c(D_{i,j}) = \frac{r_{i-1}+r_i}{2}(r_i - r_{i-1})(\theta_j - \theta_{j-1})$$

The associate Riemann sum over D with the evaluation on central points is

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\frac{r_{i-1}+r_i}{2}\cos(\frac{\theta_{i-1}+\theta_i}{2}), \frac{r_{i-1}+r_i}{2}\cos(\frac{\theta_{i-1}+\theta_i}{2}))c(D_{i,j})$$

which approaches $\iint_D f(x,y)\,dxdy$. On the other hand, the sum coincides with

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{f}(\frac{r_{i-1}+r_i}{2}, \frac{\theta_{j-1}+\theta_j}{2})\frac{r_{i-1}+r_i}{2}(r_i - r_{i-1})(\theta_j - \theta_{j-1})$$

which is a Riemann sum associate to $\iint_E f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$. The refining of the partition in the sense of Theorem 4.7 gives the equality of the two integrals of the thesis. ■

5 Iterated integrals

Until this moment we have not said how Riemann integrals in $\mathbb{R}^d$ are computed. The idea is to reduce to iterated integral in spaces of lesser dimension, which in practice means that all can be reduced to one dimensional integrals where the calculus of primitive functions is the main device for its computation.

Next result is known as Fubini theorem for Riemann integral.

Theorem 5.1. *Let $R \subset \mathbb{R}^{d_1}$ and $S \subset \mathbb{R}^{d_2}$ rectangles and $f \in \mathfrak{R}(R \times S)$. For $x \in R$ take $f_x(y) = f(x,y)$ defined on S and consider its Darboux integrals*

$$L(x) = \underline{\int_S} f_x, \qquad U(x) = \overline{\int_S} f_x.$$

Then $L, U \in \mathfrak{R}(R)$ and

$$\int_{R\times S} f = \int_R L = \int_R U$$

Moreover, $f_x \in \mathfrak{R}(S)$ for $x \in R$ except a null measure set.

Proof. Consider partitions into rectangles $\{R_i\}_{i=1}^n$ and $\{S_j\}_{j=1}^m$ of R and S respectively. Observe that $\mathrm{vol}(R_i \times S_j) = \mathrm{vol}(R_i)\mathrm{vol}(S_j)$, where each volume is understood according to the dimension of the space. If $x \in R_i$ then

$$\sup\{f_x, S_j\} \leq \sup\{f, R_i \times S_j\}$$

that implies

$$U(x) = \overline{\int_S} f_x \leq \sum_{j=1}^m \sup\{f_x, S_j\}\mathrm{vol}(S_j) \leq \sum_{j=1}^m \sup\{f, R_i \times S_j\}\mathrm{vol}(S_j)$$

Taking supremum on $x \in R_i$ we get to

$$\sup\{U, R_i\} \leq \sum_{j=1}^m \sup\{f, R_i \times S_j\}\mathrm{vol}(S_j)$$

that implies

$$\overline{\int_R} U \leq \sum_{i=1}^n \sup\{U, R_i\}\mathrm{vol}(R_i) \leq \sum_{i=1}^n \sum_{j=1}^m \sup\{f, R_i \times S_j\}\mathrm{vol}(R_i \times S_j)$$

Taking infimum on the left hand side we get that

$$\overline{\int_R} U \leq \overline{\int_{R\times S}} f = \int_{R\times S} f$$

A similar argument will show that

$$\underline{\int_R} L \geq \underline{\int_{R\times S}} f = \int_{R\times S} f$$

On the other hand, we have these obvious inequalities

$$\underline{\int_R} L \leq \underline{\int_R} U \leq \overline{\int_R} U$$

All together implies that $\underline{\int_R} U = \overline{\int_R} U$, so U is Riemann integrable on R. Similarly, we have $\underline{\int_R} L = \overline{\int_R} L$ and so the Riemann integrability of L, as well as the equality with $\int_{R\times S} f$. Now observe that $\int_R (U - L) = 0$ and the function $U - L$ is positive, so $U = L$ except a null measure set. ∎