Borel measurability and renorming in Banach spaces[1]

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1999

[1] Corrected and annotated 2019

Contents

Introducción

El presente trabajo está dedicado a la búsqueda de condiciones necesarias y suficientes para la existencia de normas equivalentes de Kadec y localmente uniformente convexas en un espacio de Banach, y también se dedica en parte al estudio de ciertos aspectos de la teoría descriptiva de conjuntos, como la caracterización de los espacios topológicos Borel absolutos. A primera vista, podría parecer que estos dos temas no guardan mucha relación entre sí, pero a lo largo de la tesis se pondrán de manifiesto las conexiones que nos han llevado a la realización de este estudio conjunto.

Antecedentes

Recordemos que una norma $\|.\|$ equivalente sobre un espacio de Banach X se dice estrictamente convexa si su esfera unidad no contiene segmentos, y se dice que $\|.\|$ es localmente uniformemente convexa (abreviadamente LUR) si para cada x, x_k en X con $\|x\| = \|x_k\| = 1$ tales que $\lim_k \|x + x_k\| = 2$ se tiene que $\lim_k \|x - x_k\| = 0$. No es muy difícil mostrar que si $\|.\|$ es LUR, entonces sobre la esfera unidad de $\|.\|$ coincide la topología débil con la topología de la norma [10]. Las normas que poseen esta propiedad se conocen como normas de Kadec. Por otra parte, existen espacios de Banach que poseen una norma Kadec pero que ninguna norma equivalente es LUR [32]. Actualmente, la obra de referencia básica sobre la teoría del renormamiento es el libro de Deville, Godefroy y Zizler [10].

En [12] Edgar demostró que si un espacio de Banach admite una norma equivalente de Kadec, entonces los conjuntos de Borel para la topología débil coinciden con los conjuntos de Borel para la norma. En un segundo articulo [13], Edgar recoge dos resultados de Schachermayer en la misma línea. Se demuestra que si un espacio de Banach X admite una norma equivalente de Kadec, entonces es un subconjunto de Borel en su bidual X^{**} para la topología débil*. El otro resultado establece que si en un espacio de Banach coinciden los conjuntos de Borel para la topología débil y para la topología de la norma sobre la esfera unidad de alguna norma equivalente, entonces estos coinciden en todo el espacio. Este último resultado, siendo más general que el teorema de Edgar, podría desviar la atención de los espacios con norma equivalente de Kadec. Sin embargo, no se conoce[1] ningún ejemplo de espacio de Banach con los mismos conjuntos de Borel para la topología débil y la topología de la norma (o que sea un subconjunto de Borel en su bidual para la topología débil*) que no admita un renormamiento Kadec. Mencionemos aquí que Talagrand probó en [64] que $l^\infty(\mathbb{N})$ no tiene los mismos conjuntos de Borel para la topología débil y para la de la norma, y que tampoco es un subconjunto de Borel en su bidual con la topología débil*.

En el año 1989 aparece un extenso trabajo de Hansell [27] que hasta la fecha está sin publicar[2]. Hansell introduce el concepto de espacio topológico descriptivo, que no detallaremos aquí por ser algo técnico (ver Definición 1.5.9), que generaliza de modo bastante natural la noción clásica de espacio analítico en el contexto general de los espacios topológicos no necesariamente metrizables ni separables. Hansell también destaca el papel que juega la existencia de una "network σ-aislada" en sustitución de las bases σ-discretas, exclusivas de los espacios metrizables por el teorema de Bing-Nagata-Smirnov. Estos conceptos encuentran aplicación en las topologías débiles de un espacio de Banach. Diremos que un espacio de Banach es descriptivo si es un

[1] Ahora sí: W. Marciszewski, R. Pol, On Banach spaces whose norm-open sets are F_σ-sets in the weak topology, *J. Math. Anal. Appl.* 350 (2009), 708–722.

[2] Descriptive sets and the topology of nonseparable Banach spaces, *Serdica Math. J.* 27 (2001), no. 1, 1—66.

espacio topológico descriptivo cuando se considera con su topología débil, lo que ocurre si y sólo si esta última tiene una "network σ-aislada", ver [27] y Teorema 2.2.8. Hansell demuestra que en un espacio de Banach descriptivo coinciden los conjuntos de Borel para la topología débil y para la topología de la norma (este resultado es más interesante si se considera que el ser descriptivo un espacio de Banach concierne exclusivamente a su topología débil) y el propio espacio es un subconjunto de Borel en su bidual para la topología débil* [27]. Como era de esperar, Hansell también demuestra que un espacio de Banach que admite una norma equivalente de Kadec es descriptivo.

Después del éxito de la noción de fragmentabilidad, sobre todo en el estudio de la propiedad de Radon-Nikodym y la obtención de selectores medibles de aplicaciones multivaluadas, Jayne, Namioka y Rogers introducen en [36] la noción de σ-fragmentabilidad de un espacio topológico respecto de una métrica. Tampoco nos detendremos en la definición de σ-fragmentabilidad, ver para más detalles la Definición 1.5.3. Diremos que un espacio de Banach es σ-fragmentable si lo es cuando se considera con su topología débil respecto de la métrica inducida por la norma. El resultado más notable de [36] establece que un espacio de Banach es σ-fragmentable si es un subconjunto de Borel en su bidual para la topología débil*. En particular, los espacios con norma equivalente Kadec son σ-fragmentables. En [35], Jayne, Namioka y Rogers introducen una nueva propiedad similar a la σ-fragmentabilidad en cuanto a su formulación pero algo más restrictiva. Se dice que un espacio de Banach X tiene un recubrimiento numerable por conjuntos de diámetro local pequeño (propiedad JNR) si para cada $\varepsilon > 0$ existe una descomposición $X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ tal que para cada $n \in \mathbb{N}$ y cada $x \in X_n^{\varepsilon}$, existe un entorno débil U de x tal que $diam(X_n^{\varepsilon} \cap U) < \varepsilon$. La propiedad JNR ocupa un lugar intermedio entre la existencia de norma equivalente de Kadec y ser el espacio un subconjunto de Borel en su bidual, es decir, no se sabe si es equivalente a alguna de estas propiedades. Por otra parte, la propiedad JNR implica la coincidencia de los conjuntos de Borel para la topología débil y para la de la norma [54].

El hecho de que un espacio de Banach X con la propiedad JNR es un subconjunto de Borel en su bidual X^{**} para la topología débil* fue establecido por Oncina [54]. Oncina también demostró que tanto la propiedad JNR como la coincidencia de los conjuntos de Borel para la topología débil y para la de la norma son propiedades que dependen exclusivamente de la topología débil, utilizando para ello una técnica de transferencia debida a Moltó, Orihuela y Troyanski [50]. Mencionemos en este punto que hay algunas soluciones parciales positivas al problema de saber si un espacio con la propiedad JNR admite una norma equivalente de Kadec. Haydon [32] ha demostrado que es cierto en los espacios de funciones continuas sobre árboles. Por su parte, Lancien [47] ha construido en espacios de Banach con índice de dentabilidad numerable funciones con la propiedad de que en sus superficies de nivel coincide la topología débil y la topología de la norma. Los resultados de Lancien en espacios de Banach duales son más satisfactorios, pues llega a construir normas duales LUR.

El resultado principal del articulo de Moltó, Orihuela y Troyanski [50] citado anteriormente es el establecer la existencia de una norma equivalente LUR en un espacio de Banach si y solo si este posee una forma particular de la propiedad JNR. Diremos que un espacio de Banach tiene la propiedad sJNR (Definición 4.1.4) cuando el abierto débil U en la definición de la propiedad JNR puede tomarse un semiespacio. La construcción de la norma LUR de estos autores emplea metodos probalísticos cuyo uso se remonta a Pisier [55], y con los cuales Troyanski [66] demostró la bella relación existente entre el renormamiento Kadec y el renormamiento LUR: un espacio de Banach que posee una norma equivalente Kadec tiene también una norma equivalente LUR si y solo si posee una norma equivalente estrictamente convexa. La caracterización del renormamiento LUR de [50] posibilitó la resolución de algunos problemas abiertos sobre la existencia de tales normas. Moltó, Orihuela, Troyanski y Valdivia prueban en [51] que si un espacio admite una norma de tipo débilmente localmente uniformemente convexa (WLUR, ver

Definición 4.3.3), entonces el espacio es renormable LUR. Para ello utilizan caracterizaciones de la propiedad JNR en términos de la topología débil exclusivamente. Al hacerlo, aparece de manera contundente la relación existente entre la propiedad JNR y el trabajo previo de Hansell: un espacio de Banach X tiene la propiedad JNR si y sólo si es descriptivo.

Desarrollo de la memoria

Nuestro estudio arranca en el primer teorema de Edgar [12]. Allí se prueba de manera explícita que si un espacio de Banach X admite una norma equivalente Kadec, entonces existe una sucesión (A_n) de convexos cerrados (precisamente las bolas de la norma Kadec con radio racional y centro en el origen) tal que para cada abierto en norma $V \subset X$ existe una sucesión de abiertos débiles (U_n) tales que $V = \bigcup_{n=1}^{\infty}(A_n \cap U_n)$. Esta propiedad nos motivó a introducir la siguiente relación de subordinación entre dos topologías τ_1 y τ_2 sobre un conjunto X. Diremos que X tiene la propiedad $P(\tau_1, \tau_2)$ si existe una sucesión (A_n) de subconjuntos de X tal que para cada punto $x \in X$ y cada $V \in \tau_2$ existen $n \in \mathbb{N}$ y $U \in \tau_2$ tales que $x \in A_n \cap U \subset V$. Nuestro propósito es mostrar que esta definición es el punto de encuentro entre los resultados mencionados arriba sobre la naturaleza de Borel de ciertos conjuntos y la existencia de normas equivalentes de Kadec y LUR en los espacios de Banach. Pasemos a describir el contenido de la memoria.

El objeto del primer capítulo es el tratamiento en el marco puramente topológico de generalizaciones del teorema de Edgar, a la vez que se exhiben las conexiones existentes con los trabajos previos de Hansell [27], Jayne, Namioka y Rogers [35], y Oncina [54]. En la sección 1.1 introducimos una versión de la propiedad P para aplicaciones entre espacios topológicos. Los resultados de esta sección no tendrán aplicaciones relevantes hasta el capítulo cuarto. Las secciones 1.2 y 1.3 se dedican al estudio de las propiedades de las aplicaciones P-Borel, que son aplicaciones con la propiedad P más un "pequeño" ingrediente que las hace medibles. En la sección 1.4 se introduce la propiedad P tal como lo hemos hecho arriba para estudiar condiciones suficientes y necesarias sobre un espacio topológico para que este sea un conjunto de Borel dentro de un espacio mayor. El resultado principal, a continuación, concierne a los espacios Borel absolutos, es decir, aquellos espacios topológicos que son subconjutos de Borel en cada inmersión dentro de un espacio topológico regular.

Teorema 1 *Sea (X, τ) un espacio topológico regular. Consideremos las siguientes afirmaciones:*

i) *(X, τ) es un espacio Borel absoluto.*

ii) *$(A, \tau|_A)$ es Borel absoluto para cada $A \in Borel(X, \tau)$.*

iii) *Existe una topología Čech-completa δ sobre X más fina que τ y una sucesión (A_n) de subconjuntos τ-Borel de X tal que para cada $x \in X$ y cada $V \in \delta$ con $x \in V$, existen $n \in \mathbb{N}$ y $U \in \tau$ tales que $x \in A_n \cap U \subset V$.*

iv) *Existe una métrica completa d sobre X más fina que τ y una sucesión (A_n) de subconjuntos τ-Borel de X tal que para cada $x \in X$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y $U \in \tau$ tales que $x \in A_n \cap U$ y $diam(A_n \cap U) < \varepsilon$.*

Entonces iv) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i). Si además (X, τ) es completamente regular, entonces i), ii) y iii) son equivalentes. Si además (X, τ) es metrizable, entonces todos las afirmaciones son equivalentes.

El único precedente, a nuestro conocimiento, sobre caracterizaciones internas de espacios Borel absolutos se debe a Marciszewski y Pelant [49], que lo hacen añadiendo la hipótesis de metrizabilidad. Recordemos aquí que los espacios metrizables Borel absolutos están caracterizados desde hace mucho tiempo como los subconjuntos de Borel de espacios métricos completos [46], lo cual se deduce inmediatamente del Teorema 1. La sección 1.5 está dedicada al estudio de las nociónes de σ-fragmentabilidad y de espacio descriptivo en relación con la propiedad P. La sección 1.6 contiene dos ejemplos relativos a pares de topologías con la propiedad P.

El capítulo segundo está destinado al estudio del renormamiento Kadec. Aunque el problema de saber si un espacio con la propiedad JNR posee una norma equivalente de Kadec sigue aun abierto, se proponen soluciones parciales, tanto en la caracterización de la existencia de una norma equivalente Kadec en términos lineales topológicos, como en la caracterización de la propiedad JNR por medio de una "función con la propiedad de Kadec" a la que sólo le falta la convexidad para ser efectivamente una norma Kadec. En la sección 2.1 se estudia en el marco de los espacios topológicos la relación existente entre la propiedad P y las funciones con la propiedad de Kadec. A pesar de lo elemental de estos resultados, se obtienen ya aplicaciones interesantes como una prueba sencilla de que la propiedad JNR es una propiedad de "tres espacios", Corolario 2.1.11, debido a Ribarska. En la sección 2.2 se examina en mismo problema pero en el contexto de los espacios de Banach. Los resultados mejoran como consecuencia de trabajar con la estructura vectorial.

Teorema 2A *Sea X un espacio de Banach. Las siguientes afirmaciones son equivalentes:*

i) X tiene la propiedad JNR.

ii) Existe una sucesión (A_n) de subconjuntos de X tal que para cada $x \in X$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y un abierto débil U tales que $x \in A_n \cap U$ y $diam(A_n \cap U) < \varepsilon$.

iii) Existe una funcion homogenea débilmente inferiormente semicontinua $F : X \to [0, \infty)$ con $\|.\| \leq F \leq 3\|.\|$ tal que la topología de la norma y la topología débil coinciden sobre el conjunto

$$S = \{x \in X : \ F(x) = 1\}$$

De la demostración del Teorema 2A se deduce que si los conjuntos A_n que aparecen en la afirmación *ii*) fuesen convexos, entonces la función F sería convexa, y así una norma equivalente Kadec. Esto permite caracterizar en la sección 2.3 la existencia de normas de Kadec en términos similares al resultado de Moltó, Orihuela y Troyanski sobre normas LUR, aunque el paralelismo es evidente cuando este último se enuncia en términos de la propiedad P (ver Teorema 3, más abajo).

Teorema 2B *Sea X un espacio de Banach. Las siguientes afirmaciones son equivalentes:*

i) X admite una norma equivalente de Kadec.

ii) Existe una sucesión (A_n) de subconjuntos convexos de X tal que para cada $x \in X$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y un abierto débil U tales que $x \in A_n \cap U$ y $diam(A_n \cap U) < \varepsilon$.

iii) Para cada abierto en norma V existen una sucesión (C_n) de subconjuntos convexos cerrados y una sucesión (U_n) de abiertos débiles tales que $V = \bigcup_{n=1}^{\infty}(C_n \cap U_n)$.

Los resultados de esta sección, como los de la anterior, los formulamos en una situación mas general, sustituyendo la topología débil por una topología vectorial τ tal que $\overline{B_X}^{\tau}$ es acotado. El manejo de este tipo de topologías nos permite dar varias aplicaciones a los espacios débilmente numerablemente determinados. La sección 2.4 se ocupa del estudio de los espacios compactos descriptivos. En la demostraciones de los resultados de esta sección se utilizan las funciones con la propiedad de Kadec, lo que justifica su ubicación en este capítulo.

El capítulo tercero de destina al estudio del renormamiento localmente uniformemente convexo, al igual que el capítulo siguiente. La división del material sobre normas LUR en dos capítulos obedece a la extensión del mismo. Las ideas y resultados de este capítulo pueden verse como una adaptación al renormamiento LUR de las ya empleadas en el anterior para el renormamiento Kadec. Ahora, nuestra preocupación está puesta en conseguir una norma equivalente LUR que sea inferiormente semicontinua respecto a una topología de la forma $\sigma(X, Z)$ donde Z es un subespacio normante del dual X^* (tal norma será además $\sigma(X, Z)$-Kadec). Denotaremos por $\mathbb{H}(Z)$ el conjunto se semiespacios afines abiertos dados por elementos de Z. La sección 3.1 sólo contiene definiciones y algunos resultados de carácter elemental. La sección 3.2 contiene una prueba simple del teorema de Troyanski mencionado en esta introducción. La prueba original, que utiliza argumentos probabilísticos, puede consultarse en [10]. El metodo geométrico que hemos utilizado arroja mas información, que nos permite dar un resultado sobre espacios con la propiedad de Mazur. La sección 3.3 contiene los resultados principales, que se resumen en el siguiente teorema:

Teorema 3 *Sea X un espacio de Banach y sea $Z \subset X^*$ un subespacio casi normante. Las siguientes afirmaciones son equivalentes:*

i) X admite una norma equivalente localmente uniformemente convexa y $\sigma(X, Z)$-semicontinua inferiormente.

ii) X admite una norma equivalente estrictamente convexa y también admite otra norma equivalente $\sigma(X, Z)$-Kadec.

iii) Existe una sucesión (A_n) de subconjuntos convexos de X tal que para cada $x \in X$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y semiespacio $\sigma(X, Z)$-abierto H tales que $x \in A_n \cap H$ y diam$(A_n \cap H) < \varepsilon$.

Una consecuencia de este resultado es que un espacio de Banach dual X^* que posee una norma w^*-Kadec (esta propiedad también se conoce como propiedad $(**)$, ver [53, 12]) tiene una norma equivalente dual LUR. Otra consecuencia es que la norma LUR que existe en un espacio débilmente numerablemente determinado se puede tomar semicontinua inferiormente respecto a una topología $\sigma(X, Z)$ donde Z es un subespacio normante prefijado del dual. La sección 3.4 tiene carácter complementario y se dedica al estudio de cierto tipo de normas estrictamente convexas.

El último capítulo continúa con el estudio de las normas localmente uniformemente convexas. Se pretende dar caracterizaciones de la existencia de una norma equivalente LUR que cada vez exijan menos de la estructura lineal del espacio, que complementadas con técnicas de transferencia permiten obtener de manera sencilla casi todos los resultados conocidos sobre renormamiento LUR, así como otros nuevos. En la sección 4.1 se elimina la hipótesis sobre la convexidad de los conjuntos A_n del Teorema 3. Ello es posible gracias a un argumento de "convexificación" basado en el famoso "superlema" de Bourgain y Namioka [4]. El siguiente resultado es una mejora de la caracterización de existencia de normas LUR de Moltó, Orihuela y Troyanski [50].

Teorema 4A *Sea X un espacio de Banach y sea $Z \subset X^*$ un subespacio casi normante. Las siguientes afirmaciones son equivalentes:*

i) X admite una norma equivalente localmente uniformemente convexa y $\sigma(X, Z)$-semicontinua inferiormente.

ii) Existe una sucesion (A_n) de subconjuntos de X tal que para cada $x \in X$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y semiespacio $\sigma(X, Z)$-abierto H tales que $x \in A_n \cap H$ y $diam(A_n \cap H) < \varepsilon$.

En el resultado mencionado de [50] no se tiene en consideración el hacer la norma semicontinua inferiormente respecto a la topología $\sigma(X, Z)$ puesto que el metodo probabilístico no lo permite: en la construcción de la norma LUR de Troyanski [10, p. 147] se pierde la semicontinuidad por el camino. Los argumentos de tipo geométrico empleados en nuestra demostración evitan este problema. El Teorema 4A contiene, en particular, una caracterización de la existencia de una norma dual LUR en un espacio de Banach dual. Sin embargo, en este caso se puede decir un poco más gracias a la compacidad.

Teorema 4B *Sea X^* un espacio de Banach dual. Las siguientes afirmaciones son equivalentes:*

i) *X^* admite una norma dual equivalente localmente uniformemente convexa.*

ii) *Existe una sucesion (A_n) de subconjuntos de X^* tal que para cada $x \in X^*$ y cada $\varepsilon > 0$, existen $n \in \mathbb{N}$ y un debil* abierto U tales que $x \in A_n \cap U$ y $diam(A_n \cap U) < \varepsilon$.*

iii) *Para cada $\varepsilon > 0$ existe una descomposición $X^* = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ tal que para cada $n \in \mathbb{N}$ y cada $x \in X_n^{\varepsilon}$, existe un entorno débil* U de x tal que $diam(X_n^{\varepsilon} \cap U) < \varepsilon$.*

La afirmación *iii*) pone de manifiesto que el problema de saber si la propiedad JNR implica que existe un renormamiento Kadec queda resuelto en espacios de Banach duales cambiando la topología débil por la débil*. La sección 4.2 contiene una caracterización práctica de las aplicaciones con la propiedad P definidas en un espacio semimétrico con valores en un espacio métrico. Este resultado fue establecido en [50] para dos métricas sobre un mismo conjunto y juega un papel fundamental en la técnica de transferencia, que estudiamos en la sección 4.4. El enunciado que damos, ligeramente más general, se aplica al estudio de las normas débilmente uniformemente convexas y sus generalizaciones, en la sección 4.3. La sección 4.5 considera el problema de saber en qué espacios de Banach la norma tiene una base σ-discreta compuesta de conjuntos convexos. Resulta ser cierto para los espacios con una norma equivalente LUR o con la propiedad de Radon-Nikodym. Como aplicación se generaliza un resultado de Frontisi [19] sobre la existencia de particiones diferenciables de la unidad.

Agradecimientos

Este trabajo no hubiera sido posible sin las valiosas indicaciones y observaciones de los profesores R. Deville y G. Vera, directores de esta tesis, a quienes deseo expresar mi más sincero agradecimiento.

Durante el tiempo que he pasado realizando este trabajo he tenido la oportunidad de discutir con muchas personas se han interesado por los resultados del mismo y me han permitido mejorarlo gracias a sus sugerencias. La generalidad que presentan los resultados principales de esta tesis no hubiese sido posible sin la ayuda de G. Godefroy, P. Holicky, J. Orihuela y M. Valdivia. Algunas aplicaciones las debo a B. Cascales, M. Jiménez, J.P. Moreno y R. Payá. Tambien quiero recordar aqui a M. Fabian, R. Hansell, R. Haydon, I. Namioka, C.A. Rogers, S. Troyanski y V. Zizler cuyo interés en mi trabajo ha sido estimulo para poder concluirlo. Sería un ingrato si no mencionase a mis compañeros de tesis H. Bachir y L. Oncina.

El Departamento de Matemáticas de la Universidad de Murcia me ha proporcionado todo tipo de facilidades para llevar a cabo mis estancias en Bordeaux. El Laboratoire de Mathématiques Pures de la Université Bordeaux I me ha acogido y me ha permitido trabajar en las mejores condiciones. La beca de investigación de la Comunidad Autonoma de la Region de Murcia ha cubierto mis gastos durante tres años de mis estudios de doctorado.

A mi familia y a todos los amigos, los que aún estan ahí y los que quedaron en el camino, gracias por vuestro apoyo y vuestra confianza.

Introduction

This work is devoted to finding necessary and sufficient conditions for the existence of equivalent norms with the Kadec property or locally uniformly rotund in a Banach space. It also deals with the study of certain aspects of descriptive set theory, such as the characterization of absolute Borel topological spaces. At first sight, it would seem that these subjects are not much related between them, but along this dissertation we shall show our motives for that joint study.

Antecedents

Recall that an equivalent norm $\|.\|$ on a Banach space X is said to be rotund or strictly convex if its unit sphere does not contain segments. The norm $\|.\|$ is said to be locally uniformly rotund (LUR for short) if for every x, x_k in X with $\|x\| = \|x_k\| = 1$ such that $\lim_k \|x + x_k\| = 2$, we have $\lim_k \|x - x_k\| = 0$. It is not difficult to show that the weak and the norm topologies coincide on the unit sphere of $\|.\|$ if the norm $\|.\|$ is LUR. The norms having this property are called Kadec norms. On the other hand, there exist Banach spaces having a Kadec norm such that no equivalent norm is LUR [32]. Nowadays, the basic reference about renorming theory is the book of Deville, Godefroy and Zizler [10].

Edgar showed in [12] that if a Banach space admits an equivalent Kadec norm, then the Borel sets for the weak topology coincide with the Borel sets for the norm topology. In a second paper [13], Edgar includes two results of Schachermayer in this line. It is shown that if a Banach space admits an equivalent Kadec norm, then it is a Borel subset of its bidual X^{**} endowed with the weak* topology. The other result establishes that if the Borel subset for the weak and the norm topologies coincide on the unit sphere of some equivalent norm, then they coincide on the whole space. This last result would deviate our attention from the Banach spaces with equivalent Kadec norm. However, it is not known[3] any example of Banach space with the same Borel sets for the weak and the norm topologies (or being a Borel subset in its bidual for the weak* topology) with no equivalent Kadec norm. Let us mention that Talagrand proved in [64] that the space $l^\infty(\mathbb{N})$ has different Borel subsets for the weak and the norm topologies, and it is not a Borel subset in its bidual endowed with the weak* topology.

In 1989 appears a long paper written by Hansell [27] and still unpublished[4]. Hansell introduced the concept of descriptive topological space, that will not be explained here because it is rather technical (see Definition 1.5.9). Descriptive topological spaces generalize, in a very natural way, the classic notion of analytic spaces in the contex of non necessary metrizable nor separable topological spaces. Hansell also pointed out the role played by the existence of a σ-isolated network replacing σ-discrete topological basis, which are exclusive of metrizable spaces after the Bing-Nagata-Smirnov theorem [43]. These concepts have applications to the weak topology of a Banach space. We shall say that a Banach space is descriptive if it is so when endowed with its weak topology, what happends if and only if it has a σ-isolated network, see [27] and Theorem 2.2.8. Hansell proves that in a descriptive Banach space

[3]Now, it is: W. Marciszewski, R. Pol, On Banach spaces whose norm-open sets are F_σ-sets in the weak topology, *J. Math. Anal. Appl.* 350 (2009), 708–722.

[4]Descriptive sets and the topology of nonseparable Banach spaces, *Serdica Math. J.* 27 (2001), no. 1, 1–66.

the Borel sets for the weak topology and the norm topology coincide (this result is more interesting having in mind that being descriptive concerns exclusively to the weak topology), and the Banach space itself is a Borel subset in its bidual endowed with the weak* topology. As expected, Hansell also proves that a Banach space admiting an equivalent Kadec norm is descriptive.

After the success of the notion of fragmentability, particularly in the study of the Radon-Nikodym property and measurable selections of multi-valuated maps, Jayne, Namioka and Rogers introduced in [36] the notion of σ-fragmentability of a topological space with respect to some metric. We shall not detail here σ-fragmentability, see Definition 1.5.3. We say that a Banach space is σ-fragmentable when endowed with its weak topology it is σ-fragmentable with respect to the norm metric. The most remarkable result of [36] establishes that a Banach space is σ-fragmentable if it is a Borel subset of its bidual endowed with the weak* topology. In particular, the spaces with an equivalent Kadec norm are σ-fragmentable. In [35], Jayne, Namioka and Rogers introduced a new property which is very similar to σ-fragmentability, but more restrictive. A Banach space X is said to have a countable cover by sets of small local diameter (property JNR) if for every $\varepsilon > 0$ there is a decomposition $X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$ such that for every $n \in \mathbb{N}$ and every $x \in X_n^{\varepsilon}$, there is a neighbourhood U of x such that $diam(X_n^{\varepsilon} \cap U) < \varepsilon$. Property JNR has a intermediate position between the existence of an equivalent Kadec norm and being the space a Borel subset in its bidual, that is, it is not known if it is equivalent to one of these two properties. On the other hand, property JNR implies the coincidence of the Borel sets for the weak and the norm topologies.

The fact that a Banach space X with the JNR property is a Borel subset of its bidual X^{**} endowed with the weak* topology was established by Oncina [54]. Oncina also proved that property JNR and the coincidence of the Borel sets for the weak and the norm topologies are both properties which depend exclusively on the weak topology, using in the proof a transfer technique due to Moltó, Orihuela and Troyanski [50]. Let us mention that there is some positive partial results to the problem of knowing if property JNR implies the existence of an equivalent Kadec norm. Haydon [32] has shown that it is true in spaces of continuous functions on trees. On the other hand, Lancien has built functions on Banach spaces with countable dentability index such that the weak topology and the norm topology coincide on the level surfaces. The results of Lancien in dual Banach spaces are more satisfactory, because he is able to build dual LUR norms.

The main result in the paper by Moltó, Orihuela and Troyanski [50] we mentioned above is the establishment of the existence of an equivalent LUR norm in a Banach space if and only if it has a particular form of property JNR. We say that a Banach space has the property sJNR (Definition 4.1.4) when the open set U in the definition of property JNR can be chosen to be a half-space. The construction of the LUR norm by these authors uses probalistic methods, techniques that go back to Pisier [55], and employed by Troyanski, allow him to prove the beautiful relation between the Kadec renormability and the LUR renormability: a Banach space having an equivalent Kadec norm has an equivalent LUR norm if and only if it has an equivalent rotund norm. The characterization of the LUR renormability of [50] allowed the resolution of some open problems about the existence of such norms. Moltó, Orihuela, Troyanski and Valdivia proved in [51] that if a Banach space has a weakly locally uniformly rotund norm (WLUR, see Definition 4.3.3), then the space is LUR renormable. To do so, they used characterizations of the JNR property exclusively in terms of the weak topology. The relation between property JNR and the previous work of Hansell appears in a striking way: a Banach space X has the JNR property if and only if it is descriptive.

Development of the dissertation

Our study starts in the first theorem of Edgar [12]. There it is proved explicitely that if a Banach space admits an equivalent Kadec norm, then there is a sequence (A_n) of closed convex sets (in fact, the balls of the Kadec norm of rational radius and centered at the origin) such that for every norm open set $V \subset X$ there is a sequence (U_n) of weak open sets such that $V = \bigcup_{n=1}^{\infty}(A_n \cap U_n)$. This property motivated us to introduce the following relation of subordination between two topologies τ_1 τ_2 on a set X. We say that X has property $P(\tau_1, \tau_2)$ if there is a sequence (A_n) of subset of X such that for every point $x \in X$ and every $V \in \tau_1$ with $x \in V$ there exists $n \in \mathbb{N}$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$. Our aim is to show that this definition is the metting point between the results we mentioned above about the Borel nature of certain sets and the existence of equivalent Kadec and LUR norms in Banach spaces. Let us describe in detail the contents of this thesis.

The aim of the first chapter is the study of generalizations of Edgar's theorem in purely topological context, exhibiting the relations with the previous work of Hansell [27], Jayne, Namioka and Rogers [35], and Oncina [54]. In section 1.1 we introduce a version of property P for maps between topological spaces. The results in this section will have no relevant applications until chapter four. Sections 1.2 and 1.3 are devoted to the study of P-Borel maps, which are maps with property P together with a "little" ingredient making them measurable. In section 1.4 we introduce property P as we did above to study sufficient and necessary conditions for a topological space to be a Borel subset in a bigger space. The main result concerns to absolute Borel topological spaces, that is, those topological spaces which are Borel subsets in every embedding into a regular topological space.

Theorem 1 *Let (X, τ) a regular topological space. Consider the following statements:*

i) (X, τ) *is absolute Borel.*

ii) $(A, \tau|_A)$ *is absolute Borel for every* $A \in Borel(X, \tau)$.

iii) There is a Čech-complete topology δ on X finer than τ and a sequence (A_n) of τ-Borel sets such that for every $x \in X$ and every $V \in \delta$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U \subset V$.

iv) There is a complete metric d on X finer than τ and a sequence (A_n) of τ-Borel sets such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$.

Then iv) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i). If (X, τ) is completely regular, then i), ii) and iii) are equivalent. If (X, τ) is metrizable, then all the statements are equivalent.

The unique precedent, to our knowledge, about inner charaterizations of absolute Borel topological spaces is due to Marciszewski and Pelant [49], under metrizability hypothesis. Let us remind here that metrizable absolute Borel spaces were characterized long time ago as the Borel subsets of complete metric spaces [46], which can be easily deduced from Theorem 1. Section 1.5 is dedicated to the study of the notions of σ-fragmentability and descriptive space in relation with property P. Section 1.6 contains two examples of pairs of topologies with property P.

The second chapter is devoted to the study of Kadec renorming. Although the problem of knowing if a Banach space with property JNR admits an equivalent Kadec norm is still open, we give partial answers for the characterization of the existence of an equivalent Kadec norm in linear topological terms and the characterization of property JNR by means of a "function with the Kadec property", which just lacks convexity to be a Kadec norm. In section 2.1 we study the relation between property P and the functions with the Kadec property in the context of topological spaces. In spite of the simplicity of these results, interesting applications are obtained, like a short proof of the "three space property" for Banach spaces with JNR, Corollary 2.1.11, due to Ribarska. In section 2.2 we go on with the same problem, but in the context of Banach spaces. Results are more satisfactory as a consequence of the vectorial framework.

Theorem 2A *Let X be a Banach space. Then the following are equivalent:*

i) *X has the property JNR.*

ii) *There is a sequence of sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and a weak open set U such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$.*

iii) *There exists a symmetric homogeneous weakly lower semicontinuous function F on X with $\|.\| \leq F \leq 3\|.\|$ such that the norm topology and the weak topology coincide on the set*

$$S = \{x \in X : \ F(x) = 1\}$$

From the proof of Theorem 2A is possible to deduce that, if the sets A_n appearing in statement *ii)* were convex, then the function F would be convex, and thus it would be an equivalent Kadec norm. This allows us to characterize in section 2.3 the existence of Kadec norms in similar terms to the result of Moltó, Orihuela and Troyanski about LUR norms, althought the parallelism is more evident when this last result is stated in terms of property P (see Theorem 3 below).

Teorema 2B *Let X be a Banach space. Then the following are equivalent:*

i) *X admits an equivalent Kadec norm.*

ii) *There is a sequence of convex sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$.*

iii) *For every norm open subset V there is a sequence (C_n) of convex subsets and a sequence (U_n) of weak open subsets such that $V = \bigcup_{n=1}^{\infty}(C_n \cap U_n)$.*

The results of this section, as the results of the previous one, are formulated in a more general situation, replacing the weak topology by some vector topology τ such that $\overline{B_X}^{\tau}$ is bounded. The use of this kind of topologies allows us to give some applications to weakly countably determined Banach spaces. Section 2.4 is devoted to the study of descriptive compact spaces. The proofs of the results of this section employ fuctions with the Kadec property, and this justifies the emplacement in this chapter of the section.

Third chapter is devoted to the study of locally uniformly rotund renorming, as the next chapter. We have separated in two chapters the material about LUR norms because of its extension. The ideas and results in this chapter can be regarded as an adaptation of the ones we have employ in the second chapter for Kadec renorming. Now, our aim is to get an equivalent LUR norm which is lower semicontinuous with respect to a topology of the kind $\sigma(X, Z)$ where Z is a norming subspace of the dual X^* (such a norm will be $\sigma(X, Z)$-Kadec). Let us denote by $\mathbb{H}(Z)$ the set of affine open half-spaces given by elements of Z. Section 3.1 just contains some definitions and elementary results. Section 3.2 contains a simple proof of Troyanski's theorem, mentioned in this introduction. The original proof, which employes probabilistic arguments, can be found in [10]. The geometrical method of our proof gives more information allowing us to prove a result about Banach spaces with Mazur's intersection property. Section 3.3 contains the main results, that can be compiled in the following theorem:

Theorem 3 *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

i) X admits an equivalent $\sigma(X, Z)$-lsc LUR norm.

ii) X admits both an equivalent rotund norm and an equivalent $\sigma(X, Z)$-Kadec norm.

iii) There is a sequence of convex sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n \cap H$ and diam$(A_n \cap H) < \varepsilon$.

A consequence of this result is that a dual Banach space X^* having a w^*-Kadec norm (this property is also known as property (**), see [53, 12]) has an equivalent dual LUR norm. Another consequence is that the LUR norm can be chosen to be lower semicontinuous with respect to topologies of the kind $\sigma(X, Z)$ where Z is a prefixed norming subspace of the dual. Section 3.4 is of complementary character and it is devoted to a certain type of rotund norms.

Last chapter goes on with the study of locally uniformly rotund norms. We intend to give characterizations of the existece of an equivalent LUR norm which require as less as possible of the linear structure of the space. These characterizations, complemented with transfer techniques, will allow to obtain easily almost all the known results about LUR renormability, as well as some new ones. In section 4.1 the convexity hypothesis of the sets (A_n) is removed from Theorem 3. That is possible after a “covexification” argument inspired in the famous “superlemma” of Bourgain and Namioka [4]. Next result is an improvement of the characterization of the existence of LUR norms of Moltó, Orihuela and Troyanski [50].

Theorem 4A *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

i) X admits an equivalent $\sigma(X, Z)$-lsc LUR norm.

ii) There is a sequence of sets $A_n \subset X$ (respectively $A_n \subset S_X$) such that for every $x \in X$ (respectively $x \in S_X$) and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n \cap H$ and $diam(A_n \cap H) < \varepsilon$.

In the result of [50] mentioned above, it is not considered to make the norm lower semicontinuous for a topology $\sigma(X, Z)$ because the probabilistic method does not allow it: in the construction of the LUR norm of Troyanski [10, p. 147] semicontinuity is lost by the way. The arguments of geometrical nature employed in our proof avoid this problem. Theorem 4A contains, in particular, a characterization of the existence of a dual LUR norm in a dual Banach space. Although, in this case is possible to say a little more thanks to compactness.

Theorem 4B *Let X^* be a dual Banach space. The following are equivalent:*

i) *X^* admits an equivalent dual LUR norm.*

ii) *There is a sequence of sets $A_n \subset X$ (respectively $A_n \subset S_X$) such that for every $x \in X$ (respectively $x \in S_X$) and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and a weak* open set U such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$.*

iii) *For every $\varepsilon > 0$ there exists a decomposition*

$$X^* = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ every point of X_n^{ε} has a relatively nonempty weak-neighbourhood in X_n^{ε} of diameter less than ε.*

Statement *iii*) shows that the problem of knowing if property JNR implies the existence of an equivalent Kadec norm is solved for dual Banach spaces changing the weak topology for the weak* one. Section 4.2 contains a useful characterization of maps with property P defined on a semimetric space with values into a metric space. That result was established in [50] for two metrics on the same space and it plays a fundamental role in the transfer technique, which we have studied in section 4.2. Our formulation, which is a bit more general, is applied to the study of weakly locally uniformly rotund norms and its generalizations, in section 4.3. In section 4.5 we consider the problem of knowing what Banach spaces have a σ-discrete topological basis for the norm topology made of convex sets. That turns to be true for the spaces having a LUR norm or with the Radon-Nikodym property. As an application, we improve a result of Frontisi [19] about the existence of differentiable partitions of unity.

Chapter 1

Descriptive part

1.1 Maps with property P

We describe a property of maps between topological spaces generalizing the continuity but not far from the measurability in a sense which will be precised in the next section. Along the chapter we shall find connections with some properties introduced and studied by several authors.

Definition 1.1.1 *Let (X, τ) and (Y, δ) be topological spaces. A map $f : X \to Y$ is said to have the property P if there is a sequence (A_n) of sets in X such that for every $x \in X$ and every $V \in \delta$ with $f(x) \in V$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $f(A_n \cap U) \subset V$.*

It is not difficult to see that $f : X \to Y$ has property P if and only if there is a sequence (A_n) of subsets of X such that f is continuous when X is endowed with the topology generated by τ and (A_n), that we denote $top(\tau, \{A_n\})$.

Example 1.1.2 *We say that $f : X \to Y$ is σ-continuous if there is a decomposition $X = \bigcup_{n=1}^{\infty} X_n$ such that f is continuous restricted to each X_n. Clearly, every σ-continuous map has property P.*

The following topological definition was introduced by Arkangels'kii [2].

Definition 1.1.3 *Let (X, τ) be a topological space. A family $\mathfrak{N}$ of subsets of X is said to be a (sub)network for τ if every open set is a union of sets (finite intersection of sets) belonging to $\mathfrak{N}$.*

Definition 1.1.1 means that $\{U \cap A_n : U \in \tau, n \in \mathbb{N}\}$ is a network for the topology $f^{-1}(\delta)$ on X.

Example 1.1.4 *Suppose that (Y, δ) has a countable network. Then every map $f : X \to Y$ has property P.*

When (Y, δ) is metrizable, to check property P is equivalent to find a sequence (A_n) of sets in X such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and a τ-open set U in X such that $x \in A_n \cap U$ and $diam(f(A_n \cap U)) < \varepsilon$. It is easy to see that for a metrizable and separable subset $A \subset X$, if f has property P, then $f(A)$ is separable as well.

Example 1.1.5 *Suppose that (Y, δ) is metrizable. Then every map $f : X \to Y$ with separable range has property P.*

Example 1.1.6 *Let τ be the usual topology on $[0, 1]$ and let δ be the discrete topology on $[0, 1]$. Then the identity map $\mathbb{I} : ([0, 1], \tau) \to ([0, 1], \delta)$ does not have the property P.*

Some properties of stability of the class of maps with property P will be proved for the particular subclass of P-Borel maps, introduced in the next section, where these properties are more interesting.

The particular case when the range of f is metrizable is most important in relation with the results of this chapter. The following result is inspired in [51] and it shows the stability of property P in a very general situation.

Proposition 1.1.7 *Let $f : X \to Y$ a map from a topological space X to a metric space (Y, d). Suppose that there is a sequence of sets $D_n \subset X$ and maps $f_n : D_n \to Y$ having property P such that*

$$f(x) \in \overline{\{f_n(x) : \ x \in D_n\}}^d$$

for every point $x \in X$. Then f has property P.

Proof. For every $m \in \mathbb{N}$, there are sets $A_n^m \subset D_m$ satisfying Definition 1.1.1 for f_m, which is defined on D_m. Given $k \in \mathbb{N}$ we define

$$E_k^m = \{x \in D_m : d(f_m(x), f(x)) \leq 1/k\}$$

We claim that the countable family of sets $\{A_n^m \cap E_k^m : m, n, k \in \mathbb{N}\}$ satisfies Definition 1.1.1 for f. Indeed, fix $x \in X$ and $\varepsilon > 0$. Take k such that $1/k < \varepsilon/3$. The hypothesis entails that there is m such that $x \in E_k^m$, in particular $x \in D_m$. Now take n and $U \in \tau$ such that $x \in A_n^m \cap U$ and $diam(f_m(A_n^m \cap U)) < \varepsilon/3$. It is clear that if $y \in A_n^m \cap E_k^m \cap U$, then

$$d(f(x), f(y)) \leq d(f(x), f_m(x)) + d(f_m(x), f_m(y)) + d(f_m(y), f(y)) \leq \varepsilon$$

which proves the claim.

The implication $i) \Rightarrow iii)$ of Theorem 1.1.10 can be regarded like a reciproque of Proposition 1.1.7.

Definition 1.1.8 *Let (X, τ) be a topological space and let $\mathcal{E} = \{E_i : i \in I\}$ be a family of subsets of X. Then $\mathcal{E}$ is said to be discrete if every point $x \in X$ has a neighbourhood that meets a member of $\mathcal{E}$ at most. The family $\mathcal{E}$ is said to be isolated if it is discrete in its union $\bigcup \mathcal{E} = \bigcup_{i \in I} E_i$ endowed with the relative topology. If there is a decomposition $I = \bigcup_{n=1}^{\infty} I_n$ such that every family $\{E_i : i \in I_n\}$ is discrete (resp. isolated), then $\mathcal{E}$ is said to be σ-discrete (resp. σ-isolated). The family $\mathcal{E}$ is said to be σ-discretely (resp. σ-isolatedly) decomposable if for every $i \in I$ there is a decomposition $E_i = \bigcup_{n=1}^{\infty} E_{i,n}$ such that the families $\{E_{i,n} : i \in I\}$ are σ-discrete for every $n \in \mathbb{N}$ (resp. σ-isolated).*

The following definition comes from Hansell [25].

Definition 1.1.9 *Let $f : X \to Y$ be a map from a set X to a topological space (Y, δ). A function base of f is a network for the topology $f^{-1}(\delta)$ on X.*

When the range of f is metrizable we have the following equivalences.

Theorem 1.1.10 *Let $f : X \to Y$ be a map from a topological space X to a metric space Y. Then the following are equivalent:*

i) f has property P.

ii) f has a σ-isolated function base.

iii) f is is uniform limit of a sequence of σ-continuous maps.

Proof. *i)* $\Rightarrow$ *ii)* By Stone's Theorem [43] there is basis $(B_i)_{i\in I}$ of Y such that $I = \bigcup_{k=1}^{\infty} I_k$ and each $(B_i)_{i\in I_k}$ is discrete. Given $i \in I$, we define for every $n \in \mathbb{N}$ the set

$$C_{n,i} = \{x \in A_n : \exists\, U \in \tau,\ x \in U,\ f(A_n \cap U) \subset B_i\}$$

It is easy to see that every $C_{n,i}$ is relatively open in the disjoint union $\bigcup_{i\in I_k} C_{n,i}$. Thus $\{(C_{n,i})_{i\in I_k}\}_{n,k}$ is σ-isolated, and by construction, a function base for f.
ii) $\Rightarrow$ *i)* Let $(C_i)_{i\in I}$ be a function base for f such that $I = \bigcup_{k=1}^{\infty} I_k$ and each family $(C_i)_{i\in I_n}$ is isolated. It is clear that Definition 1.1.1 is satisfied taking $A_n = \bigcup_{i\in I_n} C_i$.
i) $\Rightarrow$ *iii)* Fix $m \in \mathbb{N}$. Let $(B_i)_{i\in I}$ a cover of Y by open sets of diameter less than $1/m$ and such that $I = \bigcup_{k=1}^{\infty} I_k$ where every subfamily $(B_i)_{i\in I_k}$ is discrete. Take

$$E^m_{n,i} = \{x \in X : \exists\, U \in \tau,\ x \in U,\ f(A_n \cap U) \subset B_i\}$$

As above, family $(E^m_{n,i})_{i\in I_k}$ is isolated. By construction, for every $m \in \mathbb{N}$ we have

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{i\in I_k} E^m_{n,i} = X$$

Take sets $X^m_{n,i} \subset E^m_{n,i}$ such that $(X^m_{n,i})_{n\in\mathbb{N},i\in I}$ is a partition of X. Fix $x^m_{n,i} \in X^m_{n,i}$ if it is nonempty. Since every $x \in X$ belongs to just one $X^m_{n,i}$, we can define a map $f_m : X \to Y$ taking $f_m(x) = f(x^m_{n,i})$. Now, for $k, n \in \mathbb{N}$ fixed, since f_m is locally constant on $\bigcup_{i\in I_k} X^m_{n,i}$, it is continuous on this

set. Moving $k, n \in \mathbb{N}$ we cover X and thus f_m is σ-continuous. Note that $d(f(x), f_m(x)) \leq 1/m$ for every x, so (f_m) converges uniformly to f.
iii) $\Rightarrow$ *i)* It is a consequence of Proposition 1.1.7.

There are analogous results for the class of σ-fragmentable maps. Compare the above theorem with [38, Corollary 7].

Remark 1.1.11 *It is clear that the implication ii) $\Rightarrow$ i) in the preceding theorem is true even if Y is not metrizable. This shows that the class of functions with σ-isolated function base considered by Hansell in [27] is contained in the class of functions with the property P.*

1.2 A class of Borel measurable maps

If we ask the sets (A_n) of the Definition 1.1.1 to be Borel, we obtain a class between the continuous maps and the Borel measurable maps which shares many of the good properties of them.

Definition 1.2.1 *Let (X, τ) and (Y, δ) be topological spaces. A map $f : X \to Y$ is said to be P-Borel if it has property P and the sets (A_n) in Definition 1.1.1 are τ-Borel sets. That is, there is sequence (A_n) of τ-Borel sets in X such that for every $x \in X$ and every $V \in \delta$ with $f(x) \in V$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $f(A_n \cap U) \subset V$.*

To check that $f : X \to Y$ is P-Borel is enough to verify Definition 1.2.1 with $V \subset Y$ belonging to a subbasis of δ. It is not difficult to see that, in this case, f will have property P with the countable collection of finite intersections of A_n's.

Example 1.2.2 *If X is a Polish space and Y metrizable, then every Borel map $f : X \to Y$ is P-Borel*[1]*.*

[1] Under Continuum Hypothesis, sorry!

Proof. It is enough to see that $f(X)$ is separable because then the conclusion follows from Example 1.1.5. Suppose that $f(X)$ is not separable to get a contradiction. In this case, $f(X)$ must contain a discrete subset of cardinality greater than $\aleph_0$ and this implies the existence of a family of Borel subsets of $f(X)$ of cardinality greater than c. But the Borel σ-algebra of X is countably generated, so it has cardinality at most c. This contradicts the measurability of f.

We shall need the following classification of the Borel sets in a topological (not necessary metrizable) space.

Definition 1.2.3 *Let (X, τ) be a topological space. The family $\mathcal{A}_\alpha$ of the additive sets of class α and the family $\mathcal{M}_\alpha$ of the multiplicative sets of class α are constructed for every countable ordinal α by the following inductive process:*

i) The family $\mathcal{A}_0$ is composed of the τ-open sets and the family $\mathcal{M}_0$ is composed of the τ-closed sets.

ii) If $\alpha > 0$ then the sets of $\mathcal{A}_\alpha$ are of the form $\bigcup_{n=1}^{\infty}(A_n \cap B_n)$ and the sets of $\mathcal{M}_\alpha$ are of the form $\bigcap_{n=1}^{\infty}(A_n \cup B_n)$, where $A_n \in \mathcal{A}_{\alpha_n}$ and $B_n \in \mathcal{M}_{\alpha_n}$ with $\alpha_n < \alpha$.

The properties of the additive and multiplicative families are summarized in the following lemma.

Lemma 1.2.4 *Let (X, τ) be a topological space, $\mathcal{A}_\alpha$ and $\mathcal{M}_\alpha$ the families defined above. Then the following holds:*

i) $A \in \mathcal{A}_\alpha$ if and only if $X \setminus A \in \mathcal{M}_\alpha$.

ii) $\mathcal{A}_\alpha$ is stable under countable unions and $\mathcal{M}_\alpha$ is stable under countable intersections.

iii) $\mathcal{A}_\alpha$ is stable under finite intersections and $\mathcal{M}_\alpha$ is stable under finite unions.

iv) *If* $\alpha < \beta$ *then* $\mathcal{A}_\alpha \cup \mathcal{M}_\alpha \subset \mathcal{A}_\beta \cap \mathcal{M}_\beta$.

v) *If* $\alpha > 0$ *then* $\mathcal{A}_{\alpha+1} = (\mathcal{M}_\alpha)_\sigma$ *and* $\mathcal{M}_{\alpha+1} = (\mathcal{A}_\alpha)_\delta$.

vi) $\bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha = Borel(X, \tau)$.

Recall that a Borel function is said of class α if the inverse image of every open set is of additive class α, and it is said of bounded class if it is of class α for some countable ordinal α.

Theorem 1.2.5 *Every P-Borel map is Borel measurable of bounded class.*

Proof. Let V be a δ-open set in Y. We shall check that $f^{-1}(V)$ is a τ-Borel set in X. For every $x \in f^{-1}(V)$ we can find $n(x) \in \mathbb{N}$ and $U_x \in \tau$ such that $f(A_{n(x)} \cap U_x) \subset V$. Now, we have that

$$f^{-1}(V) = \bigcup_{x\in f^{-1}(V)} A_{n(x)} \cap U_x = \bigcup_{n=1}^{\infty} \bigcup_{n(x)=n} A_{n(x)} \cap U_x = \bigcup_{n=1}^{\infty} A_n \cap (\bigcup_{n(x)=n} U_x)$$

is a τ-Borel set in X. Note that if α is a countable ordinal which bounds the additive class of the sets (A_n), then for every $V \in \delta$ we have that $f^{-1}(V)$ is of additive class α.

The next result follows from the work of Hansell [25].

Theorem 1.2.6 (Hansell) *Let X be a complete metric space. Then every disjoint family with the property that arbitrary unions of sets from the family are analytic is σ-discretely decomposable.*

Corollary 1.2.7 *If X is a complete metric space and Y is metrizable, then every Borel map of bounded class $f : X \to Y$ is P-Borel.*

Proof. Suppose that f is of class α. Let $\bigcup_{n=1}^{\infty} \mathfrak{B}_n$ a basis of Y where every $\mathfrak{B}_n$ is disjoint. Since f is Borel measurable, $f^{-1}(\mathfrak{B}_n)$ satisfy the hypothesis of the theorem above, so it is σ-discretely decomposable. For every $V \in \mathfrak{B}_n$ put $f^{-1}(V) = \bigcup_{m=1}^{\infty} H(m, V)$ where $\{H(m, V) : V \in \mathfrak{B}_m\}$ is discrete. Define

$$A(n, m) = \bigcup_{V \in \mathfrak{B}_n} f^{-1}(V) \cap \overline{H(m, V)}$$

A discrete union of sets of additive class α is also of class α (see [46]), in particular, the sets $A(n, m)$ are Borel. It is easy to check that f has property P with the countable family $\{A(n, m) : n, m \in \mathbb{N}\}$, thus f is P-Borel.

The affirmation "every Borel measurable map of bounded class between metric spaces is P-Borel" is independent of the usual axioms of set theory. Hansell has pointed out us (private comunication) that under Martin's Axiom and the negation of the Continuum Hypothesis there exists a uncountable subset X of $\mathbb{R}$ such that every subset is a relative $\mathcal{F}_\sigma$. This implies in particular that the identity map into X endowed with the discrete topology is of the first Borel class, but it does not have property P by the argument used in Example 1.1.6. On the other hand, Fleissner's Axiom (see [17]) implies that every disjoint family with the property that arbitrary unions of sets from the family are analytic is σ-discretely decomposable. Assuming this, the proof of Corollary 1.2.7 shows that every Borel map of bounded class between metric spaces is P-Borel.

Now, we give some results about the stability of the class of P-Borel maps under some operations usual in measure theory.

Proposition 1.2.8 *If $f : X \to Y$ is P-Borel, and $X_0 \subset X$ and $Y_0 \subset Y$ are such that $f(X_0) \subset Y_0$. Then $f|_{X_0} : X_0 \to Y_0$ is P-Borel for the relative topologies.*

Proposition 1.2.9 *Let (X_i, τ_i) be topological spaces for $i = 1, 2, 3$. Given P-Borel maps $f : X_1 \to X_2$ and $g : X_2 \to X_3$, then $g \circ f : X_1 \to X_3$ is also a P-Borel map.*

Proof. Let $(A_n) \subset X_1$ a sequence of sets satisfying Definition 1.2.1 for f and let $(B_n) \subset X_2$ satisfying Definition 1.2.1 for g. After Theorem 1.2.5, the sets $f^{-1}(B_n)$ are τ_1-Borel. It is easy to see that the countable family $(A_n \cap f^{-1}(B_m))$ of τ_1-Borel sets satisfies Definition 1.2.1 for the map $g \circ f$.

Proposition 1.2.10 *Let (X, τ) and (Y_i, δ_i) topological spaces for $i \in I$ where I is finite or countable. Let $f_i : X \to Y_i$ be P-Borel maps for $i \in I$. Then the map $f : X \to \prod_{i \in I} Y_i$ defined by $f(x) = (f_i(x))_{i \in I}$ is P-Borel.*

Proof. Let (A_n^i) a sequence of τ-Borel sets satisfying Definition 1.2.1 for f_i. Now, for every finite subset $F \subset I$ and every finite sequence $(n_i) \subset \mathbb{N}$ for $i \in F$ consider the set $\bigcap_{i \in F} A_{n_i}^i$. In this way we obtain a countable family of τ-Borel sets. We claim that this family satisfies Definition 1.2.1. Let $x \in X$ and $V \subset \prod_{i \in I} Y_i$ an open neigbourhood of $f(x)$ of the form $\prod_{i \in I} V_i$ where $V_i = Y_i$ for $i \in I \setminus F$ and F is finite. For every $i \in F$ we set $A_{n_i}^i$ and $U_i \in \tau$ such that $f_i(A_{n_i}^i \cap U_i) \subset V_i$. Then we have that

$$f((\bigcap_{i \in F} A_{n_i}^i) \cap (\bigcap_{i \in F} U_i)) \subset \prod_{i \in I} V_i = V$$

which finishes the proof of the claim.

For two Borel maps with values in a topological vector space, it is not known if their sum will be Borel in general (see [26] for a discussion when the topologies are metrizable). For the class of P-Borel maps we do have a positive answer.

Corollary 1.2.11 *Let (X, τ) be a topological space, let $(Y, +, \delta)$ be a topological group, and let $f, g : X \to Y$ be P-Borel maps. Then their sum $f + g$ is also P-Borel.*

Proof. The map $x \to (f(x), g(x))$ is P-Borel after Proposition 1.2.10 and the composition with the sum map is P-Borel after Proposition 1.2.9.

In similar way to Proposition 1.2.10, we can prove the following.

Proposition 1.2.12 *Let (X_i, τ_i) and (Y_i, δ_i) be topological spaces for $i \in I$ finite or countable. Let $f_i : X_i \to Y_i$ be P-Borel maps for $i \in I$. Then the map $f : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ defined by $f((x_i)_{i\in I}) = (f_i(x_i))_{i\in I}$ is P-Borel.*

Proposition 1.2.13 *Let (X, τ) be a topological space, let Y be a set and let $f : X \to Y$ be a map. Let (δ_n) be a sequence of topologies on Y such that for every $n \in \mathbb{N}$, when endowed Y with δ_n, the map f is P-Borel. Then f is P-Borel when Y is endowed with $top(\{\delta_n : n \in \mathbb{N}\})$.*

The proof of the properties considered above depends mainly upon the similarity between continuity and property P. The next result shows that, when Y is metrizable, P-Borel functions, like Borel functions, are stable under pointwise limits of sequences.

Theorem 1.2.14 *Let $f : X \to Y$ be a map from a topological space X to a metric space Y. If f is the pointwise limit of a sequence (f_n) of P-Borel maps[2], then f is a P-Borel map.*

Proof. Let (A_n^i) be a sequence satifying Definition 1.2.1 for f_i. From Proposition 1.2.10 and the continuity of d we deduce that the map $d(f_i(x), f_j(x))$ is P-Borel, and thus the sets

$$E_k^i = \{x \in X : \ d(f_i(x), f_j(x)) \leq 1/k, \ \forall\, j \geq i\}$$

are τ-Borel. We claim that the countable family of sets $\{A_n^i \cap E_k^i : \ i, n, k \in \mathbb{N}\}$ satisfies Definition 1.2.1 for f. Fix $x \in X$, $\varepsilon > 0$ and $k > 3/\varepsilon$. There is $i \in \mathbb{N}$ such that $d(f_j(x), f(x)) < 1/2k$ for every $j \geq i$, thus $x \in E_k^i$. Now take $U \in \tau$ and $n \in \mathbb{N}$ such that $x \in A_n^i \cap U$ and $diam(f_i(A_n^i \cap U)) \leq \varepsilon/3$. If $y \in A_n^i \cap E_k^i \cap U$ then

$$d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)) < \varepsilon$$

which proves the claim.

[2]Misprint fixed

Remark 1.2.15 *The pointwise limit of a sequence of continuous functions has the property P with a sequence (A_n) of τ-closed sets.*

Remark 1.2.16 *With the notation of the proof, if f is limit uniformly of the sequence (f_n), then it is easy to see that f has the property P with the countable family of Borel sets $(A_n^i)_{n,i}$. In particular, this shows that a uniform limit of real Borel measurable maps of class α is also of class α.*

1.3 Images of measures

Along this section, all the measures considered will be supposed to be positive and finite.

Definition 1.3.1 *Let (X, τ) be a topological space. A measure μ defined on $Borel(X, \tau)$ is said to be*

1) *smooth if for every family (U_α) of open sets there is a countable set of indices (α_n) such that*

$$\mu(\bigcup_{\alpha} U_\alpha) = \mu(\bigcup_{n=1}^{\infty} U_{\alpha_n})$$

2) *regular if for every $A \in Borel(X, \tau)$ and every $\varepsilon > 0$ there is a τ-closed set $F \subset A$ such that*

$$\mu(A) < \mu(F) + \varepsilon$$

3) *a Radon measure if the set F in 2) can be taken τ-compact.*

The following results show that P-Borel maps have the property of carrying good measures to good measures.

Theorem 1.3.2 *Let $f : X \to Y$ be P-Borel. If μ is a smooth Borel measure on X, then the image measure $f(\mu)$ is smooth on Y.*

Proof. Suppose that f is P-Borel with some sequence (A_n) of subsets of X. Let $(V_i)_{i\in I}$ a family of open sets in Y. Reasoning as in the proof of Theorem 1.2.5, for every index $i \in I$ and every $n \in \mathbb{N}$, there is a open set $U_{n,i}$ in X such that

$$f(\bigcup_{n=1}^{\infty} A_n \cap U_{n,i}) = V_i \cap f(X)$$

For every fixed $n \in \mathbb{N}$ consider the family $(U_{n,i})_{i\in I}$ of open sets in X. Since μ is smooth we can take a countable subfamily $I_n \subset I$ such that

$$\mu(\bigcup_{i\in I} U_{n,i}) = \mu(\bigcup_{i\in I_n} U_{n,i})$$

thus

$$\mu(A_n \cap \bigcup_{i\in I} U_{n,i}) = \mu(A_n \cap \bigcup_{i\in I_n} U_{n,i})$$

and we deduce that

$$\mu(\bigcup_{n=1}^{\infty}\bigcup_{i\in I} A_n \cap U_{n,i}) = \mu(\bigcup_{n=1}^{\infty}\bigcup_{i\in I_n} A_n \cap U_{n,i})$$

Taking $I_0 = \bigcup_{n=1}^{\infty} I_n$ we obtain

$$f(\mu)(\bigcup_{i\in I} V_i) = f(\mu)(\bigcup_{i\in I_0} V_i)$$

and thus $f(\mu)$ is smooth.

Theorem 1.3.3 *Let $f : X \to Y$ be P-Borel. For every regular measure μ on X and every $\varepsilon > 0$, there is a closed set $F \subset X$ with $\mu(F) > \mu(X) - \varepsilon$ such that f restricted to F is continuous. Moreover, if μ is a Radon measure, then the closed set F can be chosen compact.*

Proof. Without loss of generality we can suppose that $\mu(X) = 1$. Let τ be the topology on X and suppose that f is P-Borel with some sequence (A_n) of subsets of X. Let $\tau_0 = top(\tau, \{A_n\}, \{X \setminus A_n\})$. Fix $\varepsilon > 0$. We claim that there is a τ-closed set $F \subset X$ with $\mu(F) > 1 - \varepsilon$ such that τ and τ_0 coincide on F. Put $A_n^0 = A_n$ and $A_n^1 = X \setminus A_n$. Let s denote a finite sequence of 0's and 1's and let $|s|$ denote its length. We define the set

$$A(s) = \bigcap_{n=1}^{|s|} A_n^{s(n)}$$

Since μ is regular, we can find closed sets $F_n^i \subset A_n^i$ for $i = 0, 1$ such that

$$\mu(F_n^0 \cup F_n^1) > 1 - \frac{\varepsilon}{2^n}$$

For every finite sequence s we define the τ-closed set (maybe empty) $F(s)$ by

$$F(s) = \bigcap_{n=1}^{|s|} F_n^{s(n)}$$

Clearly we have that $F(s) \subset A(s)$. Now define F by

$$F = \bigcap_{n=1}^{\infty} (F_n^0 \cup F_n^1)$$

By construction $\mu(F) > 1 - \varepsilon$. A neighbourhood basis for τ_0 is given by the sets of the form $A(s) \cap U$ where $U \in \tau$. It is not difficult to see that

$$F \cap A(s) \cap U = F \cap (U \setminus \bigcup \{F(r) : \ |r| = |s|, \ r \neq s\})$$

As the second set is a relative τ-open, this proves the coincidence of the topologies. Since f is τ_0-continuous then it is continuous restricted to F.

Definition 1.3.4 *It is said that f is Lusin μ-measurable if for every $\varepsilon > 0$, there is a compact set $K \subset X$ with $\mu(X \setminus K) < \varepsilon$ such that $f|_K$ is continuous.*[3]

Corollary 1.3.5 *A P-Borel map $f : X \to Y$ is Lusin μ-measurable for every Radon measure μ on X. In particular the image measure $f(\mu)$ on Y is Radon.*

[3]In relation to this notion, it is not difficult to prove that if X is a Baire space and $f : X \to Y$ is P-Borel then there exists a dense $\mathcal{G}_\delta$-set $A \subset X$ such that $f|_A$ is continuous. Moreover, if f has P with closed sets then A can be taken such that all its points are of global continuity

Recall that the family of sets of a topological space (X, τ) which are μ-measurable for every Radon measure μ on X is a σ-algebra that we denote $Univ(X, \tau)$, and called the σ-algebra of universally measurable sets.

Lemma 1.3.6 *Let X be a set and $\tau_2 \subset \tau_1$ be topologies on X. Then the following are equivalent:*

i) $Univ(X, \tau_1) = Univ(X, \tau_2)$

ii) The identity map $\mathbb{I} : (X, \tau_2) \to (X, \tau_1)$ is Lusin μ-measurable for every τ_2-Radon measure μ on X.

Proof. *i)* $\Rightarrow$ *ii)* If μ is τ_2 Radon then for every $\varepsilon > 0$ there is a τ_1-compact set K such that $\mu(K) > \mu(X) - \varepsilon$. As $\mathbb{I} : (K, \tau_2) \to (X, \tau_1)$ is continuous we get the Lusin μ-measurability.
ii) $\Rightarrow$ *i)* For a fixed τ_2-Radon measure μ we have to prove that every $A \in Univ(X, \tau_1)$ is μ-measurable. Fix $\varepsilon > 0$ and take K a τ_2-compact set with $\mu(K) > \mu(X) - \varepsilon$ such that $\mathbb{I} : (K, \tau_2) \to (X, \tau_1)$ is continuous. It is easy to see that $\mu^*(A) < \mu^*(A \cap K) + \varepsilon$. By the continuity of $\mathbb{I}|_K$ we have that K is τ_1-compact and $\mu|_K$ is τ_1-Radon. Thus $A \cap K$ is μ-measurable and $\mu^*(A \cap K) = \mu|_K(A \cap K)$. Then we have that $\mu^*(A) < \mu(A \cap K) + \varepsilon$, and thus A is μ-measurable.

From the identity of the Borel structures for two topologies on some set does not follow the identity of their universaly measurable sets. The following corollary gives a sufficient condition for a positive answer.

Corollary 1.3.7 *Let X be a set and $\tau_2 \subset \tau_1$ be topologies on X. If*

$$\mathbb{I} : (X, \tau_2) \to (X, \tau_1)$$

is P-Borel, then

$$Univ(X, \tau_1) = Univ(X, \tau_2)$$

It is well known that for a Banach space X, it is always true that $Univ(X, \|.\|) = Univ(X, w)$. But in the case of a dual Banach space X^*, the identity $Univ(X^*, \|.\|) = Univ(X^*, w^*)$ is equivalent to the Radon-Nikodym property of X^* (see [12]).

1.4 Borel sets

In this section we investigate relations between the Borel σ-algebras for two topologies on some set X, and conditions for a topological space X to be a Borel subset into some overspace Z. The following notion, introduced in [56], seems to be suitable for both aims.

Definition 1.4.1 *Let Z be a set, τ_1 and τ_2 two topologies on Z. We say that a subset $X \subset Z$ has property $P(\tau_1, \tau_2)$ with a sequence $A_n \subset Z$ of sets, if for every $x \in X$ and every $V \in \tau_1$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau_1$ such that $x \in A_n \cap U \subset V$.*

From the definition, it follows that if we do not specify the sequence (A_n), the property P only depends on X endowed with the topologies τ_1 and τ_2. In the applications that follows we shall give some properties of the sets (A_n) instead of writing explicitly the sequence.

Lemma 1.4.2 *Let X be a set and $\tau_2 \subset \tau_1$ two topologies on X. If X has $P(\tau_1, \tau_2)$ with τ_2-Borel sets, then*

$$Borel(X, \tau_1) = Borel(X, \tau_2)$$

Proof. The identity map $\mathbb{I} : (X, \tau_2) \to (X, \tau_1)$ is P-Borel.

We call σ-ring a family of subsets of a given set which is closed by countable unions and countable intersections.

Theorem 1.4.3 *Let Z a set endowed with two topologies τ_1 and τ_2. Suppose that X has $P(\tau_1, \tau_2)$ with τ_2-Borel sets. Let $M \subset Z$ be a set belonging to* *the σ-ring generated by the τ_1-open sets such that $X \subset M$. Then there is a τ_2-Borel set B such that*

$$X \subset B \subset M$$

In particular, if X belongs to the σ-ring generated by τ_1, then X is τ_2-Borel.

Proof. Let Σ be the family of those subsets $M \subset Z$ with the following property: for every $X' \subset X \cap M$ there is a τ_2-Borel set B such that $X' \subset B \subset M$. We shall prove that Σ is a σ-ring containing the τ_1-open sets.
First suppose that M is τ_1-open. Take $X' \subset X \cap M$. For every $x \in X'$ there is $n_x \in \mathbb{N}$ and $U_x \in \tau_2$ such that $x \in A_{n_x} \cap U_x \subset M$. We have that

$$X' = \bigcup_{x \in X'} \{x\} \subset \bigcup_{x \in X'} A_{n_x} \cap U_x = \bigcup_{n=1}^{\infty} (A_n \cap \bigcup_{n_x = n} U_x) = B \subset M$$

where clearly B is a τ_2-Borel set.
Let $(M_n) \subset \Sigma$ and $X' \subset X \cap \bigcup_{n=1}^{\infty} M_n$. Applying the hypothesis to the sets $X_n = X' \cap M_n$, there are τ_2-Borel sets B_n such that $X_n \subset B_n \subset M_n$. Thus

$$X' = \bigcup_{n=1}^{\infty} X_n \subset \bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} M_n$$

and we deduce that $\bigcup_{n=1}^{\infty} M_n \in \Sigma$.
Finally, let $(M_n) \subset \Sigma$ and $X' \subset X \cap \bigcap_{n=1}^{\infty} M_n$. By hypothesis, there is a τ_2-Borel B_n such that $X' \subset B_n \subset M_n$ for every $n \in \mathbb{N}$. Thus

$$X' \subset \bigcap_{n=1}^{\infty} B_n \subset \bigcap_{n=1}^{\infty} M_n$$

and so $\bigcap_{n=1}^{\infty} M_n \in \Sigma$. This ends the proof of the theorem.

Lemma 1.4.4 *Let Z be a set, τ_1 and τ_2 topologies on Z. Let X be a subset of Z such that every point of X has a τ_1-basis of neighbourhoods made up of τ_2-closed sets. If X has $P(\tau_1, \tau_2)$ with some sequence (A_n), then X also has $P(\tau_1, \tau_2)$ with the sequence $(\overline{A_n}^{\tau_2})$.*

Proof. Fix $x \in X$ and take $V \in \tau_1$ with $x \in V$. Take $V_0 \in \tau_1$ such that $x \in V_0$ and $\overline{V_0}^{\tau_2} \subset V$. There exists A_n and $U \in \tau_2$ such that $x \in A_n \cap U \subset V_0$. Thus

$$x \in \overline{A_n}^{\tau_2} \cap U \subset \overline{A_n \cap U}^{\tau_2} \subset \overline{V_0}^{\tau_2} \subset V$$

When τ_1 is metrizable, the property $P(\tau_1, \tau_2)$ is equivalent to the property SLD of Jayne, Namioka and Rogers, see Definition 1.5.5 and Proposition 1.5.6. The first corollary is the observation by these authors in [35, 37] (see also [54]) about the coincidence of Borel sets for the topology τ and a lower semicontinuous metric d in a topological space (X, τ) which has d-SLD. The second corollary extends a result of Oncina [54] which states that a Banach space X such that (X, w) has $\|.\|$-SLD is a Borel set in (X^{**}, w^*). The same result was proved previously by Hansell [27] for descriptive Banach spaces, but this hypothesis is equivalent to the property $\|.\|$-SLD of (X, w), see [51] and Theorem 2.2.8.

Corollary 1.4.5 *Let (X, τ) be a topological space and let d be a metric on X stronger than τ and such that the closed d-balls are τ-closed. If X has $P(d, \tau)$, then*

$$Borel(X, \tau) = Borel(X, d)$$

Moreover, every d-open set is a $(\mathcal{F} \cap \mathcal{G})_\sigma$-set in (X, τ).

Corollary 1.4.6 *Let (Z, τ) be a topological space and let d be a metric on Z stronger than τ such that the closed d-balls are τ-closed. Let X be a subset of Z having $P(d, \tau)$. If X is a Borel set in (Z, d), then*

$$X \in Borel(Z, \tau)$$

Moreover, if X is closed in (Z, d), then it is a $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$-set in (Z, τ).

Lemma 1.4.7 *Let X be a set and let $\mathfrak{C}$ be a class of topologies on X which satisfies these two properties:*

i) If $\tau \in \mathfrak{C}$ and S is τ-closed, then $top(\tau, \{S\}) \in \mathfrak{C}$.

ii) If $\{\tau_n\} \subset \mathfrak{C}$ *is a sequence, then* $top(\{\tau_n\}) \in \mathfrak{C}$.

Then the following holds:

1) Suppose that (τ_α) *is a transfinite sequence such that* $\tau_0 \in \mathfrak{C}$, $\tau_{\alpha+1} = top(\tau_\alpha, \{F_n^\alpha : n \in \mathbb{N}\})$ *where every* F_n^α *is* τ_α*-closed and if* α *a limit ordinal* $\tau_\alpha = top(\tau_\beta : \beta < \alpha)$*. Then* $\tau_\gamma \in \mathfrak{C}$ *and*

$$\tau_\gamma = top(\tau_0, \{F_n^\alpha : n \in \mathbb{N}, \alpha < \gamma\})$$

for every countable ordinal γ*. The sets* $\{F_n^\alpha : n \in \mathbb{N}\}$ *are of additive class* α *in* (X, τ_0)*.*

2) Given $\tau_0 \in \mathfrak{C}$ *and* $(A_n) \subset Borel(X, \tau_0)$ *of additive class* γ *there is a transfinite sequence* $(\tau_\alpha)_{\alpha \leq \gamma} \subset \mathfrak{C}$ *as in 1) such that every* A_n *is* τ_γ*-open. The sets* $\{F_n^\alpha : n \in \mathbb{N}, \alpha < \gamma\}$ *are of additive class* γ *in* (X, τ_0)*.*

Proof. It follows easily by transfinite induction.

Theorem 1.4.8 *Let* (X, d) *be a complete metric space and let* (B_n) *be a sequence of Borel subsets of* X*. There exists a sequence* (A_n) *of Borel subsets containing* (B_n) *such that* $top(d, \{A_n\})$ *is completely metrizable.*

Proof. We shall prove that the class $\mathfrak{C}$ of completely metrizable topologies on X finer than d satisfies the hypothesis of the preceding lemma.
If S is d-closed it is known that (S, d) and $(X \setminus S, d)$ can be completely metrized by metrics d_1 and d_2, respectively, which are bounded by 1. Let d_0 the metric defined on X by $d_0(x, y) = d_1(x, y)$ if $x, y \in S$, $d_0(x, y) = d_2(x, y)$ if $x, y \in X \setminus S$ and $d_0(x, y) = 1$ if else. Clearly S is d_0-clopen because the topology of d_0 is the topology generated by the union of $\{S, X \setminus S\}$ and the topology of d. Let us check that d_0 is complete. If (x_n) is a d_0-Cauchy sequence, it is easy to see that for n big enough x_n belongs either to S or to $X \setminus S$, and thus, (x_n) is d_1-convergent or d_2-convergent, respectively. This shows that (x_n) is d_0-convergent.

Let d_n be a sequence of complete metrics on X finer than d. We can suppose that the metrics d_n are bounded by 1. The distance defined by

$$d_0(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x, y)$$

is compatible with $top(\{d_n : n \in \mathbb{N}\})$. It is clear that d_0 is stronger than d. If (x_k) is a d_0-Cauchy sequence, then it is d_n-Cauchy for every n, and thus, d_n-convergent. The limit is the same for all n because it coincides with the d-limit. Thus (x_k) is d_0-convergent and (X, d_0) is complete.

Corollary 1.4.9 *Let (X, d) be a complete metric space, let Y be a topological space and let (f_n) be a sequence of P-Borel maps from X to Y. Then there is a complete metric d_0 on X such that X has $P(d_0, d)$ and every f_n is d_0-continuous.*

Proof. Let $(A^n_m)_m$ be a sequence of Borel sets satisfying Definition 1.4.1 for f_n. Then apply Theorem 1.4.8 to the countable family $(A^n_m)_{n,m}$.

The following result is well known in Polish spaces theory, see [42].

Corollary 1.4.10 *Let (X, d) be a complete separable space. Then for every Borel set $A \subset X$ (resp. every P-Borel map $f : X \to Y$) there is a metric d_0 on X such that (X, d_0) is complete separable, the d_0-Borel sets and the d-Borel sets coincide and A is d_0-clopen (resp. f is d_0-continuous). In particular (A, d_0) is complete.*

In the remaining part of the section we shall find internal conditions for a topological space to be a Borel subset into a bigger topological space.

Definition 1.4.11 *A topological space X is said to be absolute Borel if for every embedding $i : X \to Z$ into a regular topological space, then $i(X)$ is a Borel subset of Z.*

In order to characterize the absolute Borel completely regular topological spaces we shall use the notion of Čech-complete topological space. The definition of Čech-complete space that we shall use is in fact a result by Frolik [18]. We prefer this definition because it is formulated in terms of the topology of the space.

Definition 1.4.12 *A completely regular topological space (X, δ) is said to be Čech-complete if it has a complete sequence of open covers, that is, there are δ-open covers $(\mathcal{S}_n)$ of X such that every filter $\mathfrak{F}$ in X has a cluster point provided that $\mathfrak{F} \cap \mathcal{S}_n \neq \emptyset$ for every $n \in \mathbb{N}$.*

Next lemma is part of Frolik's characterizations of Čech-complete topological spaces. The proof use ideas that comes from Sierpinski's (see [43]) results about completely metrizables subspaces of a regular topological space.

Lemma 1.4.13 *A Čech-complete topological space (X, τ) is a $\mathcal{F} \cap \mathcal{G}_\delta$ set in every regular embedding. Conversely, if (X, τ) is a $\mathcal{F} \cap \mathcal{G}_\delta$ set in some compact space, then (X, τ) is Čech-complete.*

Proof. Suppose that (X, τ) is a dense subspace of a regular space $(Z, \widetilde{\tau})$. Let $(\mathcal{S}_n)$ be a complete sequence of open covers of X. We define for every $n \in \mathbb{N}$ the open sets

$$G_n = \{z \in Z : \exists U_{n,z} \in \widetilde{\tau}, z \in U_{n,z}, X \cap U_{n,z} \in \mathcal{S}_n\}$$

Clearly we have that $X \subset G_n$ for every $n \in \mathbb{N}$. We claim that $X = \bigcap_{n=1}^{\infty} G_n$. Indeed, take $z \in \bigcap_{n=1}^{\infty} G_n$. We have for every $n \in \mathbb{N}$ that $z \in U_{n,z}$ and $X \cap U_{n,z} \in \mathcal{S}_n$. Let $\mathfrak{F}$ be the filter of neighbourhoods of z. The regularity of $(Z, \widetilde{\tau})$ implies that $\bigcap_{U \in \mathfrak{F}} \overline{U}^{\widetilde{\tau}} = \{z\}$. By the density of X in Z, we have that $\mathfrak{F}|_X$ is also a filter and $X \cap U_{n,z} \in \mathfrak{F}|_X \cap \mathcal{S}_n$ for every $n \in \mathbb{N}$. Applying that $\mathcal{S}_n$ is a complete sequence of covers, we have that

$$\emptyset \neq \bigcap_{U \in \mathfrak{F}} \overline{(X \cap U)}^{\tau} \subset X \cap \bigcap_{U \in \mathfrak{F}} \overline{U}^{\widetilde{\tau}}$$

This implies that $z \in X$.
Now suppose that (X, τ) is $\mathcal{G}_\delta$-set in a compact space $(Z, \widetilde{\tau})$, that is, $X = \bigcap_{n=1}^{\infty} G_n$ where every G_n is $\widetilde{\tau}$-open. For every $n \in \mathbb{N}$ define $\mathcal{S}_n$ as the collection of the sets of the form $X \cap U$ where $U \in \widetilde{\tau}$ and $\overline{U}^{\widetilde{\tau}} \subset G_n$. Every filter $\mathfrak{F}$ must have a cluster point in Z by the compactness. If $\mathfrak{F} \cap \mathcal{S}_n \neq \emptyset$ for every $n \in \mathbb{N}$, then the cluster points must belong to G_n for every $n \in \mathbb{N}$, so $\mathfrak{F}$ has its cluster points in X and this shows that $(\mathcal{S}_n)$ is a complete sequence of open covers of X.

A well known consequence is that $\mathcal{F} \cap \mathcal{G}_\delta$ subsets of Čech-complete spaces are also Čech-complete.

Lemma 1.4.14 *The class of Čech-complete topologies on some set X which are finer than a prefixed Hausdorff topology verify the properties of Lemma 1.4.7.*

Proof. Let τ be a Čech-complete topology on X. Let $F \subset X$ a τ-closed set. Assume that X is a $\mathcal{G}_\delta$-set in some compact space $(Z, \widetilde{\tau})$. It is easy to see that the following map

$$i : X \longrightarrow \{0, 1\} \times Z$$

defined by $i(x) = (0, x)$ if $x \in F$ and $i(x) = (1, x)$ if $x \in X \setminus F$ is an embedding of X endowed with $top(\tau, \{F\})$ and $i(X)$ is a $\mathcal{F} \cap \mathcal{G}_\delta$-set in the compact space $\{0, 1\} \times Z$, and thus $top(\tau, \{F\})$ is a Čech-complete topology.
Now suppose that (τ_n) is a sequence of Čech-complete topologies on X. For every $n \in \mathbb{N}$ take a compact space $(Z_n, \widetilde{\tau}_n)$ containing (X, τ_n) as a $\mathcal{G}_\delta$ subspace. Let $\tau = top(\{\tau_n\})$ and consider the map

$$i : X \longrightarrow \prod_{n=1}^{\infty} Z_n$$

defined by $i(x) = (x)_{n=1}^{\infty}$. It is easy to see that i is an embedding of (X, τ) into a compact space. Now, $i(X)$ is a relatively closed in the $\mathcal{G}_\delta$-set $\prod_{n=1}^{\infty} X$, and thus τ is Čech-complete.

An internal characterization for absolute Borel metrizable spaces in terms of complete sequences of covers was obtained in [49]. The main result of this section provides an internal characterization of those completely regular spaces which are absolute Borel and a sufficient condition in the regular case.

Theorem 1.4.15 *Let (X, τ) a regular topological space. Consider the following statements:*

i) *(X, τ) is absolute Borel.*

ii) *$(A, \tau|_A)$ is absolute Borel for every $A \in Borel(X, \tau)$.*

iii) *There is a Čech-complete topology δ on X finer than τ such that X has $P(\delta, \tau)$ with a sequence (A_n) of τ-Borel sets.*

iv) *There is a complete metric d on X finer than τ such that X has $P(d, \tau)$ with a sequence (A_n) of τ-Borel sets.*

Then $iv) \Rightarrow iii) \Rightarrow ii) \Rightarrow i)$. If (X, τ) is completely regular, then $i)$, $ii)$ and $iii)$ are equivalent. If (X, τ) is metrizable, then all the statements are equivalent.
Moreover, if $\gamma > 0$ is countable ordinal, then (X, τ) is of multiplicative class $\gamma + 1$ in every regular embedding if and only if the sets (A_n) of $iii)$ and $iv)$ can be taken of additive class γ.

Proof. It is clear that $iv) \Rightarrow iii)$ and $ii) \Rightarrow i)$.
$iii) \Rightarrow i)$ Assume that (X, τ) is a subspace of a regular topological space $(Z, \widetilde{\tau})$. Put $A_n = A'_n \cap X$ where $A'_n \in Borel(Z, \widetilde{\tau})$ are of additive class γ. Applying Lemma 1.4.7, there is a transfinite sequence of regular topologies $(\widetilde{\tau}_\alpha)_{\alpha \leq \gamma}$ with $\widetilde{\tau}_0 = \widetilde{\tau}$ and $\widetilde{\tau}_{\alpha+1} = top(\widetilde{\tau}_\alpha, \{F_n^\alpha : n \in \mathbb{N}\})$ where F_n^α is $\widetilde{\tau}_\alpha$-closed, and every A'_n is $\widetilde{\tau}_\gamma$-open. Since δ is stronger than τ, the transfinite sequence defined by $\delta_0 = \delta$ and $\delta_{\alpha+1} = top(\delta_\alpha, \{X \cap F_n^\alpha : n \in \mathbb{N}\})$ is made up of Čech-complete topologies by Lemmata 1.4.7 and 1.4.14. We claim that $\delta_\gamma = \widetilde{\tau}_\gamma|_X$. Indeed, we have that $\widetilde{\tau}_\gamma|_X \subset \delta_\gamma$ because we add the same sets to the topologies τ and δ. On the other hand, every A'_n is $\widetilde{\tau}_\gamma$-open, and the

inclusion $\delta_\gamma \subset \widetilde{\tau}_\gamma|_X$ follows.
We have that $(X, \widetilde{\tau}_\gamma|_X)$ is Čech-complete, so by Lemma 1.4.13, X is a $\mathcal{F}\cap\mathcal{G}_\delta$-set in $(Z, \widetilde{\tau}_\gamma)$. Note that a $\widetilde{\tau}_\gamma$-open set has additive class γ in $(Z, \widetilde{\tau})$, and thus X has multiplicative class $\gamma + 1$ in $(Z, \widetilde{\tau})$.
iii) $\Rightarrow$ *ii*) It is enough to see that $(A, \tau|_A)$ satisfies the condition *iii*). If δ is a Čech-complete topology on X such that X has $P(\delta, \tau)$ with τ-Borel sets, by Lemma 1.4.7, we can construct a Čech-complete topology δ_γ on X such that A is δ_γ-open and X has $P(\delta_\gamma, \tau)$ with τ-Borel sets. Now $(A, \delta_\gamma|_A)$ is Čech-complete and A has $P(\delta_\gamma|_A, \tau|_A)$ with $\tau|_A$-Borel sets.
i) $\Rightarrow$ *iii*) Consider (X, τ) as subset of its Čech-Stone compactification $(\beta X, \widetilde{\tau})$. If X is of multiplicative class $\gamma+1$ in βX then it can be writen $X = \bigcap_{n=1}^{\infty} X_n$ where every X_n is of additive class γ. By Lemma 1.4.7, there is a Čech-complete topology on βX obtained from $\widetilde{\tau}$ by adding a countable sequence of sets of additive class γ making every X_n open. Thus X is a $\mathcal{G}_\delta$-set in a Čech-complete space and so it is Čech-complete too for that finer topology.
i) $\Rightarrow$ *iv*) It is enough to consider (X, τ) into some complete metric space and reasoning like in *i*) $\Rightarrow$ *iii*).

Corollary 1.4.16 *Let (Z, τ) be a regular topological space. Let X be a subset of Z such that there is metric d on X stronger than the restriction of τ. Suppose that X has $P(d, \tau)$ and the closed d-balls are τ-closed in X. If (X, d) is complete, then X is $(\mathcal{F}\cap\mathcal{G})_{\sigma\delta}$-set in (Z, τ). In particular, $X \in Borel(Z, \tau)$.*

We can get as corollaries some classic results, see Kuratowski [46]. The first one tell us that the Borel subsets of complete metric spaces are absolute Borel spaces.

Corollary 1.4.17 (Sierpinski) *Let X be a metrizable space. Then X is Borel subset in every embedding into a regular space if and only if X is homeomophic to a Borel subset of a complete metric space.*

Corollary 1.4.18 (Souslin) *Let X be a Polish space, Z a regular space and let $f : X \to Z$ be a one-to-one P-Borel map. Then $f(B)$ is a Borel set in Z*

for every Borel subset B of X. If Z is metrizable, it is enough to ask f to be a Borel measurable one-to-one map to get the same conclusion.

Proof. Since any Borel subset of X is a Polish space with some stronger topology, Corollary 1.4.10, we only have to prove that $f(X)$ is Borel. Again by Corollary 1.4.10, we can take a metric d such that (X, d) is complete separable and f is continuous. If τ is the topology of Z, then $f^{-1}(\tau)$ is a regular topology on X coarser than d. By Theorem 1.6.1, X has $P(d, f^{-1}(\tau))$ with Borel sets. Now apply Theorem 1.4.15 to the embedding of $(X, f^{-1}(\tau))$ into (Z, τ) to get the conclusion. If Z is metrizable, then f is P-Borel by Example 1.2.2.

1.5 Fragmentability

In this section we introduce some definitions given by Jayne, Rogers and Namioka in a series of papers [39, 52, 34, 35, 36]. We shall relate these concepts to those ones discussed before and to the notion of descriptive topological space of Hansell [27].

Definition 1.5.1 *Let (X, τ) be a topological space and d a metric on X. It is said that X is fragmentable by d if for every $\varepsilon > 0$ and every nonempty $A \subset X$ there is $U \in \tau$ such that $A \cap U \neq \emptyset$ and $diam(A \cap U) < \varepsilon$.*

Proposition 1.5.2 *Let (X, τ) be a hereditary Baire topological space and let d be a metric on X. If X has $P(d, \tau)$ with τ-closed sets then X is fragmentable by d.*

Proof. Suppose that X has $P(d, \tau)$ with a sequence of τ-closed sets (A_n). Fix $\varepsilon > 0$ and let $C \subset X$ be a nonempty τ-closed set. Define the sets

$$C_n = \{x \in C \cap A_n : \exists U \in \tau,\ x \in U,\ diam(A_n \cap U) < \varepsilon\}$$

Since $C = \bigcup_{n=1}^{\infty} C_n$, by the Baire property we have that for some $n \in \mathbb{N}$, there exists $V \in \tau$ such that $\emptyset \neq C \cap V \subset \overline{C_n}^{\tau}$. In particular we can take

$x \in C_n \cap V$. Let $U \in \tau$ such that $x \in U$ and $diam(A_n \cap U) < \varepsilon$. We have that

$$x \in C \cap V \cap U \subset \overline{C_n}^{\tau} \cap U \subset A_n \cap U$$

and thus $diam(C \cap V \cap U) < \varepsilon$.

Definition 1.5.3 *Let (X, τ) be a topological space and d a metric on X. It is said that X is σ-fragmentable by d if for every $\varepsilon > 0$, there is a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for every $n \in \mathbb{N}$ and every nonempty $A \subset X_n^{\varepsilon}$, there is $U \in \tau$ such that $A \cap U \neq \emptyset$ and $diam(A \cap U) < \varepsilon$.

Let $\mathcal{A}$ be a family of subsets of a topological space X. A subset $A \subset X$ is said to be a Souslin-$\mathcal{A}$ set if it can be written as follows

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} A(\sigma|n)$$

where $\sigma|n$ denotes the finite sequence made up from the first n terms of the sequence σ and $A(s) \in \mathcal{A}$ for every finite sequence s.

Definition 1.5.4 *A topological space X is said to be Čech-analytic if X is a Souslin-Borel subset in its Čech-Stone compactification βX. A metrizable topological space X is said to be analytic (nonseparable in general) if X is a Souslin-closed subset in a complete metric space, and this is equivalent to Čech-analytic.*

If a Banach X space is Čech-analytic in its weak topology (in particular, if X is w^*-Borel set in X^{**}), then (X, w) is σ-fragmentable by the norm (see [36]). However, all the known examples of σ-fragmentable Banach spaces satisfies the following stronger condition, introduced in [35].

Definition 1.5.5 *Let (X,τ) be a topological space and let d be a metric on X. It is said that X has a countable cover by set of small local diameter (d-SLD) if for every $\varepsilon > 0$, there exists a decomposition*

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

such that for each $n \in \mathbb{N}$ and every $x \in X_n^{\varepsilon}$, there is $U \in \tau$ such that $x \in U$ and $diam(X_n^{\varepsilon} \cap U) < \varepsilon$.

It is easy to see that if (X,τ) has d-SLD then (X,τ) is σ-fragmentable by d. The converse its not true in general. The compact space $[0,\omega_1]$ is fragmentable by the discrete metric d, but it does not have d-SLD.

The equivalence $ii) \Leftrightarrow iii)$ of the following proposition is in [51].

Proposition 1.5.6 *Let (X,τ) be a topological space and d a metric on X. The following are equivalent:*

i) X has $P(d,\tau)$.

ii) (X,τ) has d-SLD.

iii) (X,d) has a (sub)network which is σ-isolated with respect to τ.

Proof. $i) \Rightarrow ii)$ Given $\varepsilon > 0$ just define

$$X_n^{\varepsilon} = \{x \in A_n : \exists U \in \tau,\ x \in U,\ diam(A_n \cap U) < \varepsilon\}$$

$ii) \Rightarrow iii)$ Let $\mathfrak{B} = \bigcup_{m=1}^{\infty} \mathfrak{B}_m$ be a basis for the topology of d such that every $\mathfrak{B}_m$ is $1/m$ σ-discrete. Let $X = \bigcup_{n=1}^{\infty} X_n^m$ the decomposition of Definition 1.5.5 for $\varepsilon = 1/m$. Then $\{X_n^m \cap \mathfrak{B}_m\}_{m,n}$ is a network for d which is σ-isolated with respect to τ because $\mathfrak{B}_m$ is τ-σ-discrete restricted to every X_n^m.
$iii) \Rightarrow i)$ If $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ is a subnetwork with each family $\mathfrak{N}_n$ isolated, then the countable union of the families $\mathfrak{N}_{n_1} \cap \ldots \cap \mathfrak{N}_{n_k}$ is a σ-isolated network for d, thus we can suppose that $\mathfrak{N}$ is a network and each $\mathfrak{N}_n$ is isolated. Now it is easy to see that X has $P(d,\tau)$ with the sequence of sets $A_n = \bigcup \mathfrak{N}_n$.

Lemma 1.5.7 *The class of Čech-analytic topologies on some set X which are finer than a prefixed Hausdorff topology verify the properties of Lemma 1.4.7.*

Proof. It can be done like the proof of Lemma 1.4.14.

The following result is related to some ones of Junnila and Michael, see [27] and the references therein.

Theorem 1.5.8 *Let (X, τ) be a topological space. If there is a metric d finer than τ such that X has $P(d, \tau)$, then (X, τ) has a σ-isolated network. Reciprocally, if (X, τ) has a σ-isolated network, then there is a finer metric d such that X has $P(d, \tau)$ with τ-closed sets. Moreover, if (X, τ) is Čech-complete (resp. Čech-analytic), then the metric d can be chosen to be complete (resp. analytic).*

Proof. By Proposition 1.5.6, (X, d) has a network $\mathfrak{N}$ which is σ-isolated in (X, τ). Since d is finer than τ, then $\mathfrak{N}$ is a network for τ.
Reciprocally, let $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ a network with every family $\mathfrak{N}_n$ isolated. We can suppose that the network $\mathfrak{N}$ is of the following form: there is a sequence of closed sets (A_n^1) such that for every $H \in \mathfrak{N}_n$ there is an open set U_H such that $H = A_n^1 \cap U_H$. To see that, take $A_n^1 = \overline{\bigcup \mathfrak{N}_n}^{\tau}$ and take an open set U_H for every $H \in \mathfrak{N}_n$ such that $H \subset U_H$ and $G \cap U_H = \emptyset$ for every $G \in \mathfrak{N}_n$ with $G \neq H$. Define the sets $\widetilde{H} = U_H \cap A_n^1$ and

$$\widetilde{\mathfrak{N}}_n = \{\widetilde{H} : H \in \mathfrak{N}_n\}$$

It is easy to verify that every $\widetilde{\mathfrak{N}}_n$ is isolated and $\widetilde{\mathfrak{N}} = \bigcup_{n=1}^{\infty} \widetilde{\mathfrak{N}}_n$ is a network which satisfies the required condition.
Define the closed sets $A_n^2 = A_n^1 \setminus \bigcup \mathfrak{N}_n$. The following family of subsets of X

$$\{X \setminus A_n^1 : n \in \mathbb{N}\} \cup \{A_n^2 : n \in \mathbb{N}\} \cup \bigcup_{n=1}^{\infty} \mathfrak{N}_n$$

is a subbasis of a metrizable topology δ on X finer than τ. Since this topology can be obtained as the topology generated by $\tau \cup \{A_n^1 : n \in \mathbb{N}\} \cup \{A_n^2 : n \in \mathbb{N}\}$

we deduce that X has $P(\delta, \tau)$ with τ-closed sets. Finally, if τ is Čech-complete (resp. analytic), then δ is Čech-complete by Lemma 1.4.14 (resp. Čech-analytic by Lemma 1.5.7) as well, so it can completely metrized (resp. it is analytic).

Hansell introduced in [27] the concept of descriptive topological space.

Definition 1.5.9 *A topological Hausdorff space is said to be descriptive if there is a continuous surjection $T : M \to X$ from a complete metric space such that T carries discrete families in M to σ-isolatedly decomposable families in X.*

Remark 1.5.10 *Suppose that a topological space (X, τ) has d-SLD, then the identity map $\mathbb{I} : (X, d) \to (X, \tau)$ carries discrete families to σ-isolatedly decomposable families, see [54]. Thus, if the metric d is complete, then (X, τ) is a descriptive space.*

Hansell proves in [27] that a metrizable space is descriptive if and only if it is analytic, thus descriptive spaces are a natural extension of metrizable analytic spaces. He also showed that a descriptive space is Čech-analytic and it has a σ-isolated network. We will show that the reciproque also holds.

Theorem 1.5.11 *A topological space X is descriptive if and only if X is a Čech-analytic space with a σ-isolated network.*

Proof. We only have to prove the reciproque. If d is the analytic metric given by Theorem 1.5.8, then the identity map from (X, d) to (X, τ) is a continuous surjection which carries d-discrete families to σ-isolatedly decomposable families for τ by Proposition 1.5.6 and Remark 1.5.10. Since (X, d) is a descriptive space, then (X, τ) is also descriptive by [27].

Corollary 1.5.12 *A Čech-analytic subspace of a descriptive space is also descriptive.*

We need to introduce the class of hereditarily weakly θ-refinable topological spaces. See [5] for more information about them.

Definition 1.5.13 *A topological space X is said to be weakly θ-refinable if every open cover of X has a σ-isolated (non necessary open) refinement. If every subspace of X is weakly θ-refinable, then it is said that X is hereditarily weakly θ-refinable.*

Remark 1.5.14 *A topological space having a σ-isolated network is hereditarily weakly θ-refinable.*

Remark 1.5.15 *Note that if τ is a hereditarily weakly θ-refinable topology on X and (A_n) is a sequence of subsets of X, then $top(\tau, \{A_n\})$ is hereditarily weakly θ-refinable as well.*

The following is an adaptation of an argument of Hansell [27].

Proposition 1.5.16 *Let (X, τ) be a topological space which is hereditarily weakly θ-refinable, (Y, d) a metric space and $f : X \to Y$ a map. Then the following are equivalent:*

i) f has property P.

ii) There is a sequence of sets $(A_n) \subset X$ such that for every nonempty $C \subset X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $C \cap A_n \cap U$ is nonempty and $f(C \cap A_n \cap U)$ has diameter less than ε.

Proof. $i) \Rightarrow ii)$ It is trivial.
$ii) \Rightarrow i)$ After Remark 1.5.15, we know that $\tau' = top(\tau, \{A_n\})$ is hereditary weakly θ-refinable. It is enough to prove that f has property P when X is endowed with the topology τ' because then the result will follow using that X has $P(\tau', \tau)$.
For every $n \in \mathbb{N}$ define inductively a transfinite sequence $\{U^n_\alpha : \alpha < \gamma_n\}$ of τ'-open sets which covers X such that

$$diam(f(U^n_\alpha \cap (X \setminus \bigcup_{\beta<\alpha} U^n_\beta))) < 1/n$$

If we take $D^n_\alpha = U^n_\alpha \cap (X \setminus \bigcup_{\beta<\alpha} U^n_\beta) = U^n_\alpha \setminus \bigcup_{\beta<\alpha} U^n_\beta$ it is clear that any σ-isolated decomposition of $\{D^n_\alpha : n \in \mathbb{N}, \alpha < \gamma_n\}$ will be a σ-isolated function base for f.
To see that, for every $n \in \mathbb{N}$, the family $\{D^n_\alpha : \alpha < \gamma\}$ with $\gamma \leq \gamma_n$ is σ-isolatedly decomposable we proceed by induction on γ. It not difficult to see that for γ contable or non limit ordinal the induction hypothesis implies the result. Assume that γ is a limit ordinal. Since (X, τ') is hereditarily weakly θ-refinable the family of open sets $\{U^n_\alpha : \alpha < \gamma\}$ has some refinement $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$ where $\mathcal{N}_m$ is isolated for every $m \in \mathbb{N}$. Note that for every $H \in \mathcal{N}_m$, the supremun of those ordinals α such that $D^n_\alpha \cap H \neq \emptyset$ is less than γ. This fact implies easily that the family

$$\{D^n_\alpha \cap H : \alpha < \gamma, H \in \mathcal{N}_m\}$$

is σ-isolatedly decomposable, and thus $\{D^n_\alpha : \alpha < \gamma\}$ is also σ-isolatedly decomposable.

The use of the hereditarily weakly θ-refinability to turn "scattered" properties into "isolated" properties is due to Hansell, see [27, 28].

Corollary 1.5.17 (Hansell) *If (X, τ) is hereditarily weakly θ-refinable and σ-fragmented by some metric d, then (X, τ) has d-SLD (equivalently, X has property $P(d, \tau)$).*

1.6 Two examples

The first example is essentially the classic Souslin Separation Theorem for analytic sets, see [42, 46].

Theorem 1.6.1 *Let (X, d) be a complete separable metric space and let τ be a Hausdorff topology coarser than d. Then X has $P(d, \tau)$ with τ-Borel sets.*

Proof. We say that a pair of disjoint sets (E^1, E^2) is separated by some set B if $E^1 \subset B$ and $E^2 \cap B = \emptyset$. Since (X, d) is separable it is enough to see

that every d-closed set is τ-Borel, and to see that it is enough to show that every two disjoint d-closed set can be separated by a τ-Borel set.
First, observe that if we have two sequences of sets (E_n^1) and (E_n^2) such that every pair (E_n^1, E_m^2) is separated by a τ-Borel set $B_{n,m}$ then the pair $(\bigcup_{n=1}^\infty E_n^1, \bigcup_{n=1}^\infty E_n^2)$ is separated by the τ-Borel set $\bigcap_{m=1}^\infty \bigcup_{n=1}^\infty B_{n,m}$.
We proceed by reduction to the absurd. Given two disjoint d-closed sets E^1 and E^2, if they were not separated by a τ-Borel set, then there will exist d-closed sets E_1^1 and E_1^2 of diameter less than 1 and $E_1^i \subset E^i$ which are not separated by any τ-Borel set, because every E^i can be covered by countably many sets of that kind. By induction, we can construct decreasing sequences (E_n^i) for $i = 1, 2$ of nonempty closed sets such that the diameter of E_n^1 is less than $1/n$ and such that (E_n^1, E_n^2) are separated by no τ-Borel set. By completeness, $\bigcap_{n=1}^\infty E_n^i = \{e_i\}$ and $e_1 \neq e_2$. Since τ is Hausdorff, we can take disjoint neighbourhoods U_i of e_i, and being each U_i a d-open set, we have for n big enough that $E_n^i \subset U_i$, but this means that U_1 separes the pair (E_n^1, E_n^2). This contradiction shows that (E^1, E^2) are separated by a τ-Borel set.

We need the following definition for the second example.

Definition 1.6.2 *A completely regular topological space (X, τ) is said to be countably determined if for every σ-compact space $(Z, \widetilde{\tau})$ containing X as a subspace, there is a sequence (K_n) of compact subsets of Z such that for every $x \in X$ and every $y \in Z \setminus X$, there is $n \in \mathbb{N}$ such that $x \in K_n$ and $y \notin K_n$.*

It is not difficult to check that a completely regular space X is countably determined if there is sequence (A_n) of closed subsets of X such that for every open cover (U_i) of X and every $x \in X$ there is $n \in \mathbb{N}$ such that $x \in A_n$ and A_n is contained in a finite union of U_i's. It follows that X is countably determined if there is a σ-compact space $(Z, \widetilde{\tau})$ containing X as a subspace that verifies the property stated in the definition above.

The following result is part of an argument of Talagrand [63].

Theorem 1.6.3 *Let (X, τ_1) be a countably determined topological space and let τ_2 be a regular topology on X coarser than τ_1. Then X has $P(\tau_1, \tau_2)$ with the countable family*

$$\{X \cap K_{n_1} \cap \ldots \cap K_{n_j} : n_1, \ldots, n_j \in \mathbb{N}\}$$

where (K_n) is the sequence of Definition 1.6.2.

Proof. Fix $x \in X$ and let (n_j) be the set of indexes for which $x \in K_{n_j}$. We deduce that $K = \bigcap_{j=1}^{\infty} K_{n_j} \subset X$, so it is a τ_1-compact set of X. Take $V \in \tau_1$ a neighbourhood of x. Since τ_2 coincides with τ_1 on K and the regularity of τ_2, we can take $U \in \tau_2$ neighbourhood of x such that $\overline{U}^{\tau_2} \cap K \subset V$. Now K is contained in the τ_1-open set $V \cup (X \setminus \overline{U}^{\tau_2})$. By the compactness, we can take $j \in \mathbb{N}$ such that

$$X \cap K_{n_1} \cap \ldots \cap K_{n_j} \subset V \cup (X \setminus \overline{U}^{\tau_2})$$

so we have that

$$x \in X \cap K_{n_1} \cap \ldots \cap K_{n_j} \cap U \subset V$$

This proves that X has $P(\tau_1, \tau_2)$.

Problem 1.6.4 *We do not know if under the hypothesis of Theorem 1.6.3 it is possible to prove that X has $P(\tau_1, \tau_2)$ with τ_2-Borel sets. In particular, we do not know if τ_1 and τ_2 have the same Borel sets. For Banach spaces which are countably determined in its weak topology the answer is affirmative, see Theorem 2.3.5.*[4]

[4]Actually Theorem 2.3.5 contains one hypothesis that excludes topologies of pointwise convergence on total non-norming sets. Plichko (*Serdica Math. J.* 23 (1997), no. 3-4, 335–350.) proves under CH that if X is a WCG space which is not a direct sum of reflexive and separable subspaces then there is $F \subset X^*$ total such that

$$Borel(X, \sigma(X, F)) \neq Borel(X, w).$$

Chapter 2

Kadec norms

2.1 The Kadec property

In this chapter we shall study the relation between the property $P(\tau_1, \tau_2)$ and the existence of a real function with the "Kadec property". Recall that if X is a set, τ_1 and τ_2 topologies on X, we say that X has $P(\tau_1, \tau_2)$ if there is a sequence $A_n \subset X$ such that for every $x \in X$ and every $V \in \tau_1$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$. The first basic result in the context of general topological spaces is the following.

Proposition 2.1.1 *Let X be a set, τ_1 and τ_2 two topologies on X, and let (Y, δ) be a topological space having a countable basis. Suppose that there is a map $F : X \to Y$ such that for every net $(x_\omega) \subset X$ with τ_2-$\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$, then τ_1-$\lim_\omega x_\omega = x$. Then X has $P(\tau_1, \tau_2)$.*

Proof. Let (B_n) be a countable basis of Y. The condition means that for every $x \in X$ and every $V \in \tau_1$ with $x \in V$, there is $U \in \tau_2$ and $n \in \mathbb{N}$ such that if $y \in U$ and $F(y) \in B_n$, then $y \in V$. Clearly, this implies that X has $P(\tau_1, \tau_2)$ with the sequence $A_n = F^{-1}(B_n)$.

A norm on a Banach space is said to be Kadec when the norm topology and the weak topology coincide on the unit sphere. We shall need the following more general definition.

Definition 2.1.2 *Let X be a Banach space and τ a vector topology coarser than the norm topology. An equivalent norm $\|.\|$ is said τ-Kadec if the norm topology and τ coincide on the unit sphere of $\|.\|$. If τ is the weak topology of X we say that $\|.\|$ is Kadec.*

Next result appears in [1].

Proposition 2.1.3 *A τ-Kadec norm $\|.\|$ is τ-lower semicontinuous.*

Proof. Suppose that $\|.\|$ is not τ-lsc. Then there is a net (x_ω) on the unit sphere S_X and a point x out of the unit ball B_X such that τ-$\lim_\omega x_\omega = x$. Take numbers $t_\omega > 1$ such that $\|x + t_\omega(x_\omega - x)\| = \|x\|$. Let $y_\omega = x + t_\omega(x_\omega - x)$. Note that $\{t_\omega\}$ is bounded because $\inf_\omega \|x_\omega - x\| > 0$. We deduce that τ-$\lim_\omega y_\omega = x$. Since $\|y_\omega\| = \|x\|$ we should have that $\lim_\omega \|y_\omega - x\| = 0$, but this is impossible because $\|y_\omega - x\| \geq \|x_\omega - x\|$.

Proposition 2.1.4 *Let X be a vector space and τ_1, τ_2 vector topologies on X. Suppose that there exists a positive homogeneous function F on X such that τ_1 and τ_2 coincide on the set $S = \{x \in X : F(x) = 1\}$ and S is τ_1-bounded. Then X has $P(\tau_1, \tau_2)$.*

Proof. We shall check that F satisfies the hypothesis of Proposition 2.1.1. Let (x_ω) be a net with τ_2-$\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$. If $x = 0$ then τ_1-$\lim_\omega x_\omega = x$ by the τ_1-boundedness of S. Suppose $x \neq 0$. Since τ is a vector topology

$$\tau_2\text{-}\lim_\omega \frac{x_\omega}{F(x_\omega)} = \frac{x}{F(x)}$$

But by the hypothesis we have

$$\tau_1\text{-}\lim_\omega \frac{x_\omega}{F(x_\omega)} = \frac{x}{F(x)}$$

Now it is easy to see that τ_1-$\lim_\omega x_\omega = x$.

Combining the last proposition with some results from the first chapter, we deduce the following classic results of [12, 13].

Corollary 2.1.5 (Edgar & Schachermayer) *Let X be a Banach space. If X admits an equivalent τ-Kadec norm, then X has $P(\|.\|, \tau)$. In particular*

$$Borel(X, \|.\|) = Borel(X, \tau)$$

and if X admits a Kadec norm, then it is a Borel set in (X^{**}, w^*).

The following results show how property $P(\tau_1, \tau_2)$ can be characterized by means of Kadec functions (reciproque of Proposition 2.1.1).

Proposition 2.1.6 *Let X be a set, τ_1 and τ_2 two topologies on X. The following are equivalent:*

i) There is a τ_2-lower semicontinuous real function F on X such that for every net (x_ω) with $\tau_2\text{-}\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$, then $\tau_1\text{-}\lim_\omega x_\omega = x$.

ii) X has $P(\tau_1, \tau_2)$ with a sequence of τ_2-closed sets.

Proof. *i)* $\Rightarrow$ *ii)* Like in Proposition 2.1.1, for every $x \in X$ and every $V \in \tau_1$ with $x \in V$ there is $U \in \tau_2$ and $\varepsilon > 0$ such that if $y \in U$ and $|F(y) - F(x)| < \varepsilon$, then $y \in V$. Let (r_n) an enumeration of the rational numbers. Define

$$A_n = \{y \in X : F(y) \leq r_n\}$$

The sets A_n are τ_2-closed because F is τ_2-lsc. We claim that X has $P(\tau_1, \tau_2)$ with the sequence A_n. Indeed, take rationals $r_m < F(x) < r_n$ and $r_n - r_m < \varepsilon$. Consider the τ_2-open set $U' = U \setminus A_m$. Then we have that

$$x \in A_n \cap U' \subset V$$

which proves the claim.
ii) $\Rightarrow$ *i)* Let Ξ_A be the characteristic function of the set A. Consider the series

$$F(x) = \sum_{n=1}^{\infty} 4^{-n}\, \Xi_{X \setminus A_n}(x)$$

It is clear to see that F is τ_2-lsc. Let (x_ω) be a net with τ_2-$\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$. We claim that τ_1-$\lim_\omega x_\omega = x$. Indeed, a simple reasoning gives us that

$$\lim_\omega \Xi_n(x_\omega) = \Xi_n(x)$$

for every $n \in \mathbb{N}$. Now, for every τ_1-neighbourhood V of x there is n and $U \in \tau_2$ such that $x \in A_n \cap U \subset V$. Since $\Xi_{X \setminus A_n}(x_\omega)$ must be constant for ω big enough, we deduce that $x_\omega \in A_n$. Also, for ω big enough $x_\omega \in U$, thus $x_\omega \in V$ which shows the τ_1-convergence of (x_ω).

Remark 2.1.7 *Note that τ_2 is stronger than τ_1 on the sets*

$$\{x \in X : F(x) = r\} = (\bigcap_{x_0 \in A_n} A_n) \setminus (\bigcup_{x_0 \notin A_n} A_n)$$

for $r \in \mathbb{R}$ and $x_0 \in X$ such that $F(x_0) = r$.

Lancien has built functions in certain Banach spaces such that the norm topology and the weak topology coincide on the level sets (see [47]).

The proof of Proposition 2.1.6 also gives the following.

Proposition 2.1.8 *Let X be a set, τ_1 and τ_2 two topologies on X. The following are equivalent:*

i) There is a (τ_2-Borel measurable) real function F on X such that for every net (x_ω) with τ_2-$\lim_\omega x_\omega = x$ and $\lim_\omega F(x_\omega) = F(x)$ then τ_1-$\lim_\omega x_\omega = x$.

ii) X has $P(\tau_1, \tau_2)$ (with τ_2-Borel sets).

Corollary 2.1.9 *Let (X, τ) be a topological space and d a metric on X. If (X, τ) has d-SLD (equivalently, X has $P(d, \tau)$) there is a real function F on X such that every net (x_ω) in X is d-converging to some x if it is τ-converging to x and $\lim_\omega F(x_\omega) = F(x)$.*

Definition 2.1.10 *A Banach space X is said to have the JNR property if it has $P(\|.\|, w)$, or equivalently, (X, w) has $\|.\|$-SLD.*

The use of Kadec functions allows us to give a simple proof of the following result of Ribarska [62].

Corollary 2.1.11 (Ribarska) *Let X be a Banach space and $Y \subset X$ a subspace. Suppose that Y and X/Y have the JNR property. Then X also has the JNR property.*

Proof. Let f_1 and f_2 real functions defined on Y and X/Y respectively having the property stated in Corollary 2.1.9. Let $S : X/Y \to X$ be the Bartle-Graves selector of the quotien map, see [10, p. 299]. Consider the real funtions F_1 and F_2 defined on X by the formulae: $F_1(x) = f_1(x - S(x+Y))$, $F_2(x) = f_2(x+Y)$. We claim that for every net $(x_\omega) \subset X$ converging weakly to x and such that $\lim_\omega F_i(x_\omega) = F_i(x)$ for $i = 1, 2$, then (x_ω) converges to x in norm. Indeed, we have that $(x_\omega + Y)$ converges weakly to $x + Y$. Since $f_2(x_\omega + Y)$ converges to $f_2(x + Y)$, we have that $(x_\omega + Y)$ converges to $x + Y$ in norm. Then $S((x_\omega + Y))$ converges in norm to $S(x + Y)$, so $(x_\omega - S(x_\omega + Y))$ converges weakly to $x - S(x+Y)$. Since $f_1(x_\omega - S(x_\omega + Y))$ converges to $f_1(x - S(x+Y))$, we must have that $(x_\omega - S(x_\omega + Y))$ converges in the norm to $x - S(x + Y)$. We deduce that (x_ω) converges in norm to x. By Proposition 2.1.1, X has the JNR property.

2.2 The vectorial case

Given two topologies τ_1 and τ_2 on X and a family Σ of subsets of X, we shall say that Σ is good at $x \in X$ if for every $V \in \tau_1$ with $x \in \tau_1$, there exists $S \in \Sigma$ and $U \in \tau_2$ such that $x \in S \cap U \subset V$. Good family means a family good at every point of X. It is easy to see that a family Σ covering X such that on every $S \in \Sigma$ the topologies τ_1 and τ_2 coincide is good. Property $P(\tau_1, \tau_2)$ is equivalent to the existence of a countable good family.
The following lemma shows how to make a good family of "thick sets" from a good one made up of "thin sets".

Lemma 2.2.1 *Let X be a vector space, $\tau_2 \subset \tau_1$ vector topologies on X and Σ a family good at some $x \in X$. Then the family*

$$\{S + W : S \in \Sigma, 0 \in W \in \tau_1\}$$

is good at x. Thus, if Σ and Π are families of subsets of X such that for every $S \in \Sigma$ and every $W \in \tau_1$ with $0 \in W$ there exists $P \in \Pi$ such that $S \subset P \subset S + W$ then Π is good if and only if Σ is good.

Proof. Given $V \in \tau_1$ with $x \in V$, we shall find $S \in \Sigma$, $0 \in W \in \tau_1$ and $U \in \tau_2$ such that

$$x \in (S + W) \cap U \subset V$$

As $0+x \in V$, we can take $W_1, V' \in \tau_1$ with $0 \in W_1$, $x \in V'$ and $W_1+V' \subset V$. Since Σ is good at x, we can take $S \in \Sigma$ and $U' \in \tau_2$ such that $x \in S \cap U' \subset V'$. As $0+x \in U'$, we can take $W_2, U \in \tau_2$ with $0 \in W_2$, $x \in U$ and $W_2+U \subset U'$. Now take $W = W_1 \cap (-W_2) \in \tau_1$. We shall show that U and W verify the above set inclusion. If $y \in (S + W) \cap U$, then there is $z \in S$ such that $y - z \in W \subset -W_2$, so $z = (z - y) + y \in U'$, and thus $z \in S \cap U' \subset V'$. Now as $y - z \in W \subset W_1$ we have that $y = (y - z) + z \in V$.

The following is the main result of the chapter. It can be regarded like a converse of Proposition 2.1.4. We show that the Kadec function given by Proposition 2.1.6 can be made homogeneous in a normed vector space.

Theorem 2.2.2 *Let X be a Banach space and τ a vector topology coarser than the norm topology such that $\overline{B_X}^{\tau}$ is bounded. Then the following are equivalent:*

i) X has $P(\|.\|, \tau)$ (equivalently, (X, τ) has $\|.\|$-SLD).

ii) There exists a symmetric homogeneous τ-lower semicontinuous function F on X with $\|.\| \leq F(.) \leq 3\|.\|$ such that the norm topology and τ coincide on the set

$$S = \{x \in X : \; F(x) = 1\}$$

Proof. $ii) \Rightarrow i)$ It follows from Proposition 2.1.4.
$i) \Rightarrow ii)$ We shall assume that X is endowed with a τ-lower semicontinuous equivalent norm $\|.\|$. $B(0,a)$ and $B[0,a]$ will be the open and the closed balls of center 0 and radius a. As usual $B_X = B[0,1]$.
Suppose that X has $P(\|.\|,\tau)$ with the sequence (A_n). We can suppose every set A_n star shaped with respect to 0 and norm open. To see that we are going to modify the sequence in several steps.
First step. Take $A'_n = A_n \cap B_X$.
Second step. Take

$$A''_n = \{tx : \ 0 \leq t \leq 1, \ x \in A'_n\}$$

We are going to check that (A''_n) is good for the points of the unit sphere S_X. Let $x \in S_X$ and $\varepsilon > 0$. Appling Lemma 2.2.1 we can find $U \in \tau$, $n \in \mathbb{N}$ and $\delta > 0$ such that $x \in A'_n \cap U$ and $diam((A'_n + B(0,\delta)) \cap U) < \varepsilon$. Now it is clear that

$$A''_n \cap (U \setminus B[0, 1-\delta]) \subset (A'_n + B(0,\delta)) \cap U$$

Thus $U' = U \setminus B[0,1-\delta] \in \tau$ verify $x \in A''_n \cap U'$ and $diam(A''_n \cap U') < \varepsilon$.
Third step. The family

$$\{rA''_n + B(0,\delta) : \ n \in \mathbb{N}, \ r \geq 0, \ \delta > 0, \ r,\delta \in \mathbb{Q}\}$$

is good for X after Lemma 2.2.1. Renumbering the indices, this family will be the desired sequence (A_n).
Clearly the sets $\overline{A_n}^{\tau}$ are star shaped with respect to 0. Let f_n be the Minkowski's functional of $\overline{A_n}^{\tau}$. Since $\overline{A_n}^{\tau} = \{f_n \leq 1\}$ the function f_n is τ-lower semicontinuous. Let $\|f_n\|$ be the supremun of $\{|f_n(x)| : x \in B_X\}$. The function F given by the formula

$$F(x) = \|x\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n(x)}{\|f_n\|} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{f_n(-x)}{\|f_n\|}$$

is τ-lower semicontinuous and symmetric.
Let $(x_\omega) \subset S$ a net τ-converging to some $x \in S$. From the τ-lower semicontinuity of $\|.\|$ and f_n we have that

$$\|x\| \leq \liminf_{\omega} \|x_\omega\|$$

$$f_n(x) \leq \liminf_{\omega} f_n(x_\omega)$$
$$f_n(-x) \leq \liminf_{\omega} f_n(-x_\omega)$$

On the other hand, it is not difficult to see that

$$1 \geq \liminf_{\omega} \|x_\omega\| + \sum_{n=1}^{\infty} \frac{1}{2^n \|f_n\|} \liminf_{\omega} f_n(x_\omega) + \sum_{n=1}^{\infty} \frac{1}{2^n \|f_n\|} \liminf_{\omega} f_n(-x_\omega)$$

Since $F(x) = 1$, a simple reasoning with $\limsup$ gives the existence of

$$\lim_{\omega} \|x_\omega\| = \|x\|$$

$$\lim_{\omega} f_n(x_\omega) = f_n(x)$$

$$\lim_{\omega} f_n(-x_\omega) = f_n(-x)$$

for every $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. By the lower semicontinuity of $\|.\|$, there exists $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $diam(\overline{A_n}^\tau \cap U) \leq \varepsilon$. In particular, as A_n is norm open then $f_n(x) < 1$, so for ω big enough $f_n(x_\omega) < 1$, and thus $x_\omega \in \overline{A_n}^\tau$. Since for ω big enough we have that $x_\omega \in U$, we obtain that $\|x_\omega - x\| \leq \varepsilon$. This proves that (x_ω) converges to x in norm and thus the norm topology and τ coincide on S.

Remark 2.2.3 *The proof of Theorem 2.2.2 shows that if S_X has $P(\|.\|, \tau)$, then X has $P(\|.\|, \tau)$ as well.*

Remark 2.2.4 *The constant "3" in the statement ii) of Theorem 2.2.2 can be changed by any constant greater than 1. In fact every function of the form $\|.\| + aF$ with $a > 0$ has the same property. This also shows that the norm can be aproximated uniformly by homogeneous functions with the Kadec property provided the existence of one at least.*

Remark 2.2.5 *Note that S is a norm $\mathcal{G}_\delta$-set in the norm closed set*

$$B = \{x \in X : \ F(x) \leq 1\}$$

thus (S, τ) is completely metrizable, and thus a Baire space.

Problem 2.2.6 *We do not know if the function F in statement ii) of Theorem 2.2.2 can be taken to be norm continuous (even when τ is the weak topology of X).*[1]

Definition 2.2.7 *A Banach space X is said to be descriptive if (X, w) is a descriptive space. A dual Banach space X^* is said to be dual-descriptive if it has the Radon-Nikodym property and (X^*, w^*) is a descriptive space.*

The following theorem puts together some characterizations of the Banach spaces with the property JNR.

Theorem 2.2.8 *Let X be a Banach space. The following are equivalent:*

i) $(X, \|.\|)$ *has* $\|.\|$*-SLD (equivalently,* X *has* $P(\|.\|, w)$*).*

ii) X *is descriptive.*

iii) (X, w) *has a* σ*-isolated network.*

iv) *There is a metric d on X finer than the weak topology such that (X, w) has d-SLD (equivalently, X has $P(d, w)$).*

v) *There exists a symetric homogeneous w-lower semicontinuous function F on X with $\|.\| \leq F(.) \leq 3\|.\|$ such that the norm and the weak topologies coincide on the set $S = \{x \in X : \; F(x) = 1\}$.*

Proof. We already know that $v) \Leftrightarrow i) \Rightarrow ii) \Rightarrow iii) \Leftrightarrow iv)$, so we have just to prove that $iii) \Rightarrow i)$. Let $\mathfrak{N} = \bigcup_{n=1}^{\infty} \mathfrak{N}_n$ be a network of w with each $\mathfrak{N}_n$ isolated. For every $N \in \mathfrak{N}_n$ take a point $x_N \in N$. We define a function f_n on $\bigcup \mathfrak{N}_n$ as follows: $f(x) = x_N$ if $x \in N$. Clearly, we have that the functions (f_n) are weak to norm continuous because they are locally constant. We also have that

$$x \in \overline{\{f_n(x) : \; x \in dom(f_n)\}}^{w}$$

[1] The Kadec function F can be made norm continuous, see Lemma 1.3 in S. Ferrari, L. Oncina, J. Orihuela, M. Raja, Metrization theory and the Kadec property, *Banach J. Math. Anal. 10 (2016), 281–306.*

Consider the set of all the finite rational combinations of f_n's and arrange them into a sequence (g_n). As a weak cluster point of a sequence is a norm cluster point of its linear span, we have that

$$x \in \overline{\{g_n(x) : \ x \in dom(g_n)\}}^{\|.\|}$$

and this entails that X has $P(\|.\|, w)$ by Proposition 1.1.7.

Statement *i*) describes the relation between the weak topology and the norm topology. Equivalence *i*) ⇔ *iv*) is comparable to a result of Kenderov and Moors [44] which shows that a Banach space is σ-fragmentable by the norm if it is σ-fragmentable by some metric finer than the weak topology. Oncina [54] proved that the property JNR is a topological invariant of the weak topology. This result follows easily from the equivalence *i*) ⇔ *iii*). Statement *iv*) is almost the Kadec renormability of the space. The equivalence between *i*), *ii*), *iii*) and *iv*) is given to Hansell [27], but using network language in *i*) and *iv*), see Proposition 1.5.6. The equivalence *i*) ⇔ *iii*) appears in the above terminology in [51] among some others equivalent topological properties.

Remark 2.2.9 *It is not difficult to see that X can be changed by S_X in statements i), iii) and iv).*

Problem 2.2.10 *We do not know if the statements of Theorem 2.2.8 are equivalent to the existence of an equivalent Kadec norm.*

2.3 Kadec renorming

In order to apply the technique of the preceding section to characterize the existence of equivalent Kadec norms on a Banach space X, it seems to be necessary to introduce a property stronger than $P(\|.\|, \tau)$. Suppose that τ_1 and τ_2 are topologies on X. We say that X has convex-$P(\tau_1, \tau_2)$ if X has $P(\tau_1, \tau_2)$ with a sequence (A_n) of convex subsets of X.

Theorem 2.3.1 *Let X be a Banach space and let τ be a vector topology coarser than the norm topology such that $\overline{B_X}^{\tau}$ is bounded. Then X has a equivalent τ-Kadec norm if and only if X has convex-$P(\|.\|,\tau)$, that is, there exists convex sets $A_n \subset X$ such that for every $x \in X$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ and $U \in \tau$ such that $x \in A_n \cap U$ and $diam(A_n \cap U) < \varepsilon$.*

Proof. If we begin with (A_n) convex in the proof of Theorem 2.2.2 it is easily checked that all the families of sets built there are still convex. Thus F is subadditive and it is so an equivalent τ-Kadec norm.
For the converse assume that the norm of X is τ-Kadec. The proof of $i) \Rightarrow ii)$ in Proposition 2.1.6 shows that X has $P(\|.\|,\tau)$ with the sequence of closed balls of rational radius centered at 0.

Remark 2.3.2 *It follows easily that a separable Banach space has a τ-Kadec norm for every topology τ such that $\overline{B_X}^{\tau}$ is bounded, extending a remarkable theorem of Kadec stated only for topologies of convergence on quasi-norming subsets of the dual space, see [3, p. 176]. Although, under the same hypothesis, there is a τ-lower semicontinuous LUR norm, see Definition 3.1.2 and Proposition 3.1.4.*

Proposition 2.3.3 *Let X be a Banach space. The following are equivalent:*

i) X admits an equivalent Kadec norm.

ii) For every norm open $V \subset X$ there is a sequence of convex sets (C_n) and a sequence of weak open sets (U_n) such that $V = \bigcup_{n=1}^{\infty}(C_n \cap U_n)$.

Proof. $i) \Rightarrow ii)$ If X admits an equivalent Kadec norm then it has $P(\|.\|,w)$ with the countable family (C_n) of the closed balls centered at 0 and rational radius, see the proof of Theorem 2.3.1. Then use the proof of Theorem 1.2.5 to find the weak open sets (U_n).
$ii) \Rightarrow i)$ Let $\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$ be a basis for the norm topology such that every $\mathfrak{B}_n$ is discrete. Take $V_n = \bigcup \mathfrak{B}_n$. By the hypothesis we can find convex sets $(C_{m,n})$ and weak open sets $(U_{m,n})$ such that $V_n = \bigcup_{m=1}^{\infty} C_{m,n} \cap U_{m,n}$. We

claim that the countable family $(C_{m,n})$ satisfies the hypothesis of Theorem 2.3.1. Fix $x \in X$ and $\varepsilon > 0$. Then, for some $n \in \mathbb{N}$, there is $W \in \mathfrak{B}_n$ such that $x \in W \subset B(x, \varepsilon)$. Now, for some m we have that $x \in C_{m,n} \cap U_{m,n}$. Let U_0 be convex weak neighbourhood of x with $U_0 \subset U_{m,n}$. We have that $C_{m,n} \cap U_0$ is convex and it is contained in V_n. By the discreteness of $\mathfrak{B}_n$ we must have that $C_{m,n} \cap U_0 \subset W \subset B(x, \varepsilon)$.

Problem 2.3.4 *We know no example of Banach space without the JNR property such that every norm open is an $(\mathcal{F} \cap \mathcal{G})_\sigma$ for the weak topology.*

A Banach space is said to be weakly countably determined (WCD) if it is countably determined in its weak topology: a Banach space is WCD if there exists a sequence (K_n) of w^*-compact sets of X^{**} such that for every $x \in X$ and every $y \in X^{**} \setminus X$ there is $n \in \mathbb{N}$ with $x \in K_n$ and $y \notin K_n$. We shall use the well known result of Vašak [67]: *a WCD Banach space admits a LUR norm*, see Definition 3.1.2.

Theorem 2.3.5 *Let X be a WCD Banach space and let τ be a Hausdorff vector topology coarser than the weak topology of X. Then X has convex-$P(\|.\|, \tau)$. Moreover, if $\overline{B_X}^\tau$ is bounded, then X also admits an equivalent τ-Kadec norm and*

$$Borel(X, \|.\|) = Borel(X, \tau)$$

Proof. Since a WCD Banach space X admits an equivalent LUR norm, it has convex-$P(\|.\|, w)$, see Chapter 3 for the definition of LUR norm and the fact that it is Kadec. On the other hand, we have that X has $P(w, \tau)$ by Theorem 1.6.3. If we show that X has convex-$P(w, \tau)$ then X will have convex-$P(\|.\|, \tau)$ and it will be τ-Kadec renormable by Theorem 2.3.1.
Let $K_n \subset X^{**}$ be a sequence of w^*-compacts K_n satisfying Definition 1.6.2. We can assume without loss of generality that the sequence (K_n) is closed by finite intersections. We claim that the sequence of w^*-closed convex hulls $\{\overline{co}^{w^*}(K_n)\}$ also satisfies Definition 1.6.2. Indeed, fix $x \in X$ and $y \in X^{**} \setminus X$. The set $K = \bigcap_{x \in K_n} K_n$ is a weakly compact set of X containing x. Since

$\overline{co}^{w}(K)$ is a weak*-compact convex set not containing y, there is a weak*-open half space H such that $x \in H$ and $y \notin \overline{H}^{w^*}$. By compactness, there is $n \in \mathbb{N}$ such that $x \in K_n \subset H$. As $\overline{co}^{w^*}(K_n) \subset \overline{H}^{w^*}$ we have that $x \in \overline{co}^{w^*}(K_n)$ and $y \notin \overline{co}^{w^*}(K_n)$. This ends the proof of the claim. Now, Theorem 1.6.3 shows that X has convex-$P(w, \tau)$.

Note that a WCG Banach space X is a $\mathcal{F}_{\sigma\delta}$-set in every linear embedding into a normed vector space, see [12].

Corollary 2.3.6 *Let X be a WCD Banach space and let τ be a Hausdorff vector topology coarser than the weak topology of X such that $\overline{B_X}^{\tau}$ is bounded. Then (X, τ) is an absolute $(\mathcal{F} \cap \mathcal{G})_{\sigma\delta}$-set and (A, τ) is absolute Borel for every $A \in Borel(X, \|.\|)$.*

Proof. It follows from Corollary 1.4.16.

Remark 2.3.7 *Under the hypothesis of Corollary 2.3.6, we have that (X, τ) is σ-fragmentable and the τ-compact sets of X are fragmentable. See [6] for some consequences when τ is the topology of pointwise convergence on a norming subset of the dual of X.*

We shall use the following definition of [52].

Definition 2.3.8 *A Hausdorff compact space K is said a Radon-Nikodym compact if it is fragmented by τ-lower semicontinuous metric.*

If K is a Radon-Nikodym compact and d is a lower semicontinuous metric fragmenting K, then d is a complete metric stronger than the topology of K and every Radon measure on K is the restriction of a Radon measure on (K, d), see [52] and [34].

Theorem 2.3.9 *Let K be a Radon-Nikodym compact space. Then the $C(K)$ has an equivalent pointwise lower semicontinuous norm such that on its unit sphere the weak and the pointwise topologies coincide, $C(K)$ has $P(w, t_p(K))$ and*

$$Borel(C(K), w) = Borel(C(K), t_p(K))$$

Proof. A continuous function on K is d-uniformly continuous. Indeed, suppose not. Then we can take sequences (x_n) and (y_n) in K such that $\lim_n d(x_n, y_n) = 0$ while $|f(x_n) - f(y_n)| \geq \delta$ for some $\delta > 0$. If $\mathfrak{U}$ is a free ultrafilter, the sequences converge along $\mathfrak{U}$ to limits x and y respectively. But by the lower semicontinuity of d, we have $d(x, y) = 0$, so $x = y$ and we get a contradiction with the continuity of f.
Fix a d-dense set $(x_\alpha)_{\alpha\in\Gamma}$. Now we define the seminorms O_n as follows

$$O_n(f) = \sup_\alpha \sup \{|f(x) - f(x_\alpha)| : \ d(x, x_\alpha) \leq 1/n\}$$

Clearly O_n is pointwise lower semicontinuous and since every $f \in C(K)$ is d-uniformly continuous for every $\delta > 0$ there exists $n \in \mathbb{N}$ such that $O_n(f) < \delta$.
Define a new norm by the formula

$$|||f||| = \|f\| + \sum_{n=1}^{\infty} 2^{-n}\, O_n(f)$$

Evidently $\|.\| \leq |||.||| \leq 3\|.\|$. Thus $|||.|||$ is an equivalent norm in $C(K)$. It is not hard to check that the unit ball of $|||.|||$ is pointwise closed.
We are going to check that the weak and the pointwise topologies coincide on the set $S = \{f \in C(K) : |||f||| = 1\}$. Let (f_ω) be a net in S pointwise converging to $f \in S$. Take a Radon measure μ with $\|\mu\| \leq 1$ that we suppose already defined on $Borel(K, d)$ and take $\varepsilon > 0$.
From the pointwise lower semicontinuity of $\|.\|$ and O_n reasoning like in Theorem 2.1.6 we obtain that $\lim_\omega O_n(f_\omega) = O_n(f)$ for every $n \in \mathbb{N}$.
Now fix $n \in \mathbb{N}$ such that $O_n(f) \leq \varepsilon/8$. Then for ω big enough $O_n(f_\omega) \leq \varepsilon/6$.
Since μ has a d-separable d-support we can fix $F \subset \Gamma$ finite such that

$$|\mu|(\bigcup_{\alpha\in F} B[x_\alpha, 1/n]) \ > \ |\mu|(K) - \varepsilon/4$$

If ω is big enough then $|f_\omega(x_\alpha) - f(x_\alpha)| \leq \varepsilon/6$ for $\alpha \in F$. So $|f_\omega(x) - f(x)| \leq \varepsilon/2$ for every $x \in \cup_{\alpha\in F} B(x_\alpha, 1/n)$.
Having in mind that $\|f\|$ and $\|f_\omega\|$ are bounded by 1, an easy calculation gives that

$$|\mu(f_\omega - f)| \leq \int |f_\omega - f|\, d|\mu| \leq \varepsilon$$

which implies that (f_ω) converges weakly to f.
Now apply Proposition 2.1.4 to deduce that $C(K)$ has $P(w, t_p(K))$. Since the unit ball is pointwise closed, the weak and the pointwise topologies have the same Borel sets by Lemma 1.4.2. Moreover, every weak open is a countable union of differences of pointwise closed sets.

Remark 2.3.10 *Theorem 2.3.9 is still true for a continuous image of a Radon-Nikodym compact space.*

Problem 2.3.11 *We know no example of compact space K with different Borel sets for the weak and the pointwise topology.*

2.4 Descriptive compact spaces

Some results from section 1.5 about descriptive topological spaces can be sumarized in the following theorem when applied to compact spaces.

Theorem 2.4.1 *Let K be a compact Hausdorff space. Then the following statements are equivalent:*

i) K is descriptive.

ii) K has a σ-isolated network.

iii) K is fragmentable and hereditarily weakly θ-refinable.

If (K, τ) is a descriptive compact space, then there is a complete metric d such that K has $P(d, \tau)$ with τ-closed sets. If ρ is any metric which σ-fragments K, then K has $P(\rho, \tau)$.

We give some examples. Recall that a compact space K is said to be Gul'ko if the Banach space $C(K)$ is weakly countably determined, and K is said to be scattered if every nonempty closed subset has an isolated point.

Example 2.4.2 *Let X be a Banach space and $K \subset X$ a weakly compact subset. Then X has $P(\|.\|, w)$. In particular K is descriptive.*

Proof. The subspace $X_0 = \overline{span}^{\|\cdot\|}(K)$ admits an equivalent LUR norm, see [10] and Definition 3.1.2. In particular, this norm is Kadec, thus X_0 has property $P(\|.\|, w)$.

Example 2.4.3 *A Gul'ko compact space is descriptive.*

Proof. By a result of Ribarska [61] a Gul'ko compact space K is fragmentable and by a result of Gruenhage [24] it is hereditarily weakly θ-refinable, so K is descriptive. This result can also be deduced from the W*LUR renorming of the dual of a WCD Banach space, see Theorem 4.3.11.

Example 2.4.4 *If K is a scattered compact space such that $K^{(\omega_1)} = \emptyset$, then K is descriptive.*

Proof. There is a countable ordinal γ such that $K^{(\gamma)} = \emptyset$. If τ is the topology of K and d is the discrete metric on K, then K has $P(d, \tau)$ with the countable family of sets $\{K^{(\alpha)} : \alpha < \gamma\}$.

We shall study the stability of property P under continuous maps. The following result will be enough for a wide class of maps.

Theorem 2.4.5 *Let (K_i, τ_i) be compact spaces for $i = 1, 2$ and let d_i be metrics on K_i. Suppose that there is a surjection $T : K_1 \to K_2$ such that T is τ_1-τ_2 continuous and d_1-d_2 continuous. If K_1 has $P(d_1, \tau_1)$ with τ_1-closed sets, then K_2 has $P(d_2, \tau_2)$ with τ_2-closed sets.*

Proof. If K_1 has $P(d_1, \tau_1)$ with τ_1-closed sets, there is a τ_1-lsc function $F_1 : K_1 \to [0, 1]$ with the Kadec property by Proposition 2.1.6. Define a function $F_2 : K_2 \to [0, 1]$ as follows:

$$F_2(x) = \inf\{F_1(x') : T(x') = x\}$$

Since F_1 is τ_1-lsc this infimum is attained. We claim that F_2 is τ_2-lsc. Indeed, suppose that $\lim_\omega x_\omega = x$ in (K_2, τ_2) and $F_2(x_\omega) \leq r$ for every ω. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and $F_1(x'_\omega) = F_2(x_\omega)$. Let $x' \in K_1$ be a cluster point of (x'_ω). Since F_1 is τ_1-lsc we have that $F_1(x') \leq r$. On the other hand, by continuity $T(x') = x$, so $F_2(x) \leq F_1(x')$. This shows that $F_2(x) \leq r$ and the claim is proved. We claim now that F_2 has the Kadec property and then the result will follow from Proposition 2.1.6. Suppose not, that is, there is a net (x_ω) in K_2 with τ_2-limit a point x such that $\lim_\omega F_2(x_\omega) = F_2(x)$, and there is $\varepsilon > 0$ such that $d_2(x_\omega, x) > \varepsilon$. Take points $x'_\omega \in K_1$ such that $T(x'_\omega) = x_\omega$ and $F_1(x'_\omega) = F_2(x_\omega)$. Let x' be a cluster point of (x'_ω). Without loss of generality we can assume that (x'_ω) is τ_1-converging to x'. Clearly, we have that $T(x') = x$ and the following inequalities

$$\lim_\omega F_2(x_\omega) = F_2(x) \leq F_1(x') \leq \lim_\omega F_1(x'_\omega) = \lim_\omega F_2(x_\omega)$$

We deduce that $\lim_\omega F_1(x'_\omega) = F_1(x')$. By the Kadec property of F_1 we have that $\lim_\omega d_1(x'_\omega, x') = 0$, and from the d_1-d_2 continuity of T we deduce that $\lim_\omega d_2(x_\omega, x) = 0$, which is a contradiction with our supposition.

The following two lemmas will be used to prove Theorem 2.4.9. We say that a partition (D_α) of X indexed by ordinals is Montgomery if every D_α is an $\mathcal{F}_\sigma$-set and there are open sets (U_α) such that $D_\alpha = U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. A family $\mathcal{E}$ of subsets of X is said to separating if for every two different points $x, y \in X$, there is $E \in \mathcal{E}$ such that $\{x, y\} \cap E$ is a singleton.

Lemma 2.4.6 *If (X, τ) is fragmentable there exists partitions of Montgomery (D^n_α) for every $n \in \mathbb{N}$ such that (D^{n+1}_α) is finer than (D^n_α) and $\{D^n_\alpha : \alpha, n\}$ is separating.*

Proof. Since (X, τ) is completely regular every point has a basis made up of $\mathcal{F}_\sigma$-open sets. By the fragmentability it is possible to find $\mathcal{F}_\sigma$-open sets (U^n_α) such that the differences $D^n_\alpha = U^n_\alpha \setminus \bigcup_{\beta<\alpha} U^n_\beta$ have diameter less than $1/n$ and cover X. To get that (D^{n+1}_α) is finer than (D^n_α) it is enough to take intersections and to order the sets lexicographically.

Lemma 2.4.7 *If (X, τ) is fragmentable and (D^n_α) for every $n \in \mathbb{N}$ are partitions of Montgomery then there exists a metric d on X with the following properties:*

i) (X,τ) is fragmented by d.

ii) Every D^n_α is d-open.

iii) Every point of X has a d-basis made up of τ-closed sets.

Proof. After the previous lemma we can change the families (D^n_α) by finer ones which satisfies that (D^{n+1}_α) is finer than (D^n_α) and $\{D^n_\alpha : \alpha, n\}$ is separating. For every n and α take a continuous function $g^n_\alpha : X \to [0,1]$ such that g^n_α is zero out U^n_α and g^n_α is strictely positive on D^n_α (this is posible because it is an $\mathcal{F}_\sigma$-set. Define a function $f^n_\alpha(x) = g^n_\alpha(x)$ if $x \in D^n_\alpha$ and $f^n_\alpha(x) = 0$ if else. Now, we define seudometrics d_n on X by the formula:

$$d_n(x,y) = \sum_\alpha |f^n_\alpha(x) - f^n_\alpha(y)|$$

Since the family is separating, $d(x,y) = \sum_{n=1}^\infty 2^{-n} d_n(x,y)$ is a metric on X. We claim that this metric has the required properties. Indeed, to see that (X,τ) is fragmented by d it is enough to show that (X,τ) is fragmented by every d_n. Let C a nonempty subset of X. Let α be the first ordinal such that $D^n_\alpha \cap C \neq \emptyset$ and set $x \in D^n_\alpha \cap C$. Take a neighbourhood U of x such that $|g^n_\alpha(y) - g^n_\alpha(x)| < \varepsilon$ if $y \in U$. It is easy to see that $C \cap U \cap U^n_\alpha$ has d_n-diameter less than ε. If $x \in D^n_\alpha$, it is easy to check that the set $\{y \in X : |f^n_\alpha(y) - f^n_\alpha(x)| \leq \varepsilon\}$ is τ-closed and contained in D^n_α if $0 < \varepsilon < f^n_\alpha(x)$. This shows that statements ii) and iii) also holds.

Remark 2.4.8 *The metric given in Lemma 2.4.7 is not τ-lower semicontinuous in general (e.g. if X is a Gul'ko compact which is not Eberlein).*

Theorem 2.4.9 *A continuous image of a descriptive compact space is a descriptive compact space as well.*

Proof Let (K_1,τ_1) and (K_2,τ_2) compact Hausdorff spaces such that there is a continuous surjection of K_1 onto K_2. If (K_1,τ_1) has a σ-isolated network, then it is fragmentable. By [61, 15], a continuous image of a fragmentable compact space is also fragmentable, thus K_2 has a sequence of Montgomery partitions (D^n_α) by Lemma 2.4.6. If d_2 is the metric of Ribarska [61] on K_2 asociated to this partition, then $\{D^n_\alpha : \alpha, n\}$ is a basis for the d_2-topology. Let f be the continuous surjection of K_1 onto K_2. Then $(f^{-1}(D^n_\alpha))$ is a sequence of Montgomery's partitions of K_1. Let d_1 the metric given by Lemma 2.4.7. Then the sets $f^{-1}(D^n_\alpha)$ are d_1-open, and so f is d_1-d_2 continuous. Now K_1 has $P(d_1, \tau_1)$ with τ_1-closed sets since d_1 fragments K_1 and every point has a d_1-basis of τ_1-closed sets. It follows from Theorem 2.4.5 that K_2 has $P(d_2, \tau_2)$ and so (K_2, τ_2) is descriptive.

The following result can be regarded like a topological version of the transfer technique of Godefroy-Troyanski-Whitfield-Zizler [22, 10], see Theorem 4.4.5.

Theorem 2.4.10 *Let (X, τ) be a topological space and let d be a τ-lower semicontinuous metric on X. Suppose that there exists τ-compacts sets $K_n \subset X$ having $P(d, \tau)$ such that $\overline{\bigcup_{n=1}^\infty K_n}^d = X$. Then X has $P(d, \tau)$.*

Proof. We can suppose the sequence (K_n) increasing and the metric d bounded. By Proposition 2.1.6, for every $n \in \mathbb{N}$ there is a τ-lsc Kadec function $F_n : K_n \to [0, 1]$. We define the functions f_n on X as follows

$$f_n(x) = \inf\{d(x, y) + F_n(y) : y \in K_n\}$$

Remark that the infimum is attained. We claim that f_n is τ-lsc. Indeed, suppose that $\tau\text{-}\lim_\omega x_\omega = x$ and $f_n(x_\omega) \leq r$ for every ω. Take points $y_\omega \in K_n$ such that $f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega)$. Let $y \in K_n$ a cluster point of (y_ω). Then we have that

$$f_n(x) \leq d(x, y) + F_n(y) \leq r$$

because of the lower semicontinuity of d and F_n. Now we define a function F on X by the formula

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x)$$

We claim that F has the Kadec property. Indeed, suppose not. We can take a net (x_ω) in X with τ-limit a point x such that $\lim_\omega F(x_\omega) = F(x)$, and there is $\varepsilon > 0$ such that $d(x_\omega, x) > \varepsilon$. A standard argument of lower semicontinuity gives that $\lim_\omega f_n(x_\omega) = f_n(x)$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ such that $1/n < \varepsilon/3$ and $d(x, K_n) < \varepsilon/3$. We can take points $y_\omega \in K_n$ such that

$$f_n(x_\omega) = d(x_\omega, y_\omega) + F_n(y_\omega)$$

Let $y \in K_n$ a cluster point of (y_ω). Without loss of generality we can assume that τ-$\lim_\omega y_\omega = y$. Since

$$d(x, y) + F_n(y) \leq \lim_\omega f_n(x_\omega) = f_n(x) \leq d(x, y) + F_n(y)$$

we have that

$$\lim_\omega [d(x_\omega, y_\omega) + F_n(y_\omega)] = d(x, y) + F_n(y)$$

By lower semicontinuity again, we deduce that $\lim_\omega d(x_\omega, y_\omega) = d(x, y) < \varepsilon/3$ and $\lim_\omega F_n(y_\omega) = F_n(y)$. The last equality gives that $\lim_\omega d(y_\omega, y) = 0$, thus for ω big enough we have that $d(x_\omega, y_\omega) < \varepsilon/3$ and $d(y_\omega, y) < \varepsilon/3$. Since $d(x, y) < \varepsilon/3$ we have that $d(x_\omega, x) < \varepsilon$, which is a contradiction.

We introduce a particular subclass of the descriptive compact spaces.

Definition 2.4.11 *A Hausdorff compact space (K, τ) is said a Namioka-Phelps compact if there exists a lower semicontinuous metric d such that K has $P(d, \tau)$.*

We have that a compact space is Namioka-Phelps if and only if it is Radon-Nikodym and hereditarily weakly θ-refinable.

Example 2.4.12 *Let X^* be a dual Banach space with a w^*-Kadec norm (property $(**)$ in the terminology of [53]). Then (B_{X^*}, w^*) is a Namioka-Phelps compact.*

We prove in Corollary 4.4.9 that this is essentially the unique example of Namioka-Phelps compact space.

Lemma 2.4.13 *Let X be a Banach space, let $Z \subset X^*$ be a quasi-norming subset and let $K \subset X^*$ be a bounded $\sigma(X,Z)$-compact subset which has $P(\|.\|, \sigma(X,Z))$. Then $\overline{span}^{\|\cdot\|}(K)$ and $\overline{aco}^{\sigma(X,Z)}(K)$ have $P(\|.\|, \sigma(X,Z))$.*

Proof. Let $I(n,m) = [-m,m] \times \ldots \times [-m,m]$ (n times) with the usual topology of $\mathbb{R}^n$. Let $K_{n,m} = I(m,n) \times K \times \ldots \times K$ (n times). If τ is the product topology when K is endowed with $\sigma(X,Z)$, then $K_{n,m}$ is τ-compact. If K is endowed with the norm topology, then the product topology is metrized by a metric that we call d. We know that $K_{n,m}$ has $P(d,\tau)$. The map $T_{n,m}$ from $K_{n,m}$ to X defined by $T_{n,m}(\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_n) = \alpha_1 x_1 + \ldots + \alpha_n x_n$ is clearly τ-$\sigma(X,Z)$ continuous and d-$\|.\|$ continuous, thus every $\sigma(X,Z)$-compact set $T_{n,m}(K_{n,m})$ has $P(\|.\|, \sigma(X,Z))$ by Theorem 2.4.5. Since $span(K) = \bigcup_{n,m} T_{n,m}(K_{n,m})$, we have that $\overline{span}^{\|\cdot\|}(K)$ has $P(\|.\|, \sigma(X,Z))$ by Theorem 2.4.10. The result for the $\sigma(X,Z)$-closed absolutely convex hull follows from the fact $\overline{aco}^{\sigma(X,Z)}(K) = \overline{aco}^{\|\cdot\|}(K)$ because K is fragmentable by the norm, see [52].

Theorem 2.4.14 *If K is a Namioka-Phelps compact, then $(B_{C(K)^*}, w^*)$ is also a Namioka-Phelps compact.*

Proof. In the proof of [52, Theorem 5.6] it is shown that if K is a Radon-Nikodym compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*-continuous linear operator $T : C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is besides Namioka-Phelps, then $T(K)$ has $P(\|.\|, w^*)$. Then $T(B_{C(K)^*}) = \overline{aco}^{\|\cdot\|}(T(K))$ has $P(\|.\|, w^*)$ by Lemma 2.4.13, and thus, $T(B_{C(K)^*})$ is Namioka-Phelps.

Problem 2.4.15 *We do not know if $(B_{C(K)^*}, w^*)$ is descriptive for any descriptive compact space K.*[2]

Problem 2.4.16 *We do not know an example of descriptive compact space K such that the space $C(K)$ has no equivalent locally uniformly rotund norm, see Definition 3.1.2.*[3]

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[2]The answer is yes: M. RAJA, Weak* locally uniformly rotund norms and descriptive compact spaces, *J. Funct. Anal.* 197 (2003), 1–13.

[3]The answer is yes for Namioka-Phelps compacta: R. HAYDON, Locally uniformly

convex norms in Banach spaces and their duals. *J. Funct. Anal. 254* (2008), 2023–2039.

Chapter 3

LUR I

3.1 Preliminaries

The first basic notion about rotundity is the concept of strictly convex or rotund norm, defined as follows.

Definition 3.1.1 *Let X be a Banach space endowed with a norm $\|.\|$ and let S_X be the unit sphere. The norm $\|.\|$ is said to be rotund if for every $x, y \in S_X$ with $\|x + y\| = 2$, then $x = y$.*

In other words, every point of the unit sphere S_X is an extreme point in the unit ball B_X, or equivalently, the unit sphere does not contain segments.

The properties of the hilbertian norm have inspired several stronger types of rotundity of norms, which have been used in the study of the geometry of Banach spaces (for example, see [10] and the references therein).

Definition 3.1.2 *An equivalent norm $\|.\|$ on a Banach space X is said to be locally uniformly rotund at a point $x \in X$, if for every sequence $(x_k) \subset X$ with $\|x_k\| = \|x\|$ and $\lim_k \|x + x_k\| = 2\|x\|$, then $\lim_k \|x - x_k\| = 0$. If the norm $\|.\|$ is locally uniformly rotund at every point $x \in X$, then $\|.\|$ is said to be a locally uniformly rotund (LUR) norm.*

There are large classes of Banach spaces which are LUR renormable, see [10] for a survey until 1993. In these two last chapters we shall give some characterizations of the existence of equivalent LUR norms on a Banach space. We shall obtain as corollaries some of the known results about LUR renorming. Some other results cannot be deduced directely from the results of this thesis because they lie deeply on the particular structure of the Banach space. However, in many cases we shall be able to improve the LUR renormability with lower semicontinuity conditions.

The following lemma concerning to ℓ^2-sums of convex functions will replace in rotund renorming the arguments of lower semicontinuity used in section 2.2 for Kadec renorming.

Lemma 3.1.3 *Let f be a convex function on a Banach space X. Consider the symetric function*

$$Q_f(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f(\frac{x+y}{2})^2$$

Then the following properties are verified:

1) $Q_f \geq 0$.

2) *A non negative function f defined by $f^2 = \sum_{n=1}^{\infty} f_n^2$ is convex if every f_n is a non negative convex function and*

$$Q_f = \sum_{n=1}^{\infty} Q_{f_n}$$

3) *Given $x, x_k \in X$ the following are equivalent*

   a) $\lim_k f(x_k) = f(x)$ *and* $\lim_k f(\frac{x+x_k}{2}) = f(x)$.

   b) $\lim_k Q_f(x, x_k) = 0$.

Proof. Statement 2) is a straightforward calculation. To prove 1) and 3) just consider the inequalities

$$Q_f(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f(\frac{x+y}{2})^2 \geq \frac{f(x)^2 + f(y)^2}{2} - (\frac{f(x)+f(y)}{2})^2$$

$$= \frac{(f(x) - f(y))^2}{4} \geq 0$$

A classical result of Kadec (see [3, p. 178]) shows that every separable Banach space admits an equivalent LUR norm which is lower semicontinuous with respect to the topology of pointwise convergence on a given quasi-norming subset of the dual. We present below a simple new proof of the LUR renorming of a separable Banach space.

Proposition 3.1.4 *Let X be a separable Banach space and τ a vector topology on X coarser than the norm topology such that $\overline{B_X}^{\tau}$ is bounded. Then X admits an equivalent τ-lower semicontinuous LUR norm.*

Proof. Taking the Minkowski functional of $\overline{B_X}^{\tau}$ we can suppose X endowed with a τ-lower semicontinuous norm. Let (a_n) be a norm dense subset of X. Consider the τ-lower semicontinuous convex function F defined by

$$F(x)^2 = \frac{1}{3}\|x\|^2 + \frac{1}{6}\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|a_n - x\|^2}{\|a_n\|^2 + 1} + \frac{1}{6}\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|a_n + x\|^2}{\|a_n\|^2 + 1}$$

Let $B = \{x \in X : \ F(x) \leq 1\}$. An easy calculation shows that $B_X \subset B \subset 2B_X$. Let $|||.|||$ be the functional of Minkowski of B, then $|||.|||$ is a τ-lower semicontinuous equivalent norm. We claim that $|||.|||$ is a LUR norm. Indeed, let $x, x_k \in X$ with $|||x||| = |||x_k||| = 1$ and $\lim_k |||x + x_k||| = 2$. It is not difficult to see that $F(x) = F(x_k) = 1$ and $\lim_k F(\frac{x+x_k}{2}) = 1$. By Lemma 3.1.3 we have that

$$\lim_k \left(\frac{F(x)^2 + F(x_k)^2}{2} - F(\frac{x + x_k}{2})^2\right) = 0$$

and this implies for every n that

$$\lim_k \left(\frac{\|a_n - x\|^2 + \|a_n - x_k\|^2}{2} - \|a_n - \frac{x + x_k}{2}\|^2\right) = 0$$

Again by Lemma 3.1.3, we have for every $n \in \mathbb{N}$ that $\lim_k \|a_n - x_k\| = \|a_n - x\|$. Since $\|a_n - x\|$ can be made smaller than any positive number, we deduce that (x_k) must be convergent with limit x.

3.2 A theorem of Troyanski

In this section we shall prove a famous theorem of Troyanski, see [10, p. 148]. Although it is a particular case of Theorem 3.3.6 from the following section, we give an independent simple proof before.

Let X be a Banach space. We call an open affine half space defined by an element $x^* \in X^*$ a set of the form $\{x \in X : x^*(x) < \alpha\}$ where $\alpha \in \mathbb{R}$. A relatively open subset of a set defined by an intersection with some open half space is called slice. If $Z \subset X^*$ is a linear subspace of the dual, we shall denote by $\mathbb{H}(Z)$ the set of the open affine half spaces defined by elements of Z. A point x of a convex set C is said to be denting if there are slices of C containing x of arbitrarily small diameter. If the slices are given by the elements of Z we say that x is Z-denting.

Theorem 3.2.1 (Troyanski) *Let X be a Banach space. The following statements are equivalent:*

i) X admits a LUR norm.

ii) X admits both a rotund norm and a Kadec norm.

iii) X admits a norm such that every point of the unit sphere is a denting point of the unit ball.

Proof. *i)* $\Rightarrow$ *ii)* It is well known. We include a proof of a slightly more general result in the next section.
ii) $\Rightarrow$ *iii)* This is a lemma of Lin-Lin-Troyanski [48]. We include the proof for sake of completness.
First we can asume that X is endowed with a norm $\|.\|$ which is both rotund and Kadec. Indeed, if $\|.\|_1$ is rotund and $\|.\|_2$ is Kadec, then the norm $\|.\|$ defined by

$$\|.\|^2 = \|.\|_1^2 + \|.\|_2^2$$

shares both properties.
Take $x \in S_X$. We claim that x is an extreme point of $B_{X^{**}}$. Indeed, suppose

that $x = (x_1 + x_2)/2$, where $x_i \in B_{X^{**}}$ for $i = 1, 2$. Fix $\varepsilon > 0$ and set U a w^*-open neighbourhood of x such that $diam(B_{X^{**}} \cap U) < \varepsilon/2$. This is possible because of Goldstine's Theorem and the w^*-lower semicontinuity of the metric. Take U_i a w^*-open neighbourhood of x_i for $i = 1, 2$ such that $U_1 + U_2 \subset 2U$. If $y \in B_{X^{**}} \cap U_1$ then

$$\frac{y + x_2}{2} \in B_{X^{**}} \cap U$$

because of the convexity. We deduce that $diam(B_{X^{**}} \cap U_1) < \varepsilon$. Since $B_X \cap U_1$ is non empty, we obtain that x_1 can be aproximated uniformly by points of B_X. As B_X is norm complete, $x_1 \in B_X$, and in consequence $x_2 \in B_X$. As x is an extreme point in B_X, we have that $x = x_1 = x_2$. Now, since x is an extreme point of the w^*-compact set $B_{X^{**}}$, we can apply Choquet's Lemma, see [7, Proposition 25.13], to conclude that the slices of $B_{X^{**}}$ given by elements of X^* are a local basis for the norm topology at x, and thus x is denting in B_X.
iii) $\Rightarrow$ *i)* Let $\|.\|$ be a norm as in *iii)*. For $n \in \mathbb{N}$, take the set

$$B_n = \{x \in B_X : \ \forall\ H \in \mathbb{H}(X^*),\ x \in H,\ diam(B_X \cap H) > 1/n\}$$

which is a closed absolutely convex set with non empty interior. Let f_n be the functional of Minkowski of B_n. If $n > 1$, then B_n contains $\frac{1}{2}B_X$, thus we have that $f_n \leq 2\|.\|$. Define an equivalent norm $|||.|||$ on X by the formula

$$|||x|||^2 = \|x\|^2 + \sum_{n=1}^{\infty} 2^{-n} f_n(x)^2$$

We claim that $|||.|||$ is LUR. Let $x \in X$ and let $(x_k) \subset X$ such that $|||x_k||| = |||x|||$ and $\lim_k |||x + x_k||| = 2\|x\|$. We have that

$$\lim_k \left(\frac{|||x|||^2 + |||x_k|||^2}{2} - |||\frac{x + x_k}{2}|||^2\right) = 0$$

Now, using Lemma 3.1.3 it follows that

$$\lim_k \|x_k\| = \lim_k \|\frac{x + x_k}{2}\| = \|x\|$$

$$\lim_k f_n(x_k) = \lim_k f_n(\frac{x+x_k}{2}) = f_n(x)$$

for every $n \in \mathbb{N}$. We deduce that in order to prove that (x_k) converges to x we can suppose $\|x_k\| = \|x\| = 1$ by taking the sequence $(x_k/\|x_k\|)$.
Fix $\varepsilon > 0$. Since every point in S_X is a denting point of the unit ball, there is $H \in \mathbb{H}(X^*)$ such that $x \in H$ and

$$diam(B_X \cap H) \leq \varepsilon/3$$

Take $n \in \mathbb{N}$ such that $\varepsilon/3 \leq 1/n \leq \varepsilon/2$. Then $x \notin B_n$ and we have that $f_n(x) > 1$. So for $k \in \mathbb{N}$ big enough, we have that $f_n(\frac{x+x_k}{2}) > 1$, and thus $(x+x_k)/2 \in B_X \setminus B_n$. By definition of B_n, there is $H' \in \mathbb{H}(X^*)$ such that $(x+x_k)/2 \in H'$ and $diam(B_X \cap H') \leq \varepsilon/2$. But either x or x_k belongs to H', and thus, to $B_X \cap H'$. Since

$$\|x_k - x\| = 2\,\|x_k - \frac{x+x_k}{2}\| = 2\,\|x - \frac{x+x_k}{2}\|$$

we deduce that $\|x_k - x\| \leq \varepsilon$ for k big enough. Consequently x_k converges to x and the proof is finished.

The idea of considering the set obtained from a convex set after removing al the points of ε-dentability in LUR renorming of certain Banach spaces was considered by Lancien in [47].

Remark 3.2.2 *The proof of Theorem 3.2.1 also gives the following: Let X be a (dual) Banach space endowed with some (dual) norm and let $D \subset S_X$ be the set of (weak*) denting points of B_X. Then X admits an equivalent (dual) norm which is locally uniformly rotund at the points of D.*

As an application of the last remark, it is possible to prove the "three spaces property" for the renormability with norms having the Mazur intersection property.

Definition 3.2.3 *A norm on a Banach space X is said to have the Mazur intersection property if every bounded closed convex set of X is an intersection of closed balls.*

We shall use these following results from [20] and [40].

Theorem 3.2.4 (Giles, Gregory & Sims) *A norm on a Banach space X has the Mazur intersection property if and only if the sphere of its dual norm on X^* has a dense set of weak* denting points.*

Proposition 3.2.5 (Jimenez & Moreno) *Let X be a Banach space and let $Y \subset X$ be a closed subspace such that Y admits a norm whose dual norm has a dense set of locally uniformly rotund points and X/Y admits a norm with the Mazur intersection property. Then X admits an equivalent norm with the Mazur intersection property.*

A straightforward combination of all these facts gives us the following.

Corollary 3.2.6 *Let X be a Banach space and let $Y \subset X$ be a subspace such that Y and X/Y are renormable with the Mazur intersection property. Then X admits an equivalent norm with the Mazur intersection property.*

3.3 Construction of LUR norms

In this section we give some necessary and suficient conditions for the existence of an equivalent LUR norm which is lower semicontinuous with respect to the topology of convergence on a given quasi-norming set. In order to state the characterizations of the existence of a LUR renorming in terms of networks, we shall use the following extension of the definition of the property P to families of sets more general than topologies.

Definition 3.3.1 *Let Σ_1 and Σ_2 be families of subsets of a given set X. We say that X has $P(\Sigma_1, \Sigma_2)$ with a sequence (A_n) of subsets of X if for every $x \in X$ and every $V \in \Sigma_1$ with $x \in V$, there is $n \in \mathbb{N}$ and $U \in \Sigma_2$ such that*

$$x \in A_n \cap U \subset V$$

One can easily realize that this generalized property P is also transitive, that is, if X has $P(\Sigma_1, \Sigma_2)$ and $P(\Sigma_2, \Sigma_3)$ then X has $P(\Sigma_1, \Sigma_3)$. Upon that fact are based most of the "combinatorial type" results of this chapter. When X is a vector space and the sets A_n in Definition 3.3.1 can be chosen convex we shall say that X has convex-$P(\Sigma_1, \Sigma_2)$. For example, if Σ_1 is the norm topology of X and Σ_2 is a familly of half spaces $\mathbb{H}(Z)$, we shall write $P(\|.\|, Z)$ instead of $P(\Sigma_1, \Sigma_2)$.

Lemma 3.3.2 *Let X be a vector space and let $\tau_2 \subset \tau_1$ be locally convex vector topologies on X. Denote by $\mathbb{S}(\tau_i)$ a subbasis of τ_i which is given by a family of sublinear functions for $i = 1, 2$. Suppose that there exists a family Δ of sets of X with the property: for some $x \in X$ and every $V \in \mathbb{S}(\tau_1)$ with $x \in V$ there exists $A \in \Delta$ and $U \in \mathbb{S}(\tau_2)$ such that*

$$x \in A \cap U \subset V$$

Then for every $V \in \mathbb{S}(\tau_1)$ with $x \in V$, there exists $A \in \Delta$, $W \in \tau_1$ and $U \in \mathbb{S}(\tau_2)$ such that

$$x \in (A + W) \cap U \subset V$$

Proof. Suppose that $x \in X$ and $V \in \tau_1$ with $x \in V$ are given. We claim that there is $W_1, V' \in \mathbb{S}(\tau_1)$ with $0 \in W_1$, $x \in V'$ and $W_1 + V' \subset V$. Indeed, if $V = \{y \in X : f(x - y) < \varepsilon\}$ where f is a sublinear function and $\varepsilon > 0$, then just take the sets $W_1 = \{y \in X : f(y) < \varepsilon/2\}$ and $V' = \{y \in X : f(x - y) < \varepsilon/2\}$.
By the property of Δ, we can take $A \in \Delta$ and $U' \in \mathbb{S}(\tau_2)$ such that $x \in A \cap U' \subset V'$.
As above, we can take $W_2, U \in \mathbb{S}(\tau_2)$ with $0 \in W_2$, $x \in U$ and $W_2 + U \subset U'$. Now take $W = W_1 \cap (-W_2) \in \tau_1$. We shall show that U and W verify the set inclusion of the thesis. If $y \in (A + W) \cap U$, then there is $z \in A$ such that $y - z \in W \subset -W_2$, so $z = (z - y) + y \in U'$, and thus $z \in A \cap U' \subset V'$. Now as $y - z \in W \subset W_1$ we have that $y = (y - z) + z \in V$.

The following theorem is the main result in this chapter. It contains the explicit construction of the LUR norm.

Theorem 3.3.3 *Let X be a Banach space and let $Z \subset X^*$ be a quasi-norming linear subspace. The following are equivalent:*

i) X admits a $\sigma(X,Z)$-lower semicontinuous LUR norm.

ii) X admits a norm such that every point of the unit sphere is Z-denting.

iii) X has convex-$P(\|.\|, Z)$.

Proof. *i)* $\Rightarrow$ *ii)* Fix $x_0 \in S_X$ and $\varepsilon > 0$. As $\|.\|$ is a LUR norm, there is $\delta > 0$ such that if $x \in S_X$ and $\|\frac{x_0+x}{2}\| > 1-\delta$, then $\|x - x_0\| < \varepsilon$. We can suppose $\delta < 1$. Since $(1-\delta)^{-1}x_0 \notin B_X$ and B_X is $\sigma(X,Z)$-closed, by the Hahn-Banach Theorem, there is $x^* \in Z$ such that

$$\sup\{x^*(x) : \ x \in B_X\} < (1-\delta)^{-1}x^*(x_0)$$

We can suppose $\|x^*\| = 1$ and thus $(1-\delta)^{-1}x^*(x_0) > 1$. Now, define $H = \{x \in X : \ x^*(x) > 1-\delta\}$. Then $x_0 \in H$, and if $x \in B_X \cap H$, then

$$\|\frac{x_0+x}{2}\| \geq x^*(\frac{x_0+x}{2}) > 1-\delta$$

and thus $\|x - x_0\| < \varepsilon$.
ii) $\Rightarrow$ *iii)* We shall check that X has $P(\|.\|, \sigma(X,Z))$ with the set of the balls centered at 0 and rational radius. Fix $x \in X$, without loss of generality suppose $x \neq 0$. Let $B[0, \|x\|]$ be the closed ball of center 0 and radius $\|x\|$. By hypothesis, for every $\varepsilon > 0$ there is $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$B[0, \|x\|] \cap H \subset B(x, \varepsilon)$$

Fix $\varepsilon > 0$. By Lemma 3.3.2, there is $\delta > 0$ and $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$(B[0, \|x\|] + B(0,\delta)) \cap H \subset B(x, \varepsilon)$$

Now, we can find a rational $r > 0$ such that

$$B[0, \|x\|] \subset B[0, r] \subset B[0, \|x\|] + B(0, \delta)$$

and thus $x \in B[0, r] \cap H \subset B(x, \varepsilon)$.
$iii) \Rightarrow i)$ Without loss of generality we can suppose that X is already endowed with a norm $\|.\|$ which is $\sigma(X, Z)$-lower semicontinuous.
Suppose that X has $P(\|.\|, Z)$ with a sequence of convex sets (A_n). Then X also has $P(\|.\|, Z)$ with the countable family of convex sets

$$\{A_n + B(0, r) : \ n \in \mathbb{N}, \ r > 0, \ r \in \mathbb{Q}\}$$

after Lemma 3.3.2, so we can suppose that the sets A_n are norm open. Fix $a_n \in A_n$ for every $n \in \mathbb{N}$ and let f_n be the Minkowski functional with respect to the point a_n of the set $\overline{A_n}^{\sigma(X,Z)}$, that is, if g is the Minkowski functional of $\overline{A_n - a_n}^{\sigma(X,Z)}$ then $f_n(x) = g(x - a_n)$. Clearly the function f_n is convex, Lipschitz and $\sigma(X, Z)$-lsc.
Now, for $m \in \mathbb{N}$ we define the sets

$$A_{n,m} = \{x \in \overline{A_n}^{\sigma(X,Z)} : \ \forall H \in \mathbb{H}(Z), \ x \in H, \ diam(A_n \cap H) > 1/m\}$$

It is easy to see that the sets $A_{n,m}$ are empty or $\sigma(X, Z)$-closed convex. For every $p \in \mathbb{N}$ consider the sets

$$A_{n,m,p} = \overline{(A_{n,m} + B(0, 1/p))}^{\sigma(X,Z)}$$

We claim that $A_{n,m} = \bigcap_{p=1}^{\infty} A_{n,m,p}$ when $A_{n,m}$ is non empty. Indeed, if $x \notin A_{n,m}$ since that set is convex and $\sigma(X, Z)$-closed we can take $H \in \mathbb{H}(Z)$ such that $x \in H$ and $A_{n,m} \cap H = \emptyset$. Taking the halfspace given by a parallel hyperplane, we can assume that the distance between $A_{n,m}$ and H is positive. Then for some $p \in \mathbb{N}$, we have

$$(A_{n,m} + B(0, 1/p)) \cap H = \emptyset$$

and thus $x \notin A_{n,m,p}$.
Now, for every $n, m \in \mathbb{N}$ such that $A_{n,m} \neq \emptyset$ take $a_{n,m} \in A_{n,m}$. Let $f_{n,m,p}$ be

the functional of Minkowski with respect to $a_{n,m}$ of the set $A_{n,m,p}$, which is convex, Lipschitz and $\sigma(X,Z)$-lsc. If $A_{n,m}=\emptyset$, then we take $f_{n,m,p}=0$ for every $p\in\mathbb{N}$.
We define a symmetric convex function F on X by the formula

$$F(x)^2=\|x\|^2+\sum_n \alpha_n\, f_n(x)^2+\sum_{n,m,p}\beta_{n,m,p}\, f_{n,m,p}(x)^2$$

$$+\sum_n \alpha_n\, f_n(-x)^2+\sum_{n,m,p}\beta_{n,m,p}\, f_{n,m,p}(-x)^2$$

where (α_n) and $(\beta_{n,m,p})$ are positive constants taken in such a way to guarantee the uniform convergence on bounded subsets of X of the series, so F is uniformly continuous on bounded sets, and that the absolutely convex set

$$B=\{x\in X:\ F(x)\leq 1\}$$

contains 0 as a interior point. As a series of $\sigma(X,Z)$-lsc fuctions is $\sigma(X,Z)$-lsc as well, we deduce that B is $\sigma(X,Z)$-closed. Let $|||.|||$ be the functional of Minkowski of B. Then $|||.|||$ is an equivalent $\sigma(X,Z)$-lsc norm on X. We claim that for every sequence $(x_k)\subset B$ with $\lim_k |||x_k|||=1$, then $\lim_k F(x_k)=1$. Since F is $\|.\|$-continuous we deduce that

$$\{x\in X:\ F(x)<1\}\subset\{x\in X:\ |||x|||<1\}$$

because the first set is $\|.\|$-open and contained in B. Let us consider $x'_k=x_k/|||x_k|||$ and we should have $F(x'_k)=1$. We have that $\lim_k\|x'_k-x_k\|=0$. Since F is uniformly continuous on bounded sets we deduce that

$$\lim_k F(x_k)=1$$

Finally, we shall prove that $|||.|||$ is a LUR norm. Let $x\in X$ and let $(x_k)\subset X$ such that $|||x_k|||=|||x|||=1$ and $\lim_k|||x+x_k|||=2$. As we have seen above, this entails that $F(x_k)=F(x)=1$ and $\lim_k F(\frac{x+x_k}{2})=1$. We have that

$$\lim_k\left(\frac{F(x)^2+F(x_k)^2}{2}-F(\frac{x+x_k}{2})^2\right)=0$$

Now, using Lemma 3.1.3 we have that for every $n, m, p \in \mathbb{N}$

$$\lim_k f_n(x_k) = \lim_k f_n(\frac{x+x_k}{2}) = f_n(x)$$

$$\lim_k f_{n,m,p}(x_k) = \lim_k f_{n,m,p}(\frac{x+x_k}{2}) = f_{n,m,p}(x)$$

Fix $1/2 > \varepsilon > 0$. There is $n \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that $x \in A_n \cap H$ and

$$diam(\overline{A_n}^{\sigma(X,Z)} \cap H) \leq \varepsilon/3$$

because H is $\sigma(X,Z)$-open and $\|.\|$ is $\sigma(X,Z)$-lsc. Since A_n is $\|.\|$-open, we have that $f_n(x) < 1$, so for $k \in \mathbb{N}$ big enough $f_n(x_k) < 1$, and thus $x_k \in \overline{A_n}^{\sigma(X,Z)}$. Analogously, for $k \in \mathbb{N}$ big enough, $(x+x_k)/2 \in \overline{A_n}^{\sigma(X,Z)}$.
Take $m \in \mathbb{N}$ such that $2/\varepsilon < m < 3/\varepsilon$, thus $x \notin A_{n,m}$. If $A_{n,m} \neq \emptyset$ then for some $p \in \mathbb{N}$ we have that $f_{n,m,p}(x) > 1$, so for $k \in \mathbb{N}$ big enough $f_{n,m,p}(\frac{x+x_k}{2}) > 1$ and thus $(x+x_k)/2 \notin A_{n,m,p}$. If $A_{n,m} = \emptyset$ we have simply that $(x+x_k)/2 \notin A_{n,m}$.
In consequence, for $k \in \mathbb{N}$ big enough we have that

$$\frac{x+x_k}{2} \in \overline{A_n}^{\sigma(X,Z)} \setminus A_{n,m}$$

By definition of $A_{n,m}$ there is $H' \in \mathbb{H}(Z)$ such that $(x+x_k)/2 \in \overline{A_n}^{\sigma(X,Z)} \cap H'$ and $diam(\overline{A_n}^{\sigma(X,Z)} \cap H') \leq \varepsilon/2$. But either x or x_k belongs to H', and thus, to $\overline{A_n}^{\sigma(X,Z)} \cap H'$. Since

$$\|x_k - x\| = 2\,\|x_k - \frac{x+x_k}{2}\| = 2\,\|x - \frac{x+x_k}{2}\|$$

we deduce that $\|x_k - x\| \leq \varepsilon$. This ends the proof of Theorem 3.3.3.

Remark 3.3.4 *If we have the hypothesis convex-P*$(\|.\|, \sigma(X,Z))$ *instead of iii), the same proof with F defined by*

$$F(x)^2 = \|x\|^2 + \sum_n \alpha_n\, f_n(x)^2 + \sum_n \alpha_n\, f_n(-x)^2$$

suffices to prove that $|||.|||$ is an equivalent $\sigma(X,Z)$-Kadec norm on X. This provides another proof of Theorem 2.3.1.

Lemma 3.3.5 *Convex-$P(w,\sigma(X,Z))$ entails convex-$P(X^*,Z)$*

Proof. Suppose that X has $P(w,\sigma(X,Z))$ with a sequence of convex sets (A_n). We shall prove that X has $P(X^*,Z)$ with the sequence (A_n). Fix $x \in X$ and $H \in \mathbb{H}(X^*)$. Take $n \in \mathbb{N}$ and $U \in \sigma(X,Z)$ such that $x \in A_n \cap U \subset H$. Since U and the convex set $A_n \setminus H$ are disjoint, the set $\overline{A_n \setminus H}^{\sigma(X,Z)}$ cannot contain x. By the Hahn-Banach Theorem, we obtain $H' \in \mathbb{H}(Z)$ such that $x \in H'$ and $\overline{A_n \setminus H}^{\sigma(X,Z)} \cap H' = \emptyset$. Thus we have $x \in A_n \cap H' \subset H$.

Some more characterizations of the LUR renorming are contained in the following theorem. In particular, *ii)* $\Rightarrow$ *i)* generalizes the Theorem of Troyanski, Theorem 3.2.1.

Theorem 3.3.6 *Let X be a Banach space and let $Z \subset X^*$ be a quasi-norming linear subspace. The following are equivalent:*

i) X admits a $\sigma(X,Z)$-lower semicontinuous LUR norm.

ii) X admits both a rotund norm and a $\sigma(X,Z)$-Kadec norm.

iii) X admits a LUR norm and X has convex-$P(w,\sigma(X,Z))$.

Proof. *i)* $\Rightarrow$ *ii)* We shall check the Kadec property for $\sigma(X,Z)$. Take $(x_\omega) \subset S_X$ a net $\sigma(X,Z)$-converging to $x \in S_X$. As $\sigma(X,Z)$-$\lim_\omega(x+x_\omega) = 2x$ and $\|.\|$ is $\sigma(X,Z)$-lsc, we have that $\liminf_\omega \|x+x_\omega\| \geq 2\|x\| = 2$. On the other hand, $\|x+x_\omega\| \leq 2$ for every ω. Thus, $\lim_\omega \|x+x_\omega\| = 2$. Applying now that $\|.\|$ is LUR, we have that $\lim_\omega \|x-x_\omega\| = 0$.
ii) $\Rightarrow$ *iii)* By Theorem 2.3.1, if X admits a $\sigma(X,Z)$-Kadec norm, then it has convex-$P(\|.\|,\sigma(X,Z))$. In particular, X has convex-$P(w,\sigma(X,Z))$. As a $\sigma(X,Z)$-Kadec norm is Kadec, by Theorem 3.2.1 we have that X admits a LUR norm.
iii) $\Rightarrow$ *i)* To prove that X admits a $\sigma(X,Z)$-lsc LUR norm it is enough to prove that it has convex-$P(\|.\|,Z)$ by Theorem 3.3.3. Since X admits an equivalent LUR norm, then it has convex-$P(\|.\|,X^*)$ by Theorem 3.3.3. This property together with convex-$P(X^*,Z)$ entails convex-$P(\|.\|,Z)$.

Remark 3.3.7 *Note that we do not ask the rotund norm of statement ii) to be $\sigma(X, Z)$-lower semicontinuous.*

Remark 3.3.8 *In [33], it is proved that for a totally ordered set K which is compact in its order topology, the existence of rotund norm on $C(K)$ implies the existence of pointwise lower semicontinuous LUR norm on $C(K)$. This result can be obtained from Theorem 3.3.6 together with the fact that if K is such a compact, then $C(K)$ has a pointwise-Kadec norm [33].*

Corollary 3.3.9 *Let X^* be a dual Banach space such that on the unit sphere of some equivalent norm the weak and the weak* topologies coincide. Then X^* admits an equivalent dual LUR norm. In particular, if X^* has an equivalent w^*-Kadec norm, then X^* admits an equivalent dual LUR norm.*

Proof. If on the unit sphere of some equivalent norm the weak and the weak* topologies coincide, that norm is a dual norm. Indeed, take a point x^* out of the unit ball B_{X^*}. Let V be a weak neighbourhood of x^* such that $\overline{V}^w$ is disjoint with B_{X^*}. There is a weak* neighbourhood U of x^* such that $\|x^*\|S_{X^*} \cap U \subset V$. Since U is open for the weak topology we have that $\overline{\|x^*\|S_{X^*}}^w \cap U \subset \overline{\|x^*\|S_{X^*} \cap U}^w \subset \overline{V}^w$. Thus $\|x^*\|B_{X^*} \cap U \subset \overline{V}^w$ and we deduce that U is disjoint with B_{X^*}. A dual Banach space has the Radon-Nikodym property if the weak and the weak* topologies coincide on the unit sphere, see [20]. Another proof of the Radon-Nikodym property of X^* can be obtained from the P-Borel measurability of $\mathbb{I} : (X^*, w^*) \to (X^*, w)$, see Corollary 1.3.7 ant the comments below. It is well known that a dual Banach space with the Radon-Nikodym property admits a (not necessary dual) LUR norm [16]. Then apply statement *iii*) of Theorem 3.3.6.

Remark 3.3.10 *A Banach space $(X, \|.\|)$ is said to be Hahn-Banach smooth if every $x^* \in X^*$, as a functional on X, has a unique norm preserving extension to X^{**}. It follows from [29] that $(X, \|.\|)$ is Hahn-Banach smooth if and only if the weak and the weak* topologies coincide on S_{X^*}. Thus a Banach space X is Hahn-Banach smooth renormable, if and only if the dual X^* admits a dual LUR norm. In particular, every Hahn-Banach smooth Banach space admits a Frechet differentiable norm.*

Remark 3.3.11 *Note that while the fact that the dual norm is w^*-Kadec implies dual-LUR renormability, there exists Banach spaces having a Kadec norm but with no equivalent rotund norm [32]. Haydon [31] has built a Banach space X such that $(B_{X^{**}}, w^*)$ is a Corson compact (in particular X^* has RNP) and X^* admits no dual LUR norm. After Corollary 3.3.9 we have that this space X^* admits no w^*-Kadec norm.*

In WCD Banach spaces we obtain an improvement of Theorem 2.3.5 for the weak topologies.

Corollary 3.3.12 *Let X be a WCD Banach space, let $Z \subset X^*$be a quasi-norming linear subspace. Then X admits an equivalent $\sigma(X, Z)$-lower semi-continuous LUR norm.*

The result stated in Corollary 3.3.12 is implicitely showed in [22] for WCG Banach spaces. When X^* is a WCD dual space, we deduce the existence of a dual LUR norm, result of Fabian [14].

To show the relation with the work of Lancien [47] we need to introduce some notation. Given a Banach space X and a total subspace $Z \subset X^*$ consider the following construction. For a convex set B and $\varepsilon > 0$ take

$$(B)'_\varepsilon = \{x \in B : \ \forall H \in \mathbb{H}(Z), \ x \in H, \ diam(B \cap H) > \varepsilon\}$$

Now we define by transfinite induction the sets (B^α_ε) as follows

$$B^0_\varepsilon = B_X$$

$$B^{\alpha+1}_\varepsilon = (B^\alpha_\varepsilon)'_\varepsilon$$

and for α a limit ordinal

$$B^\alpha_\varepsilon = \bigcap_{\beta<\alpha} B^\beta_\varepsilon$$

Take $\delta_Z(X, \varepsilon) = \inf\{\alpha : \ B^\alpha_\varepsilon = \emptyset\}$, if it exists, and take $\delta_Z(X) = \sup_{\varepsilon>0} \delta_Z(X, \varepsilon)$. When defined, $\delta_Z(X)$ is called the Z-dentability index of X (see [1]).

Lancien has shown in [47] that for a Banach space X, the condition $\delta_{X^*}(X) < \omega_1$, where ω_1 is the first uncountable ordinal, implies the existence of a LUR norm on X. He also proved that in the case of a dual space X^*, if $\delta_X(X^*) < \omega_1$, then X^* admits a dual LUR norm. His method involves distances to sets and it seems to be difficult to apply it to other topologies.

Proposition 3.3.13 *Let X be a Banach space, let $Z \subset X^*$ be a quasi-norming linear subspace. If $\delta_Z(X) < \omega_1$, then X admits an equivalent $\sigma(X, Z)$-lower semicontinuous LUR norm.*

Proof. The method of construction of the countable family of convex sets $\bigcup_{n=1}^{\infty}\{B^{\alpha}_{1/n} : \alpha < \delta_Z(X, 1/n)\}$ implies the property convex-$P(\|.\|, Z)$ for B_X. This implies that X has convex-$P(\|.\|, Z)$. Then apply Theorem 3.3.3.

3.4 WMLUR norms

In this section we consider a class of norms whose existence in a Banach space X can be characterized by means of a network, in similar terms to those concerning the existence of LUR norms.

Definition 3.4.1 *Let X be a Banach space and τ a vector topology on X. A norm $\|.\|$ is said to be τ-midpoint locally uniformly rotund (τ-MLUR) if given a point x and sequences $(y_k), (z_k)$ in X we have $\tau\text{-}\lim_k(y_k - z_k) = 0$, whenever $\|y_k\|, \|z_k\| \leq \|x\|$ and $\lim_k \|y_k + z_k - 2x\| = 0$.*

When τ is the norm, the weak or the weak* topology we write MLUR, WMLUR or W*MLUR respectively.

Unfortunately, we do not have a characterization of the τ-MLUR norms in terms of networks, so we have to change the definition of τ-MLUR norm by a similar but stronger condition, statement *i*) below.

Proposition 3.4.2 *Let $(X, \|.\|)$ be a Banach space and $Z \subset X^*$ a norming linear subspace. The following are equivalent:*

i) *For every point x in S_X and every two nets $(y_\omega), (z_\omega)$ in B_X such that $\sigma(X,Z)\text{-}\lim_\omega(y_\omega + z_\omega - 2x) = 0$, then $\sigma(X,Z)\text{-}\lim_\omega(y_\omega - z_\omega) = 0$.*

ii) *The Z-slices at every point of the unit sphere constitute a local basis for the $\sigma(X,Z)$-topology.*

iii) *There is a dual Banach space $\widetilde{X}$ which contains isometrically X with $\sigma(X,Z) = w^*|_X$ and such that the points of S_X are extreme in $\overline{B_X}^{w^*}$.*

iv) *For any dual Banach space $\widetilde{X}$ which contains isometrically X such that $\sigma(X,Z) = w^*|_X$, then the points of S_X are extreme in $\overline{B_X}^{w^*}$.*

Proof. *i)* $\Rightarrow$ *iv)* We claim that the points of S_X are extreme in $\overline{B_X}^{w^*}$. Indeed, suppose that $x \in S_X$ can be writen as $x = (y+z)/2$ where $y, z \in \overline{B_X}^{w^*}$. Take nets $(y_\omega), (z_\omega)$ in B_X which are w^*-converging to y, z respectively. We have that $w^*\text{-}\lim_\omega(y_\omega + z_\omega)/2 = x$. Now, we have that $0 = \sigma(X,Z)\text{-}\lim_\omega(y_\omega - z_\omega) = y - z$. By the hypothesis, $x = y = z$ which proves that x is extreme.
iv) $\Rightarrow$ *iii)* Take $B = B_{X^*} \cap Z$ and consider by the canonical embedding X as a subset of $l^\infty(B) = \widetilde{X}$.
iii) $\Rightarrow$ *ii)* The Choquet's Lemma establishes that the w^*-slices at an extreme point of a w^*-compact set of a dual Banach space are a local basis of the weak* topology at x. Now note that the intersection of a w^*-slice and X is a Z-slice.
ii) $\Rightarrow$ *i)* Let U be a $\sigma(X,Z)$-neighbourhood of 0. Take U_1 a $\sigma(X,Z)$-neighbourhood of x such that $U_1 - U_1 \subset U$. Take $U_2 \subset U_1$ a $\sigma(X,Z)$-neighbourhood of x such that $2U_2 - U_2 \subset U_1$. Take a Z-half space H such that $x \in B_X \cap H \subset U_2$. For ω big enough we have that $(y_\omega + z_\omega)/2 \in H$, so either y_ω or z_ω belongs to $B_X \cap H$. Suppose, for example, that $y_\omega \in B_X \cap H \subset U_2$. Since $z_\omega = 2\frac{z_\omega + y_\omega}{2} - y_\omega$, we have that $z_\omega \in U_1$, and thus $y_\omega - z_\omega \in U$. This shows us that $\sigma(X,Z)\text{-}\lim_\omega = (y_\omega - z_\omega) = 0$.

The following proposition characterizes the existence of a norm with the properties described in Proposition 3.4.2.

Proposition 3.4.3 *Let X be a Banach space and $Z \subset X^*$ a quasi-norming linear subspace. The following are equivalent:*

i) There is an equivalent norm $|||.|||$ on X such that $(X, |||.|||)$ satisfies each statement of Proposition 3.4.2.

ii) X has convex-$P(\sigma(X,Z),Z)$.

iii) For every Banach space $\widetilde{X}$ containing isomorphically X and $\widetilde{Z} \subset \widetilde{X}^*$ such that $\widetilde{Z}|_X = Z$, there is a $\sigma(\widetilde{X},\widetilde{Z})$-lower semicontinuous equivalent norm on $\widetilde{X}$ such that the points of the unit sphere belonging to X are extreme in the unit ball.

Proof. $i) \Rightarrow ii)$ We claim that X has convex-$P(\sigma(X,Z),Z)$ with the countable family of the balls centered at 0 with rational radius. Fix $x \in X$, without loss of generality suppose $x \neq 0$. Let $B[0, \|x\|]$ the closed ball of center 0 and radius $\|x\|$. By statement $ii)$ of Proposition 3.4.2, for every $\sigma(X,Z)$-open V containing x there is $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$B[0, \|x\|] \cap H \subset V$$

By Lemma 3.3.2, for every V containing x, there is $\delta > 0$ and $H \in \mathbb{H}(Z)$ such that $x \in H$ and

$$(B[0, \|x\|] + B(0,\delta)) \cap H \subset V$$

Now we can find a rational $r > 0$ such that

$$B[0, \|x\|] \subset B[0,r] \subset B[0, \|x\|] + B(0,\delta)$$

and thus $x \in B[0,r] \cap H \subset V$.
$ii) \Rightarrow iii)$ Suppose now that X has $P(\sigma(X,Z),Z)$ with some sequence of convex sets (A_n). Clearly X has $P(\sigma(\widetilde{X},\widetilde{Z}),\widetilde{Z})$ as a subset of $\widetilde{X}$. By the addition of balls $B(0,r)$ where r is rational and using Lemma 3.3.2, we can suppose that the sets (A_n) are norm open in $\widetilde{X}$. Take a point $a_n \in A_n$ and

consider f_n the Minkowski functional of $\overline{A_n}^{\sigma(\widetilde{X},\widetilde{Z})}$ with respect to a_n. Define a positive convex function F by the formula

$$F(x)^2 = \|x\|^2 + \sum_{n=1}^{\infty} \alpha_n f_n(x)^2 + \sum_{n=1}^{\infty} \alpha_n f_n(-x)^2$$

where (α_n) are positive numbers chosen in such a way that the series converges uniformly on bounded subsets of $\widetilde{X}$ and such that the symmetric convex set $B = \{x \in X : \ F(x) \leq 1\}$ contains 0 as an interior point. From the $\sigma(\widetilde{X}, \widetilde{Z})$-lower semicontinuity of F we deduce that B is the unit ball of an equivalent $\sigma(X, Z)$-lsc norm $|||.|||$ on $\widetilde{X}$.
We claim that $|||.|||$ is rotund at the points of X. Given $x \in X$ with $|||x||| = 1$, suppose that $x = (x_1 + x_2)/2$ and that $|||x_1||| = |||x_2||| = 1$. Now, we have that

$$\frac{F(x_1)^2 + F(x_2)^2}{2} - F(\frac{x_1 + x_2}{2})^2 = 0$$

Using Lemma 3.1.3 we deduce that $f_n(x_1) = f_n(x_2) = f_n(\frac{x_1+x_2}{2})$ for every $n \in \mathbb{N}$. If x_1 and x_2 were different then we would find a $\sigma(\widetilde{X}, \widetilde{Z})$-closed $\sigma(\widetilde{X}, \widetilde{Z})$-neighbourhood V of x that does not contains x_1 neither x_2. By property P we can find $n \in \mathbb{N}$ and $H \in \mathbb{H}(\widetilde{Z})$ such that $x \in A_n \cap H \subset V$ In particular $f_n(x) < 1$. Now, x_1 or x_2 belongs to H. Suppose that x_1 does. Since $\overline{A_n}^{\sigma(\widetilde{X},\widetilde{Z})} \cap H \subset V$ we deduce that x_1 cannot belong to $\overline{A_n}^{\sigma(\widetilde{X},\widetilde{X})}$, and thus, $f_n(x_1) > 1$. The contradiction shows that must be $x_1 = x_2$.
iii) $\Rightarrow$ *i)* Taking $\widetilde{X} = X^{**}$, statement *iii)* of Proposition 3.4.2 is verified.

In the case of a dual Banach space X^*, we have the following characterization of the renormability by a dual rotund norm.

Corollary 3.4.4 *Let X^* be a dual Banach space. Then X^* admits an equivalent dual rotund norm if and only if X^* has convex-$P(w^*, X)$.*

For the weak topology, we deduce that the property of being WMLUR is equivalent to those ones discused in Proposition 3.4.2.

Theorem 3.4.5 *Let X be a Banach space. The following are equivalent:*

i) X admits an equivalent WMLUR norm.

ii) X has convex-$P(w, X^*)$.

iii) For every Banach space $\widetilde{X}$ containing isomorphically X and $\widetilde{Z} \subset \widetilde{X}^*$ such that $\widetilde{Z}|_X = X^*$, there is a $\sigma(\widetilde{X}, \widetilde{Z})$-lower semicontinuous equivalent norm on $\widetilde{X}$ such that the points of the unit sphere belonging to X are extreme in the unit ball.

Proof. We just have to prove that if X is WMLUR, then X satisfies the statements of Proposition 3.4.2. Assume that X is WMLUR and suppose that X does not satisfy statement *iv*) of Proposition 3.4.2. Then there are points $x \in S_X$ and $y, z \in B_{X^{**}}$ such that $y + z = 2x$ and $y \neq z$. Take U^* and V^* weak* neighbourhoods of y and z respectively such that $0 \notin U^* - V^*$. Now, take $U = B_X \cap U^*$ and $V = B_X \cap V^*$. By Goldstine Theorem, $y \in \overline{U}^{w^*}$ and $z \in \overline{V}^{w^*}$. Since $2x \in \overline{U + V}^{w^*}$, we deduce that $2x \in \overline{U + V}^{w} = \overline{U + V}^{\|\cdot\|}$. That implies that we can take sequences $(y_n) \subset U$ and $(z_n) \subset V$ such that $\lim_n \|y_n + z_n - 2x\| = 0$. By construction we have that $(y_n - z_n)$ does not converge to 0 and this contradicts that X is WMLUR.

Remark 3.4.6 *In [45] is implicitely proved that a Banach space is WMLUR if and only if every point of S_X is an extreme point in $B_{X^{**}}$.*

An interpretation of Theorem 3.4.5 is that the Banach spaces X which are "universally rotund" are exactly the WMLUR Banach spaces. For the class of rotund Banach spaces we have not found a satisfactory characterization by means of networks. We say that a set A cuts another set B if $A \cap B$ is a proper subset of B. It is not difficult to prove the following using the method of the proof of Proposition 3.4.3.

Proposition 3.4.7 *A Banach space X has an equivalent rotund norm if and only if there is a sequence of closed convex sets (A_n) such that for every two points $x, y \in X$ with $x \neq y$, there is $n \in \mathbb{N}$ such that A_n cuts $\{x, y, \frac{x+y}{2}\}$.*

Proposition 3.4.8 *Let X be a Banach space and Y a subspace of X. The following are equivalent:*

i) There is an equivalent norm $|||.|||$ on X such that for every $x, y \in X$ with $x + y \in Y$ and $|||x||| = |||y||| = |||\frac{x+y}{2}|||$, then $x, y \in Y$.

ii) There is a sequence A_n of closed convex sets of X such that for every $x, y \in X \backslash Y$ with $x+y \in Y$, there is $n \in \mathbb{N}$ such that A_n cuts $\{x, y, \frac{x+y}{2}\}$.

Chapter 4

LUR II

4.1 Main LUR renorming results

In this section we shall improve Theorem 3.3.3. The difference with the results of the preceding chapter is that we no longer assume that the sets A_n are convex. This will allow us to give in the next sections sufficient conditions to the LUR renormability in terms of fragmentability and existence of maps into LUR spaces.

Theorem 4.1.1 *Let X be a Banach space and let $Z \subset X^*$ be a quasi-norming linear subspace. The following statements are equivalent:*

i) X admits a $\sigma(X,Z)$-lower semicontinuous LUR norm.

ii) X (resp. S_X) has $P(\|.\|, Z)$.

iii) X (resp. S_X) has both $P(\|.\|, w)$ and $P(w, Z)$.

The idea to prove the theorem is to use some convexity arguments coming from the study of the Radon-Nikodym Property. The Bourgain-Namioka Lemma (see [4]) is the master key to prove that if a set has a point of ε-dentability, then its convex envelope also has points of 3ε-dentability. But the original point may not be a point of "small" dentability in the convex envelope. The following lemma establishes that using some kind of iteration "eating" at each step points of small dentability in convex envelopes, we can reach a given point of ε-dentability.

Lemma 4.1.2 *Let M and ε be positive constants and let $H \in \mathbb{H}(Z)$ be an open halfspace. For every $E \subset X$ with $diam(E) \leq M$ and $diam(E \cap H) < \varepsilon$, there is a set $H[E] \subset E$ with the following properties:*

i) If $F \subset E$, then $H[F] \subset H[E]$.

ii) For every point $x \in E \setminus H[E]$, there is $H' \in \mathbb{H}(Z)$ such that $x \in H'$, $H[E] \cap H' = \emptyset$ and $diam(co(E) \cap H') < 3\varepsilon$.

iii) If $(H^n[E])$ is the sequence defined inductively by $H^1[E] = H[E]$ and $H^{n+1}[E] = H[H^n[E]]$, then $\bigcap_{n=1}^{\infty} H^n[E] \cap H = \emptyset$.

Proof. We can assume without loss of generality that X is already endowed with a $\sigma(X, Z)$-lsc norm, thus every subset of X has the same diameter that its $\sigma(X, Z)$-closure. Suppose that $H = \{x \in X : x^*(x) > 1\}$ with $x^* \in Z$ and $E \cap H \subset B$ where B is a closed ball of diameter less than 2ε. Define the convex sets $C = \overline{co}^{\sigma(X,Z)}(E)$, $C_0 = C \cap B$ and $C_1 = C \setminus H$. For every $0 \leq r \leq 1$ consider the convex set

$$D_r = \{(1 - \lambda)x_0 + \lambda x_1 : \ r \leq \lambda \leq 1, \ x_0 \in C_0, \ x_1 \in C_1\}$$

Note that if $x \in D_0 \setminus D_r$ then $x = (1 - \lambda)x_0 + \lambda x_1$ with $x_0 \in C_0$, $x_1 \in C_1$ and $0 \leq \lambda < r$, thus $\|x - x_0\| = \lambda\|x_0 - x_1\| \leq Mr$. Since $C = \overline{D_0}^{\sigma(X,Z)}$, we have that

$$diam(C \setminus \overline{D_r}^{\sigma(X,Z)}) \leq diam(D_0 \setminus D_r) \leq 2Mr + diam(C_0) \leq 2Mr + 2\varepsilon$$

Fix the parameter $r > 0$ small enough to have $diam(C \setminus \overline{D_r}^{\sigma(X,Z)}) \leq 3\varepsilon$ and remark that r depends only on M and ε.
Take $H[E] = E \cap \overline{D_r}^{\sigma(X,Z)}$. Statement i) trivially holds. To check statement ii) take $x \in E \setminus H[E]$. We have that $x \in C \setminus \overline{D_r}^{\sigma(X,Z)}$ and this set has diameter less than 3ε. By the Hahn-Banach Theorem, we can find $H' \in \mathbb{H}(Z)$ such

that $x \in H'$ and $H' \cap \overline{D_r}^{\sigma(X,Z)} = \emptyset$. Thus $diam(C \cap H') < 3\varepsilon$. Finally, we shall check statement *iii*). Note that

$$\sup\{x^*(x) : \ x \in H[E]\} \leq \sup\{x^*(x) : \ x \in D_r\}$$

$$\leq (1-r)\sup\{x^*(x) : \ x_0 \in C_0\} + r\sup\{x^*(x_1) : \ x_1 \in C_1\}$$

$$\leq (1-r)\sup\{x^*(x) : \ x \in E\} + r$$

So if $s_n = \sup\{x^*(x) : \ x \in H^n[E]\}$. We have that $1 \leq s_{n+1} \leq (1-r)s_n + r$. This shows that s_n converges to 1. If $x \in H$, as $x^*(x) > 1$, then $x \notin H^n[E]$ for some n big enough.

Lemma 4.1.3 *Let $A \subset X$ be a bounded set and $\varepsilon > 0$. There exists a sequence of convex sets (C_n) such that, for every point x contained in a Z-slice of A of diameter less than ε, then there is $n \in \mathbb{N}$ such that x is contained in a Z slice of C_n of diameter less than 3ε.*

Proof. Given $E \subset A$, for every $H \in \mathbb{H}(Z)$ such that $diam(E \cap H) < \varepsilon$, let $H[E]$ the set given by the preceding lemma and take $E' = \bigcap_H H[E]$. We claim that for every $x \in E \setminus E'$ there is $H' \in \mathbb{H}(Z)$ such that $x \in co(E) \cap H'$ and $diam(co(E) \cap H') < 3\varepsilon$. Indeed, if $x \in E \setminus E'$ then there is $H \in \mathbb{H}(Z)$ such that $x \in E \setminus H[E]$. Then, apply *ii*) of Lemma 4.1.2.
Now, we define a sequence (E_n) by induction: $E_0 = A$ and $E_{n+1} = (E_n)'$ if $E_n \neq \emptyset$. We claim that for every point x contained in a Z-slice of A of diameter less than ε, there is some n such that $x \notin E_n$. Indeed, suppose that $x \in A \cap H$ with $H \in \mathbb{H}(Z)$ and $diam(H \cap A) < \varepsilon$. Clearly we have $E_n \subset H^n[E]$. Condition *iii*) of Lemma 4.1.2 shows that for some n, $x \notin E_n$. From the properties of the sequence E_n, it is clear that the convex sets $C_n = co(E_n)$ satisfies the required condition.

Proof of Theorem 4.1.1.
i) $\Rightarrow$ *ii*) It is already known, Theorem 3.3.3.
ii) $\Leftrightarrow$ *iii*) It is a consequence of the transitive property of P.

ii) $\Rightarrow$ *i)* Assume that S_X has $P(\|.\|, Z)$ with a sequence of sets $S_n \subset S_X$. We claim that X has $P(\|.\|, Z)$. Indeed, consider the countable family

$$\{rS_n + B(0, s) : \ r, s \in \mathbb{Q}^+, \ n \in \mathbb{N}\}$$

Fix $\varepsilon > 0$ and take $x \neq 0$ then $x/\|x\| \in S_X$. By the hypothesis and Lemma 3.3.2 there are $n, m \in \mathbb{N}$ and $H \in \mathbb{H}(Z)$ such that

$$x/\|x\| \in (S_n + B(0, 1/m)) \cap H \subset B(x/\|x\|, \varepsilon/2\|x\|)$$

There is a rational $0 < r < \|x\|$ such that $x/r \in (S_n + B(0, 1/m)) \cap H$, so $x \in (rS_n + B(r/m)) \cap rH$ and $diam((rS_n + B(r/m)) \cap rH) \leq \varepsilon$.
Let (A_n) be a sequence such that X has $P(\|.\|, Z)$ with it. That means that for every $x \in X$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that x is contained in a Z-slice of a A_n of diameter less than ε. Without loss of generality, we can assume that every A_n is bounded. For every $n \in \mathbb{N}$ and every $\varepsilon > 0$ consider the sequence of convex sets $(C^{\varepsilon}_{n,m})_{m=0}^{\infty}$ given by Lemma 4.1.3 for $A = A_n$. According to Lemma 4.1.3, the countable family of convex sets

$$\{C^{1/k}_{n,m} : \ k, n, m \in \mathbb{N}\}$$

satisfies statement *iii)* of Theorem 3.3.3, thus X admits a $\sigma(X, Z)$-lower semicontinuous LUR norm.

Definition 4.1.4 *Let E be a subset of a Banach space X. It is said that A has countable cover by sets having slices of small diameter (sJNR) if for every $\varepsilon > 0$ we have $E = \bigcup_{n=1}^{\infty} E_n$ and for every $x \in E_n$, $n \in \mathbb{N}$, there is a slice of E_n of diameter less than ε and containing x.*

It is easy to prove with ideas from Proposition 1.5.6 that a subset $E \subset X$ has sJNR if and only if it has $P(\|.\|, X^*)$. Definition 4.1.4 is used in [50] to give the following characterization of the LUR renormability of a Banach space. Their proof is based in probabilistic arguments.

Corollary 4.1.5 (Moltó, Orihuela & Troyanski) *For a Banach space X the following conditions are equivalent:*

i) X admits an equivalent LUR norm.

ii) X (resp. S_X) has sJNR.

iii) X (resp. S_X) has JNR and an equivalent WMLUR norm

Proof. Recall that JNR, sJNR and WMLUR renormability are equivalent to $P(\|.\|, w)$, $P(\|.\|, X^*)$ and convex-$P(w, X^*)$ respectively.

In the case of a dual Banach space X^*, Theorem 4.1.1 implies that X^* has a dual LUR norm if and only if it has $P(\|.\|, X)$. The following result shows that for dual spaces "slices" are no longer necessary: there exists an equivalent dual LUR norm on X^* if and only if it has $P(\|.\|, w^*)$. Thus the renormability by a dual LUR norm is a topological property.

Theorem 4.1.6 *Let X^* be a dual Banach space. The following conditions are equivalent:*

i) X^* admits an equivalent dual LUR norm.

ii) X^* (resp. S_{X^*}) has $P(\|.\|, w^*)$.

iii) X^* is dual-descriptive.

iv) X is Asplund and S_{X^*} is hereditarily weakly θ-refinable.

v) X^* has $P(w, w^*)$ with w^*-Borel sets.

Proof. We already know that $i) \Rightarrow ii) \Rightarrow iii)$ and $i) \Rightarrow v)$.
$ii) \Rightarrow i)$ If a dual Banach space X^* has $P(\|.\|, w^*)$, then X^* has the Radon-Nikodym property, see Corollary 1.3.7 and the comments below. We know by Theorem 2.2.2 that there is a w^*-lower semicontinuous real function F on X^* with $\|.\| \leq F(.) \leq 3\|.\|$ such that the norm and the w^*-topology coincide on the set $S = \{x^* \in X^* : F(x^*) = 1\}$. Let $K = \{x^* \in X^* : F(x^*) \leq 1\}$. Since X^* has the Radon-Nikodym property, $\overline{co}^{\|\cdot\|}(K)$ will be a w^*-compact set, symmetric and with nonempty norm interior, that is the unit ball of some

equivalent dual norm on X^*. Without loss of generality we can suppose X^* endowed with that norm, namely $B_{X^*} = \overline{co}^{\|\cdot\|}(K)$. We will show that the norm and the w^*-topology coincide on S_{X^*}.

Suppose not, that is, there is some $\varepsilon > 0$ and some net (x^*_ω) in S_{X^*} w^*-converging to a point $x^* \in S_{X^*}$ such that $\|x^*_\omega - x^*\| > \varepsilon$. Take Radon probabilities μ_ω on K such that $x^*_\omega = \int_K \mathbb{I}\, d\mu_\omega$ (integrals are taken in the sense of Bochner, see [11]). Without loss of generality we can suppose that (μ_ω) converges in $(C(K)^*, w^*)$ to a Radon probability μ on K. We must have that $x^* = \int_K \mathbb{I}\, d\mu$.

Since $\|x^*_\omega\| = \|x^*\| = 1$, we have that μ_ω and μ are supported by $S_{X^*} \cap K \subset S$. We can take disjoint norm compact sets $K_i \subset S$ for $i = 1, \ldots, n$ with diameter less than $\varepsilon/7$ such that $\mu(\bigcup_{i=1}^n K_i) > 1 - \varepsilon/12$. We can take a norm compact set $K_0 \subset S$ disjoint from $\bigcup_{i=1}^n K_i$ such that $\mu(\bigcup_{i=0}^n K_i) > 1 - \varepsilon/12n$. Take disjoint norm open sets V_i for $i = 0, \ldots, n$ with $K_i \subset V_i$ and the diameter of V_i for $i = 1, \ldots, n$ less that $\varepsilon/6$. Since the norm and the w^*-topology coincide on S, we can take w^*-open sets U_i such that $U_i \cap S = V_i \cap S$.

By Urysohn's Lemma, we can take w^*-continuous functions f_i for $i = 0, \ldots, n$ from B_{X^*} to $[0, 1]$ such that $f_i|_{K_i} = 1$ and $f_i|_{X^* \setminus U_i} = 0$. Since $\int_K f_i d\mu_\omega$ converges to $\int_K f_i d\mu \geq \mu(K_i)$ for $i = 0, \ldots, n$ we will have for ω big enough that $\mu_\omega(V_i) = \mu_\omega(U_i) \geq \int_K f_i d\mu_\omega > \mu(K_i) - \varepsilon/12n^2$ for $i = 0, \ldots, n$. On the other hand, it must be $\mu_\omega(V_i) < \mu(K_i) + \varepsilon/6n$. If not, then $\mu_\omega(V_j) \geq \mu(K_j) + \varepsilon/6n$ for some j. Summing with the above inequalities for $i \neq j$ we will have $\mu_\omega(\bigcup_{i=0}^n V_i) > \mu(\bigcup_{i=0}^n K_i) + \varepsilon/6n - n\varepsilon/12n^2 > 1 - \varepsilon/12n + \varepsilon/6n - \varepsilon/12n = 1$ which is a contradiction. Thus we have that $|\mu_\omega(V_i) - \mu(K_i)| < \varepsilon/6n$ and $\mu_\omega(\bigcup_{i=1}^n V_i) > 1 - \varepsilon/6$.

Fix any $i = 1, \ldots, n$. We can take points $x^*_1, x^*_2 \in \overline{co}^{\|\cdot\|}(V_i)$ such that $\mu(K_i)x^*_1 = \int_{K_i} \mathbb{I}\, d\mu$ and $\mu_\omega(V_i)x^*_2 = \int_{V_i} \mathbb{I}\, d\mu_\omega$. Since the diameter of V_i is less than $\varepsilon/6$, then $\|x^*_1 - x^*_2\| \leq \varepsilon/6$. We have that

$$\| \int_{V_i} \mathbb{I}\, d\mu_\omega - \int_{K_i} \mathbb{I}\, d\mu\| = \|\mu_\omega(V_i)x^*_2 - \mu(K_i)x^*_1\|$$

$$\leq |\mu_\omega(V_i) - \mu(K_i)|.\|x^*_2\| + \mu(K_i)\|x^*_1 - x^*_2\| \leq (1/n + \mu(K_i))(\varepsilon/6)$$

We will show that $\|x^*_\omega - x^*\| < \varepsilon$ to get the final contradiction

$$\|x^*_\omega - x^*\| = \| \int_K \mathbb{I}\, d\mu_\omega - \int_K \mathbb{I}\, d\mu\|$$

$$\leq \| \int_{K\setminus\bigcup_{i=1}^n V_i} \mathbb{I}\, d\mu_\omega - \int_{K\setminus\bigcup_{i=1}^n K_i} \mathbb{I}\, d\mu\| + \sum_{i=1}^n \| \int_{V_i} \mathbb{I}\, d\mu_\omega - \int_{K_i} \mathbb{I}\, d\mu\|$$

$$\leq \varepsilon/6 + \varepsilon/12 + \sum_{i=1}^n (1/n + \mu(K_i))(\varepsilon/6) < \varepsilon$$

This shows that the norm $\|.\|$ is w^*-Kadec. By Corollary 3.3.9, X^* has an equivalent dual LUR norm.
iii) $\Rightarrow$ *iv)* If X^* is dual-descriptive, then X is Asplund. Since (X^*, w^*) has a σ-isolated network, it is hereditarily weakly θ-refinable.
iv) $\Rightarrow$ *ii)* If X is Asplund, then (X^*, w^*) is σ-fragmentable by the norm [36]. Then apply Corollary 1.5.17 to deduce that S_{X^*} has $P(\|.\|, w^*)$.
v) $\Rightarrow$ *ii)* Property $P(w, w^*)$ with w^*-Borel sets implies that

$$Univ(X^*, w^*) = Univ(X^*, w) = Univ(X^*, \|.\|)$$

by Corollary 1.3.7. This implies that X^* has the Radon-Nikodym property by [12]. So X^* is LUR renormable by [16], and it has $P(\|.\|, w)$. This last property together with $P(w, w^*)$ gives that X^* has $P(\|.\|, w^*)$.

It is well known that if a Banach space X has an equivalent Fréchet differentiable norm, then it is an Asplund space [10]. A norm on X is Fréchet differentiable if the dual norm in X^* is LUR, see [10]. The dual of an Asplund space, that is, a dual Banach space with the Radon-Nikodym property always has an equivalent LUR norm [16], but this norm is not a dual norm in general.

Corollary 4.1.7 *Let X be an Asplund Banach space such that (S_{X^*}, w^*) is hereditarily weakly θ-refinable. Then X has an equivalent Fréchet differentiable norm.*

Let X be an Asplund Banach space. We shall consider the following construction on its dual X^*. For any weak*-compact convex subset $B \subset X^*$ and $\varepsilon > 0$ take

$$(B)'_\varepsilon = \{x^* \in B : \ \forall U \in w^*, \ x^* \in U, \ diam(B \cap U) > \varepsilon\}$$

Now we define by transfinite induction the sets (B_ε^α) as follows

$$B_\varepsilon^0 = B_{X^*}$$

$$B_\varepsilon^{\alpha+1} = (B_\varepsilon^\alpha)'_\varepsilon$$

and for α a limit ordinal

$$B_\varepsilon^\alpha = \bigcap_{\beta<\alpha} B_\varepsilon^\beta$$

Now take $Sz(X,\varepsilon) = \inf\{\alpha : \ B_\varepsilon^\alpha = \emptyset$, and $Sz(X) = \sup_{\varepsilon>0} \delta_Z(X,\varepsilon)$. The ordinal number $Sz(X)$ is called the Szlenk index of X. The following result has proved by Lancien [47] using a Kunen-Martin type argument.

Corollary 4.1.8 (Lancien) *Let X be an Asplund Banach space such that $Sz(X) < \omega_1$. Then X^* has an equivalent dual LUR norm.*

4.2 Symmetrics

We shall need the following generalization of metric spaces.

Definition 4.2.1 *We say that a function $\rho : X \times X \to \mathbb{R}$ is a symmetric if verifies the following conditions:*

i) $\rho(x,y) \geq 0$

ii) $\rho(x,y) = 0$ *implies that* $x = y$

iii) $\rho(x,y) = \rho(y,x)$

A topology τ on X is said to be symmetrizable if there is a symmetric ρ such that for every point $x \in X$, the family of "open balls" of radius $r > 0$

$$B_\rho(x, r) = \{y \in X : \ \rho(x, y) < r\}$$

is a basis of τ at x.
A topological space endowed with a symmetric which symmetrizes the topology will be called semi-metric space.

Since a symmetric does not satisfy the triangular inequality, the balls are not necessary open. Moreover, given a symmetric ρ on a set, there does not necessary exist a topology which is symmetrized by ρ, see [23].

The following theorem is a version for maps of results from [50, 51].

Theorem 4.2.2 *Let (X, ρ) be a semi-metric space, let (Y, d) be a metric space and let $f : X \to Y$ be a map. Consider the following statements:*

i) f has property P.

ii) For every $x \in X$ there is separable subset $S(x) \subset Y$ with the following property: for every sequence (x_n) converging to a point $x \in X$ then

$$f(x) \in \overline{\bigcup_{n=1}^{\infty} S(x_n)}^{d}$$

Then ii) $\Rightarrow$ i). If ρ is a metric, then i) and ii) are equivalent.

Proof. *ii) $\Rightarrow$ i)* Fix $n \in \mathbb{N}$ and consider points x_α and ρ-open sets U_α such that $x_\alpha \in U_\alpha \subset B_\rho(x_\alpha, 1/n)$ and such that $\bigcup_\alpha U_\alpha = X$. For every $m > n$ define

$$D^m_\alpha = \{x \in U_\alpha : \ B_\rho(x, 1/m) \subset U_\alpha\}$$

Let A^m_α defined by

$$A^m_\alpha = D^m_\alpha \setminus \bigcup_{\beta < \alpha} U^n_\beta$$

We claim that $\{A^m_\alpha\}_\alpha$ is a isolated family. Indeed, if $\alpha < \beta$, $x \in A^m_\alpha$ and $y \in A^m_\beta$ then $x \in D^m_\alpha$ and $y \notin U_\alpha$ and thus $\rho(x, y) \geq 1/m$.
Now, we may assume that $S(x)$ is countable for every $x \in X$. Put $S(x) = (y_k(x))_k$. For every m, k define a function $f_{n,m,k}$ on the set

$$dom(f_{n,m,k}) = \bigcup_\alpha A^m_\alpha$$

defined by $f_{n,m,k}(x) = y_k(x_\alpha)$ if $x \in A^m_\alpha$. Since $f_{n,m,k}$ is locally constant it is continuous on its domain. From the construction of $f_{n,m,k}$ and the hypothesis, it follows easily that

$$f(x) \in \overline{\{f_{n,m,k}(x) : x \in dom(f_{n,m,k})\}}^d$$

for every $x \in X$. By Proposition 1.1.7, we obtain that f has property P.
$i) \Rightarrow ii)$ We assume that ρ is a metric. Let $(A_n) \subset X$ a sequence verifiying the property P with f. Fix $x \in X$ and take, if possible, a point $x_{n,m} \in A_n \cap B(x, 1/m)$ and put $S(x) = \bigcup_{n,m} f(x_{n,m})$. We claim that the sets $S(x)$ satisfies statement $ii)$. Suppose that (x_k) converges to x. Fix $\varepsilon > 0$ and take $n \in \mathbb{N}$ and $\delta > 0$ such that $x \in A_n \cap B(x, \delta)$ and $f(A_n \cap B(x, \delta)) \subset B(f(x), \varepsilon)$. Take k big enough to have $\rho(x, x_k) < \delta/3$. Then there is $x^k_{n,m} \in A_n \cap B(x_k, 1/m)$ with $\delta/3 < 1/m < 2\delta/3$. Since $x^k_{n,m} \in A_n \cap B(x_k, 2\delta/3) \subset A_n \cap B(x, \delta)$ we have that $d(f(x), f(x^k_{n,m})) < \varepsilon$. This ends the proof.

Remark 4.2.3 *If (Y, d) is a metric vector space, it is possible to replace statement $ii)$ of the theorem by the weaker following one:*

iii) For every $x \in X$ there is separable subset $S(x) \subset Y$ with the following property: for every sequence (x_n) converging to a point $x \in X$, we have

$$f(x) \in \overline{span}^d\{\bigcup_{n=1}^{\infty} S(x_n)\}$$

From the proof of the preceding theorem it is easy to obtain the following, see also [5, 23].

Proposition 4.2.4 *If (X, τ) is a symmetrizable topological space, then it is hereditarily weakly θ-refinable, that is, every family of open sets has a σ-isolated refinement.*

4.3 WLUR norms

The following theorem gives suficient conditions for a Banach space to be LUR renormable in terms of a particular symmetric.

Theorem 4.3.1 *Let* $(X, \|.\|)$ *be a rotund Banach space, let* $Z \subset X^*$ *be a norming subspace and let* ρ *be the symmetric defined by the formula*

$$\rho(x, y) = \frac{\|x\|^2 + \|y\|^2}{2} - \|\frac{x + y}{2}\|^2$$

Suppose that ρ *symmetrizes some topology on* X *(resp.* S_X*). Then* X *admits an equivalent* $\sigma(X, Z)$*-lower semicontinuous LUR norm if one of the two following conditions is satified:*

a) If $x = \rho\text{-}\lim_n x_n$*, then* $x \in \overline{span}^{\|\cdot\|}(\{x_n\}))$

b) (X, ρ) *(resp.* (S_X, ρ)*) is* σ*-fragmentable by* $\|.\|$

Proof. An easy calculation shows that $\rho(x, y) \geq (\|x\| - \|y\|)^2/4$ so the rotundity of $\|.\|$ implies that ρ is a symmetric. Fix $x \in X$ and $\varepsilon > 0$. Find two rational numbers $r > s > 0$ such that $r > \|x\| > s$ and $r^2 - s^2 < \varepsilon$. By the Hahn-Banach Theorem, we can find $H \in \mathbb{H}(Z)$ such that $x \in H$ and $B[0, s] \cap H = \emptyset$. Now, if $y \in B[0, r] \cap H$, then $\|\frac{x+y}{2}\| > s$, so $\rho(x, y) < \varepsilon$. This proves that X has $P(\rho, Z)$.
There is an equivalent $\sigma(X, Z)$-lsc LUR norm on X if and only if X has $P(\|.\|, Z)$, so it is enough to show that X has $P(\|.\|, \rho)$. In case a), Theorem 4.2.2 implies that X has $P(\|.\|, \rho)$. In case b), by Proposition 4.2.4 and Corollary 1.5.17 we have that (X, ρ) has $\|.\|$-SLD and this is equivalent to $P(\|.\|, \rho)$. This ends the proof of the theorem.

Remark 4.3.2 *Note that if* $\|.\|$ *is already a LUR norm, then the symmetric defined in the theorem satisfies conditions* $a)$ *and* $b)$ *trivially, because* ρ *symmetrizes the norm topology.*

In order to give some applications of Theorem 4.3.1, we introduce the following notion of rotundity.

Definition 4.3.3 *A norm $\|.\|$ on X is said to be τ-locally uniformly rotund (τ-LUR) if it is τ-lower semicontinuous and for every x_n, x with $\|x_n\| = \|x\| = 1$ and $\lim_n \|x + x_n\| = 2$, then τ-$\lim_n x_n = x$.*

When τ is the norm we obtain the LUR norms. In case of the weak or the weak* topology we write WLUR or W*LUR respectively.

It is easy to see that the symmetric ρ of Theorem 4.3.1 for a $\sigma(X, Z)$-LUR norm symmetrizes the topology $\sigma(X, Z)$ on the unit sphere. Thus, applying a) of Theorem 4.3.1 we arrive to the main result in [51].

Corollary 4.3.4 (Moltó, Orihuela, Troyanski & Valdivia) *Let X be a WLUR Banach space. Then X is LUR renormable.*

Corollary 4.3.5 *Let X be a Banach space such that X^* is W*LUR. If either X is Asplund or has the Grothendieck property, then X^* admits an equivalent dual LUR norm.*

Proof. A Banach space X is Asplund if and only if (X^*, w^*) is σ-fragmentable by the norm [36], thus we can apply condition b) of Theorem 4.3.1. If X has the Grothendieck property, then the w^*-convergent sequences in X^* are w-convergent, so the result follows from condition a) of Theorem 4.3.1.

We shall use that the dual of a WCD Banach space has an equivalent W*LUR norm [10, p. 288] to give an alternative proof of the following well known result, see [10] for details.

Corollary 4.3.6 (Fabian & Troyanski) *Let X be an Asplund space. If either X or X^* is WCD, then X admits a LUR norm such that the dual norm is also LUR.*

Proof. If X is an Asplund WCD Banach space, then X is renormable LUR and X^* has an equivalent dual LUR norm by Corollary 4.3.5. If X^* is WCD, then its bidual X^{**} has an equivalent W*LUR norm. The restriction of this norm to X is a WLUR norm, and thus X is LUR renormable. On the other hand X^* admits a dual LUR norm, Corollary 3.3.12. The result follows from Asplund's averaging technique [10, p. 55].

Since bounded pointwise convergent sequences in $C(K)$ spaces are also weakly convergent, we obtain the following.

Corollary 4.3.7 *Let K be a Hausdorff compact space. If $C(K)$ has an equivalent $t_p(K)$-LUR norm, then it has an equivalent $t_p(K)$-lower semicontinuous LUR norm.*

The following answers a question of Haydon [30] and generalizes a result Moltó, Orihuela and Troyanski [50].

Theorem 4.3.8 *Let K be a compact space and let $K_n \subset K$ be compact subsets such that every space $C(K_n)$ has an equivalent (pointwise-lower semicontinuous) LUR norm. If there is a lower semicontinuous metric d on K such that $K = \overline{\bigcup_{n=1}^{\infty} K_n}^d$, then $C(K)$ has an equivalent (pointwise-lower semicontinuous) LUR norm.*

Proof. Let $\|.\|_n$ an equivalent LUR norm on $C(K_n)$ bounded by the supremun norm. Define the oscilation functions as follows

$$O_n(f) = \sup\{|f(x) - f(y)| : x, y \in K, d(x, y) \leq 1/n\}$$

and consider the equivalent norm $|||.|||$ on $C(K)$ defined by the formula

$$|||f|||^2 = \|f\|^2 + \sum_{n=1}^{\infty} 2^{-n} \|f|_{K_n}\|_n^2 + \sum_{n=1}^{\infty} 2^{-n} O_n(f)^2$$

If we prove that $|||.|||$ is a WLUR norm, then the result will follow from Corollary 4.3.4. To see that, suppose that $|||f_k||| = |||f|||$ and $\lim_k |||f_k + f||| = 2|||f|||$. Lemma 3.1.3 gives that (f_k) converges to f uniformly on every K_n. We claim that $(f_k(x))$ converges to $f(x)$ for every $x \in K$. Fix $\varepsilon > 0$ and take n big enough to have that $O_n(f) < \varepsilon/3$ (this is posible because continuous functions on K are d-uniformly continuous, see the proof of Theorem 2.3.9). Now take $y \in \bigcup_{m=1}^{\infty} K_m$ such that $d(x,y) < 1/n$. If k is big enough, then $O_n(f_k) < \varepsilon/3$ and $|f_k(y) - f(y)| < \varepsilon/3$. We have that

$$|f_k(x) - f(x)| \leq |f_k(x) - f_k(y)| + |f_k(y) - f(y)| + |f(y) - f(x)| < \varepsilon$$

and this end the proof of the claim. Thus we have that (f_k) converges to f weakly by Lebesgue's Theorem and the norm $|||.|||$ is WLUR. If every norm $\|.\|_n$ is $t_p(K)$-lsc, then the norm $|||.|||$ is also $t_p(K)$-lsc. The proof shows in this case that $|||.|||$ is $t_p(K)$-LUR, and then $C(K)$ has an equivalent $t_p(K)$-lsc LUR norm by Corollary 4.3.7.

Corollary 4.3.9 [1] *Let $K \subset X^*$ be a norm fragmentable w^*-compact subset and take $H = \overline{conv}^{w^*}(K)$. If every $C(K^n)$ has an equivalent (pointwise-lower semicontinuous) LUR norm for every*[2] *$n \in \mathbb{N}$, then $C(H)$ has an equivalent (pointwise-lower semicontinuous) LUR norm.*

Proof. If L is a compact Hausdorff space such that $C(L)$ has an equivalent (pointwise lower semicontinuous) LUR norm, then any continuous image of L has the same property. Convex combinations of points of K with rational coefficients, which are norm dense in H, can be regarded as a countable union of continuous images of K^n's, see proof of Lemma 2.4.13, which are norm dense by the fragmentability.

Lemma 4.3.10 *Let X be a Banach space and $Z \subset X^*$ a norming subspace. Suppose that $\|.\|$ is $\sigma(X,Z)$-LUR. Then the symmetric*

$$\rho(x,y) = \frac{\|x\|^2 + \|y\|^2}{2} - \|\frac{x+y}{2}\|^2$$

symmetrizes the topology $\tau = top(\sigma(X,Z), \{B(0,r) : r > 0\})$

[1] The original version of this Corollary was wrong or, at least, not provable with the given arguments

[2] Now we know that it is enough to ask that for $n = 1$ thanks to N. Ribarska, A stability property for locally uniformly rotund renorming, *J. Math. Anal. Appl.* 350 (2009), 811–828.

Proof. A net (x_ω) is τ-convergent to a point $x \in X$ if and only if it is $\sigma(X,Z)$-convergent to x and $\lim_\omega \|x_\omega\| = \|x\|$. This is equivalent to $\lim_\omega \|x_\omega\| = \|x\|$ and $\lim_\omega \|x + x_\omega\| = 2\|x\|$, using that $\|.\|$ is $\sigma(X,Z)$-LUR. By Lemma 3.1.3 we obtain the equivalence to $\lim_\omega \rho(x_\omega, x) = 0$.

Theorem 4.3.11 *If X^* is a dual Banach space having an equivalent W*LUR norm, then (B_{X^*}, w^*) is a descriptive compact space. In particular, if K is a Hausdorff compact space such that $C(K)^*$ has an equivalent W*LUR norm, then K is descriptive.*

Proof. We can suppose that X^* is already endowed with such a W*LUR norm. By a result of Ribarska, see [15], (B_{X^*}, w^*) is fragmentable by some metric d. The topology $\tau = top(w^*, \{B(0,r) : r > 0\})$ on B_{X^*} is symmetrizable after Lemma 4.3.10. Since (B_{X^*}, τ) is fragmentable by d, we obtain from Remark 4.2.4 and Corollary 1.5.17 that B_{X^*} has $P(d, \tau)$. On the other hand, B_{X^*} has $P(\tau, w^*)$, and thus X^* has $P(d, w^*)$.

From the W*LUR renorming of X^* when X is WCD Banach space, we can deduce that Gul'ko compact spaces are descriptive, Example 2.4.3.

4.4 Transfer technique

We give a adaptation of the transfer technique developed in [50] which involves topologies of type $\sigma(X,Z)$.

Theorem 4.4.1 *Let X and Y be Banach spaces, let $Z \subset X^*$ be a quasi-norming subspace and let $T : X \to Y$ be a bounded one-to-one operator. Suppose that Y is LUR and that there is a norming subspace $W \subset Y^*$ such that T is $\sigma(X,Z)$-$\sigma(Y,W)$-continuous. If the map*

$$T^{-1} : (T(X), \|.\|) \longrightarrow (X, \|.\|)$$

has the property P, then X admits an equivalent $\sigma(X,Z)$-lower semicontinuous LUR norm.

Proof. Let $\|.\|_T$ the norm on X defined by $\|x\|_T = \|T(x)\|$. Suppose that $W \subset Y^*$ is a norming subspace such that T is $\sigma(X,Z)$-$\sigma(Y,W)$-continuous. Then Y has $P(\|.\|, W)$ with some sequence $A_n \subset Y$. Since the elements of $\mathbb{H}(W)$ can be lifted by T to elements of $\mathbb{H}(Z)$, we deduce that X has $P(\|.\|_T, Z)$. The hypothesis about T^{-1} implies that X also has $P(\|.\|, \|.\|_T)$. Then X has $P(\|.\|, Z)$, so it is renormable by a $\sigma(X,Z)$-lsc LUR norm.

For the applications of the transfer method, the following proposition from [50] will be very usefull.

Proposition 4.4.2 *Let X and Y be Banach spaces and let $T : X \to Y$ be a bounded one-to-one operator. Then the map*

$$T^{-1} : (T(X), \|.\|) \longrightarrow (X, \|.\|)$$

has the property P if and only if, for every $x \in X$, there is a separable subspace $S(x) \subset X$ with the following property: for every sequence (x_n) such that $T(x_n)$ converges to $T(x)$, then

$$x \in \overline{span}^{\|\cdot\|}\{\bigcup_{n=1}^{\infty} S(x_n)\}$$

Proof. It follows from Theorem 4.2.2 and Remark 4.2.3.

The hypothesis of the proposition is fulfilled, in particular, if (x_n) converges weakly to x whenever that $(T(x_n))$ converges to $T(x)$.

Once we know that $c_0(\Gamma)$ has a pointwise-lsc LUR norm, it can be used to renorm those Banach spaces X which injects into $c_0(\Gamma)$ in a reasonable way. The following is a version of a result of Troyanski [66].

Proposition 4.4.3 *Let X be a Banach space and let $T : X \to c_0(\Gamma)$ be a bounded one-to-one linear operator. For every $\Gamma_0 \subset \Gamma$, let $X(\Gamma_0)$ be the subspace of those elements $x \in X$ such that $supp(T(x)) \subset \Gamma_0$. Suppose that $X(\Gamma_0)$ is separable when Γ_0 is countable. If for every $x \in X$ there is a countable subset $\Gamma(x) \subset \Gamma$ verifying the following properties* *i) supp*$(T(x)) \subset \Gamma(x)$

ii) for every sequence $(x_n) \subset X$

$$X(\bigcup_{n=1}^{\infty} \Gamma(x_n)) \subset \overline{span}^{\|\cdot\|}\{\bigcup_{n=1}^{\infty} X(\Gamma(x_n))\}$$

then X *is renormable* LUR*. Moreover, if every coordinate of* T *is* $\sigma(X,Z)$*-continuous for some quasi-norming* $Z \subset X^*$*, then* X *admits a* $\sigma(X,Z)$*-lower semicontinuous* LUR *norm.*

Proof. Take for every $x \in X$ the separable subspace $S(x) = X(\Gamma(x))$. Conditions $i)$ and $ii)$ imply that $x \in \overline{span}^{\|\cdot\|}\{\bigcup_{n=1}^{\infty} S(x_n)\}$. Then the subspaces $S(x)$ satify Proposition 4.4.2 and X is renormable LUR (with lower semicontinuity) by Theorem 4.4.1.

As an inmediate corollary we obtain that $l^1(\Gamma)$ admits a dual LUR norm. This is a particular case of a Banach space having a strong Markushevich basis which are LUR renormable by the same argument. More generally we have the following.

Corollary 4.4.4 (Troyanski) *A Banach space having a separable projective resolution of the identity is LUR renormable.*

Proof. There is a long sequence of projections $T_\alpha : X \to X$ with $\alpha \in [\omega, \gamma)$ such that $T_\alpha(X)$ is separable, $(\|T_\alpha(x)\|) \in c_0[\omega_0, \gamma)$ and every $x \in X$ lies in $\overline{span}^{\|\cdot\|}\{T_\alpha(x)\}$, see for example [15]. We can take countable disjoint sets Γ_α and equibounded linear injections $j_\alpha : T_\alpha(X) \to c_0(\Gamma_\alpha)$. Take $\Gamma = \bigcup_\alpha \Gamma_\alpha$ and define a linear operator $T : X \to c_0(\Gamma)$ by gluing the maps $j_\alpha \circ T_\alpha$. Then the hypothesis of Corollary 4.4.3 are fulfilled if we take as $\Gamma(x)$ the union of those Γ_α such that $T_\alpha(x) \neq 0$.

The result about the LUR renormability of a Banach space having “many” projections of Zizler [68] can also be deduced from the transfer method of

Moltó, Orihuela and Troyanski. See [41] for more details.

The following a transfer result for LUR renorming of Godefroy-Troyanski-Whitfield-Zizler [22, 10]. A topological version of it is Theorem 2.4.10.

Theorem 4.4.5 (Godefroy, Troyanski, Whitfield & Zizler) *Let X be a Banach space, let $Z \subset X^*$ be a quasi-norming subset, let Y^* be a dual Banach space having a dual LUR norm and let $T : Y^* \to X$ be a bounded linear operator w^*-$\sigma(X, Z)$ continuous. Then X has an equivalent $\sigma(X, Z)$-lower semicontinuous norm which is LUR at the points of $\overline{T(Y^*)}^{\|\cdot\|}$.*

Notice the differences between Theorem 4.4.5 and Theorem 4.4.2. We shall prove the following interpolation result in the spirit of Davis-Figiel-Johnson-Pelczyński, see for example [8], that can be regarded as a reciproque of Theorem 4.4.5.

Theorem 4.4.6 *Let X a Banach space, let $Z \subset X^*$ be a quasi-norming subset and let $K \subset X$ be a bounded $\sigma(X, Z)$-compact subset which has $P(\|.\|, \sigma(X, Z))$. Then there exists a dual Banach space Y^* having a dual LUR norm and a bounded one-to-one linear operator $T : Y^* \to X$ which is w^*-$\sigma(X, Z)$ continuous such that $K \subset T(B_{Y^*})$.*

Proof. After Lemma 2.4.13 we know that $K_0 = \overline{aco}^{\|\cdot\|}(K)$ is an absolutely convex compact set with $P(\|.\|, \sigma(X, Z))$. Thus K_0 is fragmented by the norm. Following Namioka [52], there is an Asplund space Y and a bounded injective linear operator $T : Y^* \to X$ which is w^*-$\sigma(X, Z)$ continuous such that $K_0 \subset T(B_{Y^*}) \subset 2^n K_0 + B[0, 1/2^n]$ for every $n \in \mathbb{N}$. By Theorem 2.4.10 we have that $T(B_{Y^*})$ will be a descriptive $\sigma(X, Z)$-compact subset of X. Since T is an homeomorphism when restricted to B_{Y^*}, we deduce that (B_{Y^*}, w^*) is hereditarily weakly θ-refinable. Thus Y^* has an equivalent dual LUR norm by Theorem 4.1.6.

Corollary 4.4.7 *Let X be a Banach space, let $Z \subset X^*$ be a quasi-norming subset and let $K \subset X$ be a bounded $\sigma(X, Z)$-compact subset which has*

$P(\|.\|, \sigma(X,Z))$. Then X has an equivalent $\sigma(X,Z)$-lower semicontinuous norm which is LUR at the points of $\overline{span}^{\|\cdot\|}(K)$,

Proof. Apply the Theorems 4.4.6 and 4.4.5.

The following extends a well known result of Deville [9], see also [10].

Corollary 4.4.8 *Let K be a Hausdorff compact space. Then $C(K)^*$ has an equivalent dual LUR norm if and only if K is a countable union of relatively discrete subsets.*

Proof. Suppose that $C(K)^*$ has an equivalent dual LUR norm. Then $C(K)^*$ has $P(\|.\|, w^*)$, and in particular, K has $\|.\|$-SLD. The decomposition of Definition 1.5.5 for $\varepsilon < 1$ will give K as a countable union of relatively discrete subsets. Conversely, assume that K is a countable union of relatively discrete subsets. Then K has d-SLD where d is the discrete metric. Since K is d-fragmentable, then K must be scattered. Regarding K as a subset of $C(K)^*$, it has $P(\|.\|, w^*)$ and $C(K)^* = \overline{span}^{\|\cdot\|}(K)$ has an equivalent dual LUR norm by Corollary 4.4.7.

A Hausdorff compact space K satisfies the hypothesis of Corollary 4.4.8 if and only if K is scattered and hereditarily weakly θ-refinable. Recall that a Hausdorff compact space is Namioka-Phelps (see Definition 2.4.11) if and only if it is Radon-Nikodym and hereditarily weakly θ-refinable.

Corollary 4.4.9 *A Hausdorff compact space K is Namioka-Phelps if and only if it is homeomorphic to a w^*-compact subset of a dual Banach space which has a dual LUR norm.*

Proof. Let d be a lower semicontinuous metric on (K, τ) such that K has $P(d, \tau)$. There is a dual space X^* containing K as w^*-compact subset in such a way that the metric d is induced by the norm [34]. Then the result will follow from Theorem 4.4.6.

Theorem 4.4.10 *Let K be a Namioka-Phelps compact space. Then $C(K)^*$ has an equivalent W*LUR norm. In particular, $C(K)$ has an equivalent Gâteaux differentiable norm.*

Proof. The proof of [52, Theorem 5.6] shows that if K is a Radon-Nikodym compact, then there is a dual Banach space X^* and a bounded injective w^*-w^*-continuous linear operator $T : C(K)^* \to X^*$ such that $T(K)$ is fragmented by the norm $\|.\|$ of X^*. If K is Namioka-Phelps we have that $T(K)$ has $P(\|.\|, w^*)$. By Corollary 4.4.7, we can suppose X^* that is endowed with a dual norm which is LUR at the points of $T(C(K)^*)$. Define an equivalent dual norm $|||.|||$ on $C(K)^*$ by the formula $|||x|||^2 = \|x\|^2 + \|T(x)\|^2$. We claim that $|||.|||$ is W*LUR. To see that, take points x, x_n in $C(K)^*$ with $|||x||| = |||x_n||| = 1$ and $\lim_n |||x_n + x||| = 2$. By Lemma 3.1.3, we have that $\lim_n \|T(x_n)\| = \|T(x)\|$ and $\lim_n \|T(x_n) + T(x)\| = 2\|T(x)\|$. Since $\|.\|$ is LUR at $T(x)$, we have that $\lim_n \|T(x_n) - T(x)\| = 0$. In particular, $T(x_n)$ is w^*-convergent to $T(x)$, and then (x_n) is w^*-convergent to x because of the w^*-continuity of T^{-1} on $T(B_{C(K)^*})$.

Problem 4.4.11 *We do not know if $C(K)^*$ has an equivalent W*LUR norm for any descriptive compact space K.*[3]

4.5 Partitions of unity

In this section we show that Banach spaces with a LUR norm or the Radon-Nikodym property have a σ-discrete basis (see Definition 1.1.8) of the norm topology made up of convex sets. We apply this to the existence of smooth partitions of unity.

Theorem 4.5.1 *Let X be a Banach space. Suppose that X has either an equivalent LUR norm or the Radon-Nikodym property. Then the norm topology of X has a σ-discrete basis made up of convex sets.*

[3]The answer is yes: M. RAJA, Weak* locally uniformly rotund norms and descriptive compact spaces, *J. Funct. Anal.* 197 (2003), 1–13.

Proof. Suppose that the norm of X is already LUR. Fix $\varepsilon > 0$ and define by transfinite induction a a family of convex sets $\{B_\alpha\}$ as follows: $B_0 = B_X$, $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$ if α is a limit ordinal and

$$B_{\alpha+1} = B_\alpha \setminus \{x \in X : x_\alpha^*(x) > a_\alpha\}$$

where $x_\alpha^* \in S_{X^*}$ and $a_\alpha \in \mathbb{R}$ are such that

$$diam(B_X \cap \{x \in X : x_\alpha^*(x) > a_\alpha\}) < \varepsilon$$

The process ends when $B_\gamma \subset B(0,1)$ for some ordinal γ.
Take $\delta > 0$. We define convex sets

$$C(\alpha, \varepsilon, \delta) = B_\alpha \cap \{x \in X : x_\alpha^*(x) \geq a_\alpha + \delta\}$$

We have $S_X \subset \bigcup_{\delta>0} \bigcup_{\alpha<\gamma} C(\alpha, \varepsilon, \delta)$ for every $\varepsilon > 0$. We claim that the family $\{C(\alpha, \varepsilon, \delta) : \alpha < \gamma\}$ is δ-discrete. Indeed, if $\alpha < \beta$ then for every $x \in C(\alpha, \varepsilon, \delta)$ and every $y \in C(\beta, \varepsilon, \delta)$ we have $x_\alpha^*(x) \geq a_\alpha + \delta$ and $x_\alpha^*(y) \leq a_\alpha$ since $y \in B_{\alpha+1}$. We deduce that $x_\alpha^*(x - y) \geq \delta$ and thus $\|x - y\| \geq \delta$.
Take $m, n \in \mathbb{N}$ with $n > 3m$. Put $\varepsilon = 1/m$ and $\delta = 3/n$. Now, we define the open convex sets $U(\alpha, m, n) = C(\alpha, \varepsilon, \delta) + B(0, 1/n)$. A calculation shows that $\{U(\alpha, m, n) : \alpha < \gamma\}$ is $1/n$-discrete family of sets of diameter less than $1/m$. We claim that

$$\{rU(\alpha, m, n) : \ \alpha < \gamma, \ r \in \mathbb{Q}^+ \ m, n \in \mathbb{N}\} \cup \{B(0, 1/n) : \ n \in \mathbb{N}\}$$

is a σ-discrete basis of the norm topology of X. Indeed, fix $x \in X$ and $\varepsilon > 0$. We can assume that $x \neq 0$. Take $m > \|x\|/\varepsilon$. For some α and n big enough, $x/\|x\| \in U(\alpha, m, n)$. There is $r \in \mathbb{Q}$ with $0 < r < \|x\|$ such that $x/r \in U(\alpha, m, n)$. Thus $x \in rU(\alpha, m, n)$ and $diam(rU(\alpha, m, n)) < r/m \leq \varepsilon$.
If X has the Radon-Nikodym property, then every closed convex bounded set has nonempty slices of arbitrary small diameter. So we can define by transfinite induction a family of convex sets $\{B_\alpha\}$ as follows: $B_0 = B_X$, $B_\alpha = \bigcap_{\beta<\alpha} B_\beta$ if α is a limit ordinal and

$$B_{\alpha+1} = B_\alpha \setminus \{x \in X : x_\alpha^*(x) > a_\alpha\}$$

where $x^*_\alpha \in S_{X^*}$ and $a_\alpha \in \mathbb{R}$ are such that

$$diam(B_\alpha \cap \{x \in X : x^*_\alpha(x) > a_\alpha\}) < \varepsilon$$

and the process ends when $B_\gamma = \emptyset$ for some ordinal γ. The rest of the proof can be done as the LUR case.

Recall that a family of subsets is said to be locally finite if every point of the space has a neighbourhood that meets only a finite number of members of the family. A family that is a countable union of locally finite families is said to be σ-locally finite.

Problem 4.5.2 *We know no example of Banach space without a σ-locally finite basis made up with convex sets or a σ-discrete basis made up with convex sets.*[4]

Definition 4.5.3 *A collection $\{\psi_\alpha : \alpha \in \Lambda\}$ of C^k-smooth real functions on a Banach space is called a partition of unity if $\{supp(\psi_\alpha)\}_{\alpha\in\Lambda}$ is locally finite and $\sum_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for every $x \in X$.*
A Banach space X is said to have C^k-smooth partitions of unity if for any open cover $(U_i)_{i\in I}$ of X there is a C^k-smooth partition of unity $\{\psi_\alpha : \alpha \in \Lambda\}$ such that $\{supp(\psi_\alpha)\}_{\alpha\in\Lambda}$ is a refinement of $(U_i)_{i\in I}$.

Proposition 4.5.4 *Let X be a Banach space having a σ-locally finite basis made up with convex sets. If X^* admits a dual LUR norm, then X has a C^1-smooth partition of unity.*

Proof. Suppose that X is already endowed with a norm such that the dual norm is LUR. Let $\mathfrak{B}$ a σ-locally finite basis of convex sets of X. For every $B \in \mathfrak{B}$ define

$$B_n = \{x \in B : \ d(X \setminus B, x) \geq 1/n\}$$

[4]I have been asking this problem since then in many conferences. Now my question appears in an open problems book, yet it is attributed to other people.

It is easy to see that B_n is a closed convex subset of B.
A lemma of Toruńczyk [65] establishes that X has a C^1-smooth partition of unity if and only if X has a σ-locally finite basis of sets which are the form $f^{-1}(a, b)$ with f a C^1-differentiable real function. We shall construct a basis of that kind.
Since the dual norm is LUR, the function $f(x) = d(C, x)^2$ is C^1-smooth for any convex set C, see [10, p. 365]. Fix $B \in \mathfrak{B}$. Taking $n \in \mathbb{N}$ big enough to have $B_n \neq \emptyset$, we can define

$$B_n^o = \{x \in X : \ d(B_n, x)^2 < 1/n^2\}$$

We have that $B_n \subset B_n^o \subset B$. This implies that the family

$$\mathfrak{B}^o = \{B_n^o : \ B \in \mathfrak{B}, \ n \in \mathbb{N}\}$$

is a σ-locally finite basis satisfying the condition of Toruńczyk.

Corollary 4.5.5 *Let X be a Banach space such that X^* has an equivalent w^*-Kadec norm. If either a) X is LUR renormable or b) X has the Radon-Nikodym property, then X admits a C^1-smooth partition of unity.*

Case *a*) of the above corollary is due to Vanderwerff, see [10, p. 364]. His proof is different.

Remark 4.5.6 *If we ask to convex functions on X to be aproximated uniformly on bounded subsets by C^k-smooth functions in Proposition 4.5.4, instead of a dual LUR norm, we shall get the existence C^k-smooth partitions of unity.*

Remark 4.5.7 *Frontisi [19] has shown that if X has a LUR norm and if every equivalent norm in X can be aproximated by C^k-smooth functions uniformly on bounded subsets, then X has C^k-smooth partitions of unity. The result of Frontisi can be deduced from the proof of Theorem 4.5.1, by symmetrizing the construction to have the sets B_α absolutely convex. This also gives that the LUR hypothesis can be changed by the Radon-Nikodym property to obtain the same conclusion.*

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