MEAN PROJECTION AND SECTION RADII OF CONVEX BODIES

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Abstract. In this paper we introduce new series of mean outer and inner radii, which are defined as the outer (respectively, inner) radius of, either the projection of the convex body onto an i-dimensional subspace, or the i-dimensional section, $1 \leq i \leq n$, averaged over the Grassmannian manifold, and with respect to the Haar probability measure. We study some properties of these new functionals, establishing inequalities among them, as well as their relation with other measures as the volume or the quermassintegrals.

1. Introduction and notation

The setting of this paper will be the family of compact and convex sets with non-empty interior in the n-dimensional Euclidean space $\mathbb{R}^n$. We will call them convex bodies, and the family of all convex bodies will be denoted by $\mathcal{K}^n$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively. We denote the n-dimensional unit ball by B_n and its boundary, i.e., the $(n-1)$-dimensional unit sphere, by $\mathbb{S}^{n-1}$.

The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$ (or $\mathrm{vol}_n(M)$ if the distinction of the dimension is needed) and, in particular, we write $\kappa_n = \mathrm{vol}(B_n)$.

The set of all i-dimensional (linear) subspaces of $\mathbb{R}^n$ will be denoted by $\mathcal{L}_i^n$; in the same way, for $L \in \mathcal{L}_i^n$, we will write for short $\mathcal{L}_j^n(L)$, $j < i$, to denote the set of all j-dimensional (linear) planes of $\mathbb{R}^n$ which are contained in L. We will denote by $\nu_{n,i}$ the unique Haar probability measure on $\mathcal{L}_i^n$ invariant under orthogonal maps. Moreover, we will write $\nu_{L,j}$ if we work with the Grassmannian manifold $\mathcal{L}_j^n(L)$ restricted to a fixed subspace L. Further, if $K \in \mathcal{K}^n$, the orthogonal projection of K onto L will be denoted by $K|L$, and by $L^{\perp} \in \mathcal{L}_{n-i}^n$ we will represent the orthogonal complement of L. By $\mathrm{lin}\{u_1, \ldots, u_m\}$ we denote the linear hull of the vectors $u_1, \ldots, u_m$.

A convex body $K \in \mathcal{K}^n$ can be represented by real functions in several ways. Two of them are the *support function*, of great importance in the Brunn-Minkowski theory, and the *radial function*, a crucial notion in the

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dual Brunn-Minkowski theory. They are defined in the following way: for $u \in \mathbb{S}^{n-1}$, the support function of $K \in \mathcal{K}^n$ in the direction u is given by

$$h(K,u) = \max\bigl\{\langle x,u\rangle : x \in K\bigr\},$$

whereas, if K contains the origin 0, its radial function is

$$\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}.$$

The normalized average of the support function on the sphere

$$\mathrm{b}(K) = \frac{2}{n\kappa_n}\int_{\mathbb{S}^{n-1}} h(K,u)\,\mathrm{d}u,$$

where $\mathrm{d}u$ stands for the $(n-1)$-dimensional (spherical) Lebesgue measure, is the *mean width* of K ([13, p. 50]).

If $0 \in K$ and we replace in the above integral the support function by the radial function, we get the normalized average length $\ell(K)$ of chords of K through the origin, namely,

$$\ell(K) = \frac{2}{n\kappa_n}\int_{\mathbb{S}^{n-1}} \rho(K,u)\,\mathrm{d}u.$$

Within the (dual) Brunn-Minkowski theory, $(\kappa_n/2)\mathrm{b}(K)$ is (up to normalization) the $(n-1)$-st quermassintegral of K ([13, (5.57)]), whereas $(\kappa_n/2)\ell(K)$ coincides with the so-called $(n-1)$-st dual quermassintegral of K. We refer the reader to Section 3, where we will introduce these notions.

The diameter, minimal width, circumradius and inradius of a convex body K are denoted by $\mathrm{diam}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For information on these functionals and their properties we refer to [2, pp. 56–59]. If K is contained in an affine subspace A of dimension k, we write $\mathrm{r}(K;A)$ and $\omega(K;A)$ to denote the inradius and the minimal width of K calculated in the corresponding ambient space $\mathbb{R}^k$ (where we identify A with $\mathbb{R}^k$).

In this paper we introduce four new families of averages of (appropriate) radii associated to a convex body $K \in \mathcal{K}^n$.

Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1,\dots,n$, the* i-th mean outer radius *and the* i-th mean inner radius of K with respect to projections *are defined, respectively, as*

$$\widetilde{\mathrm{R}}_i^{\pi}(K) = \int_{\mathcal{L}_i^n} R(K|L)\,\mathrm{d}\nu_{n,i}(L), \qquad \widetilde{\mathrm{r}}_i^{\pi}(K) = \int_{\mathcal{L}_i^n} r(K|L;L)\,\mathrm{d}\nu_{n,i}(L).$$

Definition 1.2. *For $K \in \mathcal{K}^n$ and $i = 1,\dots,n$, the* i-th mean outer radius *and the* i-th mean inner radius of K with respect to sections *are defined, respectively, as*

$$\widetilde{\mathrm{R}}_i^{\sigma}(K) = \int_{\mathcal{L}_i^n} \max_{x\in L^{\perp}} \mathrm{R}\bigl(K\cap(x+L)\bigr)\,\mathrm{d}\nu_{n,i}(L),$$

$$\widetilde{\mathrm{r}}_i^{\sigma}(K) = \int_{\mathcal{L}_i^n} \max_{x\in L^{\perp}} \mathrm{r}\bigl(K\cap(x+L);x+L\bigr)\,\mathrm{d}\nu_{n,i}(L).$$

For the sake of brevity we will refer to $\widetilde{\mathrm{R}}_i^{\pi}(K)$ (respectively, $\widetilde{\mathrm{r}}_i^{\pi}(K)$) as *mean projection outer (respectively, inner) radii*. Analogously, $\widetilde{\mathrm{R}}_i^{\sigma}(K)$ (and $\widetilde{\mathrm{r}}_i^{\sigma}(K)$) will be referred to as *mean section outer (inner) radii.*

We observe that the above definitions make sense because the functionals appearing inside the integrals are continuous, and so we can integrate over the Grassmannian. Definitions 1.1 and 1.2 are the natural notions that may arise from some series of outer and inner radii which are well-known in the literature: for $K \in \mathcal{K}^n$ and $i = 1, \dots, n$, we write

$$\mathrm{R}_i^{\pi}(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L), \qquad \mathrm{r}_i^{\pi}(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{r}(K|L; L) \tag{1.1}$$

and, using sections,

$$\begin{aligned} \mathrm{R}_i^{\sigma}(K) &= \min_{L \in \mathcal{L}_i^n} \max_{x \in L^{\perp}} \mathrm{R}\big(K \cap (x+L)\big), \\ \mathrm{r}_i^{\sigma}(K) &= \min_{L \in \mathcal{L}_i^n} \max_{x \in L^{\perp}} \mathrm{r}\big(K \cap (x+L); x+L\big). \end{aligned} \tag{1.2}$$

From the definition it trivially follows that $\mathrm{R}_n^{\pi}(K) = \mathrm{R}_n^{\sigma}(K) = \mathrm{R}(K)$, $\mathrm{r}_n^{\pi}(K) = \mathrm{r}_n^{\sigma}(K) = \mathrm{r}(K)$ and $\mathrm{R}_1^{\pi}(K) = \mathrm{R}_1^{\sigma}(K) = \omega(K)/2 = \mathrm{r}_1^{\pi}(K) = \mathrm{r}_1^{\sigma}(K)$.

Replacing in the above definitions the min-condition by a max-condition, four additional series of successive outer and inner radii can be obtained, say, $\overline{\mathrm{R}}_i^{\pi}(K)$, $\overline{\mathrm{r}}_i^{\pi}(K)$, $\overline{\mathrm{R}}_i^{\sigma}(K)$ and $\overline{\mathrm{r}}_i^{\sigma}(K)$, respectively. We observe that now, replacing 'min' by 'max' in (1.1), leads to $\overline{\mathrm{R}}_1^{\pi}(K) = \mathrm{diam}(K)/2 = \overline{\mathrm{r}}_1^{\pi}(K)$ (and analogously for sections).

As we did in Definitions 1.1 and 1.2, the notation here has been chosen so that a π symbol indicates that we are using projections, whereas a σ means that we are dealing with sections.

It is easy to check that all the above types of outer radii are increasing in i, whereas the inner radii are decreasing in i for $1 \leq i \leq n$. The first systematic study of these families of successive radii was developed in [1]. For more information on these radii and their relation with other measures, we refer, among others, to [1, 3, 5, 6, 8, 9] and the references inside.

In view of (1.1), (1.2) and the corresponding notions via maxima, it was natural to consider the mean outer and inner radii provided in Definitions 1.1 and 1.2. The immediate relations

$$\begin{aligned} &\mathrm{r}_i^{\pi}(K) \leq \widetilde{\mathrm{r}}_i^{\pi}(K) \leq \overline{\mathrm{r}}_i^{\pi}(K), \quad \mathrm{R}_i^{\pi}(K) \leq \widetilde{\mathrm{R}}_i^{\pi}(K) \leq \overline{\mathrm{R}}_i^{\pi}(K) \quad \text{and} \\ &\mathrm{r}_i^{\sigma}(K) \leq \widetilde{\mathrm{r}}_i^{\sigma}(K) \leq \overline{\mathrm{r}}_i^{\sigma}(K), \quad \mathrm{R}_i^{\sigma}(K) \leq \widetilde{\mathrm{R}}_i^{\sigma}(K) \leq \overline{\mathrm{R}}_i^{\sigma}(K), \end{aligned}$$

$i = 1, \dots, n$, are obvious, and thus, one of the main aims for introducing these new notions is the possibility to strengthen existing inequalities among the classical radii.

In this paper we investigate basic properties of these new radii, aiming to use them to improve or better understand some of the already known inequalities, which involve radii, or to prove new ones.

2. First properties of the mean radii

It follows from Definitions 1.1 and 1.2 that

$$\widetilde{\mathrm{R}}_n^{\pi}(K) = \widetilde{\mathrm{R}}_n^{\sigma}(K) = \mathrm{R}(K), \quad \widetilde{\mathrm{r}}_n^{\pi}(K) = \widetilde{\mathrm{r}}_n^{\sigma}(K) = \mathrm{r}(K)$$

and, moreover,

$$\widetilde{\mathrm{R}}_1^{\pi}(K) = \frac{1}{2}\mathrm{b}(K) = \widetilde{\mathrm{r}}_1^{\pi}(K).$$

The latter follows from the observation that for any one dimensional linear subspace $L \in \mathcal{L}_1^n$, $\mathrm{r}(K|L) = \mathrm{R}(K|L) = \omega(K|L)/2$.

The case of $\widetilde{\mathrm{R}}_1^{\sigma}$ and $\widetilde{\mathrm{r}}_1^{\sigma}$ will need a further definition. We recall that the *difference body* of a convex body K is the Minkowski sum $\mathrm{D}K := K - K = K + (-K)$, where $-K = \{-x \in \mathbb{R}^n : x \in K\}$.

Lemma 2.1. *Let $K \in \mathcal{K}^n$ be a convex body. Then*

$$\widetilde{\mathrm{R}}_1^{\sigma}(K) = \widetilde{\mathrm{r}}_1^{\sigma}(K) = \frac{1}{4}\,\ell(\mathrm{D}K).$$

Proof. Let $K \in \mathcal{K}^n$ be a convex body and let $L = \mathrm{lin}\{u\} \in \mathcal{L}_1^n$, with $u \in \mathbb{S}^{n-1}$. It is known ([13, p. 529]) that the length of a longest chord of K in direction u coincides with $\rho(\mathrm{D}K, u)$. In other words, we have

$$\rho(\mathrm{D}K, u) = \max_{x \in L^{\perp}} \mathrm{vol}_1\bigl(K \cap (x+L)\bigr) = 2 \max_{x \in L^{\perp}} \mathrm{R}\bigl(K \cap (x+L)\bigr).$$

Then, integrating over the Grassmannian $\mathcal{L}_1^n$, we get

$$\begin{aligned}
\widetilde{\mathrm{R}}_1^{\sigma}(K) &= \int_{\mathcal{L}_1^n} \max_{x \in L^{\perp}} \mathrm{R}\bigl(K \cap (x+L)\bigr)\,\mathrm{d}\nu_{n,1}(L)\\
&= \frac{1}{2}\int_{\{\mathrm{lin}\{u\} : u \in \mathbb{S}^{n-1}\}} \rho(\mathrm{D}K, u)\,\mathrm{d}\nu_{n,1}\bigl(\mathrm{lin}\{u\}\bigr)\\
&= \frac{1}{2\,n\kappa_n}\int_{\mathbb{S}^{n-1}} \rho(\mathrm{D}K, u)\,\mathrm{d}u = \frac{1}{4}\,\ell(\mathrm{D}K).
\end{aligned}$$

The case of the mean section inner radius $\widetilde{\mathrm{r}}_1^{\sigma}(K)$ is analogous, just using that $\rho(\mathrm{D}K, u) = 2\max_{x \in L^{\perp}} \mathrm{r}\bigl(K \cap (x+L); x+L\bigr)$. □

Next we consider the monotonicity of the families of mean projection and section outer and inner radii in the dimension i of the Grassmannian where the average is taken.

Proposition 2.1. *Let $K \in \mathcal{K}^n$ be a convex body. Then, for any $2 \le i \le n$, $\widetilde{\mathrm{R}}_{i-1}^{\pi}(K) \le \widetilde{\mathrm{R}}_i^{\pi}(K)$ and $\widetilde{\mathrm{R}}_{i-1}^{\sigma}(K) \le \widetilde{\mathrm{R}}_i^{\sigma}(K)$. Moreover $\widetilde{\mathrm{r}}_{i-1}^{\pi}(K) \ge \widetilde{\mathrm{r}}_i^{\pi}(K)$ and $\widetilde{\mathrm{r}}_{i-1}^{\sigma}(K) \ge \widetilde{\mathrm{r}}_i^{\sigma}(K)$.*

Proof. First we deal with the mean projection radii. Let $L \in \mathcal{L}_i^n$, with $i \ge 2$. Then, for any $(i-1)$-dimensional subspace L' of L, it is clear that $\mathrm{R}(K|L') \le \mathrm{R}(K|L)$. Thus

$$\int_{\mathcal{L}_{i-1}^n(L)} \mathrm{R}(K|L')\,\mathrm{d}\nu_{L,i-1}(L') \le \mathrm{R}(K|L),$$

and hence

$$\int_{\mathcal{L}_i^n}\int_{\mathcal{L}_{i-1}^n(L)} \mathrm{R}(K|L')\,\mathrm{d}\nu_{L,i-1}(L')\,\mathrm{d}\nu_{n,i}(L)$$
$$\leq \int_{\mathcal{L}_i^n} \mathrm{R}(K|L)\,\mathrm{d}\nu_{n,i}(L) = \widetilde{\mathrm{R}}_i^\pi(K).$$

By the uniqueness of the Haar probability measure on $\mathcal{L}_{i-1}^n$, the integral on the left hand side of the previous inequality is

$$\int_{\mathcal{L}_{i-1}^n} \mathrm{R}(K|L')\,\mathrm{d}\nu_{n,i-1}(L') = \widetilde{\mathrm{R}}_{i-1}^\pi(K).$$

The inradius case is analogous just noticing that $\mathrm{r}(K|L';L') \geq \mathrm{r}(K|L;L)$ for any $(i-1)$-dimensional subspace L' of L.

In the case of the mean section outer radii, we observe that if $L \in \mathcal{L}_i^n$, $i \geq 2$, and $L' \in \mathcal{L}_{i-1}^n(L)$, then for every vector $x \in (L')^\perp$ there exists $z_x \in L^\perp$ such that $x + L' \subset z_x + L$. Thus

$$\max_{x\in(L')^\perp} \mathrm{R}\bigl(K\cap(x+L')\bigr) \leq \max_{x\in L^\perp} \mathrm{R}\bigl(K\cap(x+L)\bigr)$$

and hence, using again the uniqueness of the Haar probability measure on $\mathcal{L}_{i-1}^n$, we get

$$\begin{aligned}
\widetilde{\mathrm{R}}_{i-1}^\sigma(K) &= \int_{\mathcal{L}_{i-1}^n} \max_{x\in(L')^\perp} \mathrm{R}\bigl(K\cap(x+L')\bigr)\,\mathrm{d}\nu_{n,i-1}(L')\\
&= \int_{\mathcal{L}_i^n}\int_{\mathcal{L}_{i-1}^n(L)} \max_{x\in(L')^\perp} \mathrm{R}\bigl(K\cap(x+L')\bigr)\,\mathrm{d}\nu_{L,i-1}(L')\,\mathrm{d}\nu_{n,i}(L)\\
&\leq \int_{\mathcal{L}_i^n} \max_{x\in L^\perp} \mathrm{R}\bigl(K\cap(x+L)\bigr)\,\mathrm{d}\nu_{n,i}(L) = \widetilde{\mathrm{R}}_i^\sigma(K).
\end{aligned}$$

The same argument works for the inner radii $\widetilde{\mathrm{r}}_i^\sigma$, just observing that, following the above notation,

$$\max_{x\in(L')^\perp} \mathrm{r}\bigl(K\cap(x+L');x+L'\bigr) \geq \max_{x\in L^\perp} \mathrm{r}\bigl(K\cap(x+L);x+L\bigr).$$

Indeed, if $x_0 + \mathrm{r}\bigl(K\cap(x+L);x+L\bigr)B_i \subset K\cap(L+x)$, with $x_0 \in L^\perp$, then there exists $\bar{x}_0 \in (L')^\perp$ so that $x_0 + \mathrm{r}\bigl(K\cap(x+L);x+L\bigr)B_{i-1} \subset K\cap(L'+\bar{x}_0)$, which shows the above inequality. □

Therefore we have the chains of inequalities

$$\mathrm{r}(K) = \widetilde{\mathrm{r}}_n^\pi(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1^\pi(K) = \frac{1}{2}\mathrm{b}(K) = \widetilde{\mathrm{R}}_1^\pi(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n^\pi(K) = \mathrm{R}(K)$$

and

$$\mathrm{r}(K) = \widetilde{\mathrm{r}}_n^\sigma(K) \leq \cdots \leq \widetilde{\mathrm{r}}_1^\sigma(K) = \frac{1}{2}\,\ell(\mathrm{D}K) = \widetilde{\mathrm{R}}_1^\sigma(K) \leq \cdots \leq \widetilde{\mathrm{R}}_n^\sigma(K) = \mathrm{R}(K).$$

At this point we would like to observe that in [15], Zindler provided an example of a 3-dimensional convex body so that

$$\min_{L\in\mathcal{L}_2^3}\max_{x\in L^\perp}\mathrm{R}\big(K\cap(x+L)\big)<\min_{L\in\mathcal{L}_2^3}\mathrm{R}(K|L),$$

which raises the question whether there exists $K\in\mathcal{K}^n$ with

$$\widetilde{\mathrm{R}}_i^\sigma(K)<\widetilde{\mathrm{R}}_i^\pi(K).$$

Next property relates the mean outer and inner radii when the projections are taken onto subspaces of appropriate dimensions (analogously for sections). The corresponding result for classical radii was proved in [1, Lemma 2.1].

Lemma 2.2. *Let $K\in\mathcal{K}^n$ be a convex body. Then $\widetilde{\mathrm{R}}_i^\pi(K)\geq\widetilde{\mathrm{r}}_{n-i+1}^\pi(K)$ and $\widetilde{\mathrm{R}}_i^\sigma(K)\geq\widetilde{\mathrm{r}}_{n-i+1}^\sigma(K)$ for $1\leq i\leq n$.*

Proof. Let $L_i\in\mathcal{L}_i^n$ and $L_{n-i+1}\in\mathcal{L}_{n-i+1}^n$ be such that $\widetilde{\mathrm{R}}_i^\pi(K)=\mathrm{R}(K|L_i)$ and $\widetilde{\mathrm{r}}_{n-i+1}^\pi(K)=\mathrm{r}(K|L_{n-i+1};L_{n-i+1})$. We observe that these subspaces exist because the Grassmannian is a connected space and the functions

$$\mathcal{L}_i^n\longrightarrow\mathcal{K}^n,\quad L\mapsto K|L,$$

for $i=1,\dots,n$, are continuous with respect to the usual metric in the Grassmannian.

Then, since there exists $L_1\in\mathcal{L}_1^n(L_{n-i+1}\cap L_i)$,

$$\begin{aligned}\widetilde{\mathrm{R}}_i^\pi(K)&=\mathrm{R}(K|L_i)\geq\mathrm{R}(K|L_1)=\mathrm{r}(K|L_1;L_1)\geq\mathrm{r}(K|L_{n-i+1};L_{n-i+1})\\&=\widetilde{\mathrm{r}}_{n-i+1}^\pi(K).\end{aligned}$$

Using that the functions $\mathcal{L}_i^n\longrightarrow\mathcal{K}^n$, $L\mapsto\max_{x\in L^\perp}\mathrm{R}\big(K\cap(L+x)\big)$ and $L\mapsto\max_{x\in L^\perp}\mathrm{r}\big(K\cap(L+x);x+L\big)$ are continuous, the same argument yields the statement for mean section radii. □

3. Inequalities for the mean radii involving other functionals

Given a convex body $K\in\mathcal{K}^n$ and a non-negative real number λ, the volume of the Minkowski sum (vectorial addition) $K+\lambda B_n$ is expressed as a polynomial of degree n in λ, namely,

$$\mathrm{vol}(K+\lambda B_n)=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K)\lambda^i,$$

which is called the (classical) *Steiner formula* of K (see [14]). The coefficients $\mathrm{W}_i(K)$ are the *quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [13, Section 5.1]. In particular, $\mathrm{W}_0(K)=\mathrm{vol}(K)$, $n\mathrm{W}_1(K)=\mathrm{S}(K)$ is the classical surface area of K, $(2/\kappa_n)\mathrm{W}_{n-1}(K)=\mathrm{b}(K)$ and, moreover, $\mathrm{W}_n(K)=\kappa_n$.

The so-called *dual quermassintegrals* of a convex body K containing the origin arise when the volume of the *radial sum* of K and a ball λB_n is considered. We observe that the radial sum is defined for more general sets than just convex bodies containing the origin, although we will not work in this setting here. Given $K, L \in \mathcal{K}^n$ containing the origin, the radial sum $K\tilde{+}L$, is the set, not necessarily convex, whose radial function is given by $\rho(K\tilde{+}L, u) = \rho(K, u) + \rho(L, u)$ for every $u \in \mathbb{S}^{n-1}$. Then the volume of $K\tilde{+}\lambda B_n$ is also a polynomial, namely

$$\mathrm{vol}(K\tilde{+}\lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{\mathrm{W}}_i(K)\lambda^i,$$

and the coefficients $\widetilde{\mathrm{W}}_i(K)$ are the *dual quermassintegrals* of K. Further, $\widetilde{\mathrm{W}}_0(K) = \mathrm{vol}(K)$, $\widetilde{\mathrm{W}}_n(K) = \kappa_n$ and $2/\kappa_n \widetilde{\mathrm{W}}_{n-1}(K) = \ell(K)$. The so-called dual Brunn-Minkowski theory, was first introduced by Lutwak in [10, 11]. We refer the reader to [13, s. 9.3] for more details on this theory.

In the following, we will also use the inequalities

$$\mathrm{r}(K)\mathrm{W}_i(K) \leq \mathrm{W}_{i-1}(K) \leq \mathrm{R}(K)\mathrm{W}_i(K), \tag{3.1}$$

for $1 \leq i \leq n$. Since, up to translations, $\mathrm{r}(K)B_n \subset K$ and $K \subset \mathrm{R}(K)B_n$ these inequalities are a direct consequence of the monotonicity and the homogeneity of the quermassintegrals (cf. e.g. [7, Theorem 6.1.3]). We observe that the analogous inequalities for dual quermassintegrals require the assumption that one of the largest balls contained in K, as well as the smallest ball containing K, are centered at 0. We will not make use of this fact.

First, using Urysohn's inequality

$$\mathrm{vol}(K) \leq \kappa_n \left(\frac{\mathrm{b}(K)}{2}\right)^n$$

(see e.g. [13, (7.21)]) and the identity $\mathrm{b}(K)/2 = \widetilde{\mathrm{R}}_1^\pi(K)$, the volume and the first mean projection outer radius can be related. Now, using the monotonicity in i of the mean projection radii, proven in Proposition 2.1, as well as the relation $\mathrm{W}_{n-1}(K)/\kappa_n = \mathrm{b}(K)/2$, we can directly link the volume and the last but one quermassintegral with all mean projection outer radii. The equality cases follow from the equality case in Urysohn's inequality.

Corollary 3.1. *Let $K \in \mathcal{K}^n$. Then*

$$\mathrm{vol}(K) \leq \kappa_n \widetilde{\mathrm{R}}_1^\pi(K) \dots \widetilde{\mathrm{R}}_n^\pi(K),$$
$$\mathrm{W}_{n-1}(K) \leq \frac{\kappa_n}{n}\left(\widetilde{\mathrm{R}}_1^\pi(K) + \dots + \widetilde{\mathrm{R}}_n^\pi(K)\right).$$

Equality holds in the first inequality if and only if K is a ball. Balls also attain equality in the second inequality.

In view of this corollary we conjecture the following result. We just introduce an additional notation: for $x_1, \dots, x_m \in \mathbb{R}$ let

$$\mathrm{s}_i(x_1, \dots, x_m) = \sum_{\substack{J \subseteq \{1,\dots,m\} \\ \#J = i}} \prod_{j \in J} x_j$$

denote the i-th elementary symmetric function of $x_1,\dots,x_m$, $1\le i\le m$, setting $\mathrm{s}_0(x_1,\dots,x_m)=1$.

Conjecture 3.1. *Let $K\in\mathcal{K}^n$. Then, for any $i=0,\dots,n$,*

$$\mathrm{W}_i(K)\le\frac{\kappa_n}{\binom{n}{i}}\,\mathrm{s}_{n-i}\left(\widetilde{\mathrm{R}}_1^{\pi}(K),\dots,\widetilde{\mathrm{R}}_n^{\pi}(K)\right).$$

Equality holds if and only if K is a ball.

In the case of the mean projection inner radii, we obtain lower bounds for all quermassintegrals of K. We notice that the case $i=n-1$ is indeed an equality (cf. Lemma 2.1), and hence we exclude it.

Proposition 3.1. *Let $K\in\mathcal{K}^n$. Then, for any $i=0,\dots,n-2$,*

$$\mathrm{W}_i(K)\ge\kappa_n\widetilde{\mathrm{r}}_{n-i}^{\pi}(K)^{n-i}. \tag{3.2}$$

Moreover, if K is 0-symmetric (i.e., so that $K=-K$), then

$$\widetilde{\mathrm{W}}_i(K)\ge\kappa_n\widetilde{\mathrm{r}}_{n-i}^{\sigma}(K)^{n-i}. \tag{3.3}$$

Equality holds in both inequalities if and only if K is a ball.

Proof. The case $i=0$, namely, $\mathrm{vol}(K)\ge\kappa_n\mathrm{r}(K)^n$ is well-known (cf.(3.1)). Therefore we assume that $1\le i\le n-2$.

Kubota's integral recursion formula (see e.g. [13, (5.72)]) ensures that

$$\mathrm{W}_i(K)=\frac{\kappa_n}{\kappa_{n-i}}\int_{\mathcal{L}_{n-i}^n}\mathrm{vol}_{n-i}(K|L)\,\mathrm{d}\nu_{n,n-i}(L),$$

and since by (3.1) we have $\mathrm{vol}_{n-i}(K|L)\ge\mathrm{r}(K|L;L)^{n-i}\kappa_{n-i}$, then

$$\mathrm{W}_i(K)\ge\kappa_n\int_{\mathcal{L}_{n-i}^n}\mathrm{r}(K|L;L)^{n-i}\,\mathrm{d}\nu_{n,n-i}(L).$$

Finally, applying Hölder's inequality (see e.g. [7, Corollary 1.5]) we can conclude that

$$\mathrm{W}_i(K)\ge\kappa_n\left(\int_{\mathcal{L}_{n-i}^n}\mathrm{r}(K|L;L)\,\mathrm{d}\nu_{n,n-i}(L)\right)^{n-i}=\kappa_n\widetilde{\mathrm{r}}_{n-i}^{\pi}(K)^{n-i}.$$

Equality holds, in particular, if and only if $\mathrm{vol}_{n-i}(K|L)=\mathrm{r}(K|L;L)^{n-i}\kappa_{n-i}$ for all $L\in\mathcal{L}_{n-i}^n$, i.e., if and only if $K|L$ is an $(n-i)$-ball for all $L\in\mathcal{L}_{n-i}^n$. It is equivalent to the fact that K is a ball (see [4, Corollary 3.1.6]).

In analogy to Kubota's formula, the dual quermassintegrals $\widetilde{\mathrm{W}}_i(K)$ admit also an integral geometric representation as the means of the volumes of sections (see e.g. [13, (9.38)]):

$$\widetilde{\mathrm{W}}_i(K)=\frac{\kappa_n}{\kappa_{n-i}}\int_{\mathcal{L}_{n-i}^n}\mathrm{vol}_{n-i}(K\cap L)\,\mathrm{d}\nu_{n,n-i}(L),\quad i=1,\dots,n.$$

Again, by (3.1) we have

$$\mathrm{vol}_{n-i}(K\cap L)\ge\mathrm{r}(K\cap L;L)^{n-i}\kappa_{n-i},$$

and since K is 0-symmetric, the section $K\cap(x+L)$, $x\in L^{\perp}$, having the largest inradius is the central slice. It can be deduced as Brunn's theorem (see e.g. [12, Theorem 12.2.1]), because an even concave function is largest at 0. Therefore we have

$$\max_{x\in L^{\perp}} \mathrm{r}\bigl(K\cap(x+L);x+L\bigr)=\mathrm{r}(K\cap L;L)$$

which, together with Kubota's formula yields

$$\widetilde{\mathrm{W}}_i(K)\geq\kappa_n\int_{\mathcal{L}^n_{n-i}}\max_{x\in L^{\perp}}\mathrm{r}\bigl(K\cap(x+L);x+L\bigr)^{n-i}\,\mathrm{d}\nu_{n,n-i}(L).$$

Applying again Hölder's inequality we conclude that

$$\widetilde{\mathrm{W}}_i(K)\geq\kappa_n\widetilde{\mathrm{r}}^{\sigma}_{n-i}(K)^{n-i}.$$

Equality holds if and only if $\mathrm{vol}_{n-i}(K\cap L)=\mathrm{r}(K\cap L;L)^{n-i}\kappa_{n-i}$ for all $L\in\mathcal{L}^n_{n-i}$, i.e., if and only if $K\cap L$ is an $(n-i)$-ball for all $L\in\mathcal{L}^n_{n-i}$. It is equivalent to the fact that K is a ball (see [4, Corollary 7.1.4]). $\square$

We observe that (3.3) (for convex bodies not-necessarily symmetric) cannot be deduced from (3.2), because dual quermassintegrals and quermassintegrals of convex bodies (containing the origin) are related by the inequality

$$\widetilde{\mathrm{W}}_i(K)\leq\mathrm{W}_i(K)$$

(see e.g. [10, Corollary 1.4]).

Corollary 3.2. *Let $K\in\mathcal{K}^n$. Then, for any $i=0,\dots,n-1$,*

$$\mathrm{vol}(K)\geq\kappa_n\widetilde{\mathrm{r}}^{\pi}_{n-i}(K)^{n-i}\,\widetilde{\mathrm{r}}^{\pi}_{n}(K)^{i} \tag{3.4}$$

and

$$\mathrm{vol}(K)\geq\kappa_n\widetilde{\mathrm{r}}^{\sigma}_{n-i}(K)^{n-i}\,\widetilde{\mathrm{r}}^{\sigma}_{n}(K)^{i}.$$

Equality holds in both inequalities if and only if K is a ball.

Proof. (3.4) is a direct consequence of the inequality $\mathrm{vol}(K)\geq\mathrm{r}(K)^i\mathrm{W}_i(K)$, $i=0,\dots,n$, (cf. (3.1)) and (3.2). Finally, since $\widetilde{\mathrm{r}}^{\pi}_i(K)\geq\widetilde{\mathrm{r}}^{\sigma}_i(K)$, $1\leq i\leq n$, from (3.4) it follows the analogous result for the mean section inner radii. $\square$

In particular, if $n=2$ and $i=1$ then (3.4) becomes

$$\mathrm{vol}(K)\geq\kappa_2\,\widetilde{\mathrm{r}}^{\pi}_1(K)\,\widetilde{\mathrm{r}}^{\pi}_2(K).$$

We conjecture that this also holds in any dimension. Namely:

Conjecture 3.2. *Let $K\in\mathcal{K}^n$. Then*

$$\mathrm{vol}(K)\geq\kappa_n\widetilde{\mathrm{r}}^{\pi}_1(K)\dots\widetilde{\mathrm{r}}^{\pi}_n(K).$$

Equality holds if and only if K is the ball.

4. Mean radii and the Minkowski addition

As we have already seen, the difference body $\mathrm{D}K$ of a convex body K happens to play a role in the definition of $\widetilde{\mathrm{R}}_1^\sigma(K)$ and $\widetilde{\mathrm{r}}_1^\sigma(K)$. This motivates the next proposition, in which we obtain an inequality relating the mean section inner radii with the volume of the difference body.

Proposition 4.1. *Let $K \in \mathcal{K}^n$. Then,*

$$\mathrm{vol}(\mathrm{D}K) \geq 2^n \kappa_n \widetilde{\mathrm{r}}_1^\sigma(K)^n, \quad \textit{and}$$

$$\mathrm{vol}(\mathrm{D}K) \geq 2^n \kappa_n \widetilde{\mathrm{r}}_1^\sigma(K) \dots \widetilde{\mathrm{r}}_n^\sigma(K). \tag{4.1}$$

Equality holds if and only if K is a ball.

Proof. One form of the so-called dual isoperimetric inequality (see e.g. [4, (B.28)]) relates the volume of a convex body K (containing the origin) with its the average length of chords: it states that

$$\mathrm{vol}(K) \geq \frac{\kappa_n}{2^n} \ell(K)^n,$$

with equality if and only if K is a (centered) ball. Then, applying the above inequality to $\mathrm{D}K$ (which contains the origin) and using Lemma 2.1, we get

$$\mathrm{vol}(\mathrm{D}K) \geq \kappa_n \left(\frac{\ell(\mathrm{D}K)}{2} \right)^n = 2^n \kappa_n \widetilde{\mathrm{r}}_1^\sigma(K)^n.$$

Finally, the monotonicity of the mean inner radii (see Proposition 2.1) yields inequality (4.1). The equality characterization follows from the equality case in the dual isoperimetric inequality. □

In [5, Proposition 4.2] it was proven, among other results, that the classical radii R_i^π satisfy

$$\sqrt{2}\sqrt{\frac{i+1}{i}} \mathrm{R}_i^\pi(K) \leq \mathrm{R}_i(\mathrm{D}K) \leq 2\mathrm{R}_i^\pi(K).$$

Here we extend this type of result to some mean outer and inner radii.

Proposition 4.2. *Let $K \in \mathcal{K}^n$. Then, for any $i = 1, \dots, n$,*

i) $\sqrt{\frac{2(i+1)}{i}}\, \widetilde{\mathrm{R}}_i^\pi(K) \leq \widetilde{\mathrm{R}}_i^\pi(\mathrm{D}K) \leq 2\, \widetilde{\mathrm{R}}_i^\pi(K)$,

ii) $2\, \widetilde{\mathrm{r}}_i^\pi(K) \leq \widetilde{\mathrm{r}}_i^\pi(\mathrm{D}K) \leq c(i)\, \widetilde{\mathrm{r}}_i^\pi(K)$,

where

$$c(i) = \begin{cases} 2\sqrt{i} & \textit{for } i \textit{ odd}, \\ \frac{2(i+1)}{\sqrt{i+2}} & \textit{for } i \textit{ even}. \end{cases}$$

Proof. The behavior of the circumradius and the inradius with respect to the Minkowski addition is well known (see e.g. [5, (2)]), namely,

$$\mathrm{R}(K + K') \leq \mathrm{R}(K) + \mathrm{R}(K'), \quad \mathrm{r}(K + K') \geq \mathrm{r}(K) + \mathrm{r}(K'), \tag{4.2}$$

for $K, K' \in \mathcal{K}^n$. Let $L \in \mathcal{L}_i^n$. Then we trivially get

$$\mathrm{R}(\mathrm{D}K|L) = \mathrm{R}(K|L - K|L) \leq 2\mathrm{R}(K|L) \quad \text{and}$$
$$\mathrm{r}(\mathrm{D}K|L; L) = \mathrm{r}(K|L - K|L; L) \geq 2\mathrm{r}(K|L; L),$$

and integrating over the Grassmannian $\mathcal{L}_i^n$ we obtain

$$\widetilde{\mathrm{R}}_i^\pi(\mathrm{D}K) \leq 2\widetilde{\mathrm{R}}_i^\pi(K) \quad \text{and} \quad \widetilde{\mathrm{r}}_i^\pi(\mathrm{D}K) \geq 2\widetilde{\mathrm{r}}_i^\pi(K).$$

In order to prove the lower bound in i) and the upper bound in ii) we observe that, since the diameter (respectively, the minimal width) of a 0-symmetric set equals twice its circumradius (respectively, inradius), then,

$$\mathrm{R}(\mathrm{D}K|L) = \mathrm{R}\big(\mathrm{D}(K|L)\big) = \frac{1}{2}\mathrm{diam}\big(\mathrm{D}(K|L)\big) = \mathrm{diam}\left(\frac{1}{2}\mathrm{D}(K|L)\right)$$

and

$$\mathrm{r}(\mathrm{D}K|L; L) = \mathrm{r}\big(\mathrm{D}(K|L); L\big) = \frac{1}{2}\omega\big(\mathrm{D}(K|L); L\big) = \omega\left(\frac{1}{2}\mathrm{D}(K|L); L\right).$$

It is well-known (see e.g. [2, p. 79]) that both, the diameter and the minimal width, are preserved under the transformation $\mathrm{D}(\cdot)/2$. Then, applying the well-known Jung and Steinhagen inequalities (see e.g. [2, pp. 84 and 86]) in dimension i to the above identities yields

$$\mathrm{R}(\mathrm{D}K|L) = \mathrm{diam}\left(\frac{1}{2}\mathrm{D}(K|L)\right) = \mathrm{diam}(K|L) \geq \sqrt{\frac{2(i+1)}{i}}\mathrm{R}(K|L)$$
$$\mathrm{r}(\mathrm{D}K|L; L) = \omega\left(\frac{1}{2}\mathrm{D}(K|L); L\right) = \omega(K|L; L) \leq c(i)\mathrm{r}(K|L; L).$$

Integrating over the Grassmannian $\mathcal{L}_i^n$ we get the required inequalities. $\square$

In the case of the mean section radii, the situation is more involved, and we have only been able to settle the case $i = 1$ (the case $i = n$ is trivial). For a convex body $K \in \mathcal{K}^n$ we always have, by the definition of difference body, that $\mathrm{DD}K = 2\mathrm{D}K$. Since $\widetilde{\mathrm{r}}_1^\sigma(K) = \ell(\mathrm{D}K)/4$ (cf. Lemma 2.1), in order to compute $\widetilde{\mathrm{r}}_1^\sigma(\mathrm{D}K)$ it suffices to use the homogeneity of $\ell(K)$ to observe that

$$\widetilde{\mathrm{r}}_1^\sigma(\mathrm{D}K) = \frac{1}{4}\ell(\mathrm{DD}K) = \frac{1}{4}\ell(2\mathrm{D}K) = \frac{1}{2}\ell(\mathrm{D}K) = 2\widetilde{\mathrm{r}}_1^\sigma(K).$$

Analogously we get that

$$\widetilde{\mathrm{R}}_1^\sigma(\mathrm{D}K) = 2\widetilde{\mathrm{R}}_1^\sigma(K).$$

We conclude the paper considering the Minkowski addition of two arbitrary convex bodies K, K', rather than $K + (-K)$. In [5, Theorem 1.1], the relation between certain classical radii and the Minkowski sum of convex bodies was established. Here we extend that property to the mean radii.

Proposition 4.3. *Let $K, K' \in \mathcal{K}^n$. Then, for any $i = 2, \dots, n$,*

$$\text{i) } \widetilde{\mathrm{R}}_i^\pi(K) + \widetilde{\mathrm{R}}_i^\pi(K') \geq \widetilde{\mathrm{R}}_i^\pi(K+K') \geq \frac{\sqrt{2}}{2}\left(\widetilde{\mathrm{R}}_i^\pi(K) + \widetilde{\mathrm{R}}_i^\pi(K')\right) \quad \textit{and}$$

$$\text{ii) } \widetilde{\mathrm{r}}_i^\pi(K) + \widetilde{\mathrm{r}}_i^\pi(K') \leq \widetilde{\mathrm{r}}_i^\pi(K+K') \leq \frac{\sqrt{2}}{2}\left(\widetilde{\mathrm{r}}_i^\pi(K) + \widetilde{\mathrm{r}}_i^\pi(K')\right).$$

Moreover,

$$\widetilde{\mathrm{R}}_1^\pi(K+K') = \widetilde{\mathrm{R}}_1^\pi(K) + \widetilde{\mathrm{R}}_1^\pi(K'),$$
$$\widetilde{\mathrm{r}}_1^\pi(K+K') = \widetilde{\mathrm{r}}_1^\pi(K) + \widetilde{\mathrm{r}}_1^\pi(K').$$

Proof. Let $L \in \mathcal{L}_i^n$, $2 \leq i \leq n$. On the one hand, using (4.2) we have that

$$\mathrm{R}\big((K+K')|L\big) = \mathrm{R}(K|L + K'|L) \leq \mathrm{R}(K|L) + \mathrm{R}(K'|L),$$
$$\mathrm{r}\big((K+K')|L;L\big) = \mathrm{r}(K|L + K'|L;L) \geq \mathrm{r}(K|L;L) + \mathrm{r}(K'|L;L).$$

On the other hand, in [5, Proofs of Theorems 1.1 and 1.2] it was proven that

$$\mathrm{R}\big((K+K')|L\big) \geq \frac{\sqrt{2}}{2}\left(\mathrm{R}(K|L) + \mathrm{R}(K'|L)\right).$$

and

$$\mathrm{r}\big((K+K')|L;L\big) \leq \frac{\sqrt{2}}{2}\left(\mathrm{r}(K|L;L) + \mathrm{r}(K'|L;L)\right).$$

Thus, integrating over the Grassmannian $\mathcal{L}_i^n$ the above four inequalities, we get the result.

The last identities for the case $i = 1$ are a direct consequence of the fact that the mean width is Minkowski additive. □

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