Estimates for the integrals of powered *i*-th mean curvatures

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Abstract We show (upper and lower) estimates for the integrals of powered i-th mean curvatures, $i = 1,\dots,n-1$, of compact and convex hypersurfaces, in terms of the quermassintegrals of the corresponding C^2_+-convex bodies. These bounds are obtained as consequences of a most general result for functions defined on a general probability space. Moreover, similar estimates for the integrals of powers of the elementary symmetric functions of the radii of curvature of C^2_+-convex bodies are proved. This probabilistic result will also allow to get new inequalities for the dual quermaßintegrals of starshaped sets, via further estimates for the integrals of the composition of a convex/concave function with the (powered) radial function.

1 Introduction

As usual in the literature we will write $\mathbb{R}^n$ for the n-dimensional Euclidean space, endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the Euclidean norm $\|\cdot\|$.

Moreover, $\mathscr{H}^k$, $0 \le k \le n$, will denote the k-dimensional Hausdorff measure on $\mathbb{R}^n$, and thus, if M is a subset of a k-plane or a k-dimensional sphere $\mathbb{S}^k$, then $\mathscr{H}^k(M)$ coincides, respectively, with the k-dimensional Lebesgue measure of M in $\mathbb{R}^k$ or with the k-dimensional spherical Lebesgue measure in $\mathbb{S}^k$.

A classical isoperimetric type result in differential geometry of curves due to Gage [6] states that if $\gamma : I \longrightarrow \mathbb{R}^2$ is a planar, regular, closed and convex curve with curvature k, length L and enclosing an area A, then

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$$\int_\gamma k^2 \mathrm{d}s \geq \pi \frac{\mathrm{L}}{\mathrm{A}}. \tag{1}$$

In [8] Green & Osher provided a general method in order to obtain inequalities of the type

$$\int_\gamma k^m \mathrm{d}s \geq f(\mathrm{L}, \mathrm{A}),$$

i.e., lower bounds for the integral of powers of the curvature in terms of some relation between the area and the length of the curve. In particular, Gage's inequality can also be derived with their method.

Moving now on to 2-dimensional surfaces in $\mathbb{R}^3$, there are two relevant curvatures to consider: the Gauss curvature κ and the mean curvature H. Then, in the spirit of (1), we find the famous Gauss-Bonnet theorem (see e.g. [5]) and the Willmore theorem (see [14]). Gauss-Bonnet's theorem shows that if $M \subset \mathbb{R}^3$ is a compact (smooth) surface which is homeomorphic to the sphere, then

$$\int_M \kappa \, \mathrm{d}\mathscr{H}^2 \geq 4\pi;$$

Willmore's inequality states that for any compact (smooth) surface $M \subset \mathbb{R}^3$ having curvature H positive everywhere,

$$\int_M H^2 \, \mathrm{d}\mathscr{H}^2 \geq 4\pi.$$

The above inequalities have their analogues for compact hypersurfaces $M \subset \mathbb{R}^n$: the Gauss-Bonnet theorem rewrites

$$\int_M \kappa \, \mathrm{d}\mathscr{H}^{n-1} \geq n|B_n|,$$

whereas Willmore's inequality becomes

$$\int_M H^{n-1} \, \mathrm{d}\mathscr{H}^{n-1} \geq n|B_n|. \tag{2}$$

Here $|\cdot|$ stands for the volume, i.e., the Lebesgue measure, and B_n denotes the Euclidean unit ball centered at the origin. Willmore's inequality in an arbitrary dimension was proved by Chen, see [3, 4]. In addition, Ros [11] proved that

$$\int_M \frac{1}{H} \, \mathrm{d}\mathscr{H}^{n-1} \geq n|M|. \tag{3}$$

Besides the major importance that these results have by themselves, they are specially interesting because they imply isoperimetric inequalities (see e.g. [10]).

Since a compact hypersurface $M \subset \mathbb{R}^n$ has associated $n-1$ relevant curvatures, the so-called i-th mean curvatures H_i, $i = 1, \ldots, n-1$, the above results motivate the following question:

Main problem: To obtain (lower and/or upper) estimates for the integrals

$$\int_M H_i^\alpha \, \mathrm{d}\mathscr{H}^{n-1} \quad \text{and} \quad \int_M \frac{1}{H_i^\alpha} \, \mathrm{d}\mathscr{H}^{n-1}$$

for $\alpha \geq 0$ and any $i = 1, \dots, n-1$, as well as improvements of Chen's and Ros' inequalities, in the convex case.

Next we introduce the notation and main concepts that will be needed throughout the paper, as well as our main results.

2 Notation and previous results

Let $\mathscr{K}_0^n$ be the set of all convex bodies, i.e., compact convex sets with non-empty interior, in $\mathbb{R}^n$ containing the origin 0. A convex body $K \in \mathscr{K}_0^n$ is said to be of class C^2 if its boundary hypersurface $\mathrm{bd}K$ is a regular submanifold of $\mathbb{R}^n$, in the sense of differential geometry, which is twice continuously differentiable. Moreover, we say that K is of class C^2_+ if K is of class C^2 and the Gauss map $\nu_K : \mathrm{bd}K \longrightarrow \mathbb{S}^{n-1}$, mapping a boundary point $x \in \mathrm{bd}K$ to the (unique) normal vector of K at x, is a diffeomorphism. Thus, in this case, we can consider the $n-1$ principal curvatures $k_1, \dots, k_{n-1}$ of $\mathrm{bd}K$ and, as usual in the literature, we will denote by

$$H_i = \frac{1}{\binom{n-1}{i}} \sum_{1 \leq j_1 < \cdots < j_i \leq n-1} k_{j_1} \cdots k_{j_i}, \quad i = 1, \dots, n-1,$$

the *i-th mean curvature*, setting $H_0 = 1$. In particular, $H_1 = H$ is the classical *mean curvature* and $H_{n-1} = \kappa$ is the *Gauss-Kronecker curvature*.

The pursued estimates for the integrals of powered i-th mean curvatures will be given in terms of the so-called quermaßintegrals of the convex bodies, which are special geometric measures associated to the set. We define them next.

In [13], Steiner proved that given $K \in \mathscr{K}_0^n$ and a non-negative real number λ, the volume of the Minkowski sum (vectorial addition) $K + \lambda B_n$ is expressed as a polynomial of degree n in λ, namely,

$$|K + \lambda B_n| = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K) \lambda^i,$$

which is called the (classical) *Steiner formula* of K. The coefficients $\mathrm{W}_i(K)$ are the *quermaßintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [12, Section 5.1]. In particular, $\mathrm{W}_0(K) = |K|$ (the area $\mathrm{A}(K)$ in the planar case), $n\mathrm{W}_1(K) = \mathrm{S}(K)$ is the surface area (the perimeter $\mathrm{L}(K)$ in the plane) and $2\mathrm{W}_{n-1}/|B_n|$ is the mean width of K. Moreover, $\mathrm{W}_n(K) = |B_n|$.

In order to state the main results of the paper, we need an additional definition. The *outer radius* and the *inner radius* of $K \in \mathscr{K}_0^n$ are defined as the quantities

$$\begin{aligned} \overline{\mathrm{R}}(K) &= \min\{R>0 : K \subseteq RB_n\} = \max\bigl\{\|x\| : x \in \mathrm{bd}K\bigr\}, \\ \overline{\mathrm{r}}(K) &= \max\{r \geq 0 : rB_n \subseteq K\} = \min\bigl\{\|x\| : x \in \mathrm{bd}K\bigr\}. \end{aligned}$$

Clearly, the value $\overline{\mathrm{R}}(K) - \overline{\mathrm{r}}(K)$ is not translation invariant, but since K is compact, there exists a (unique) point $c_K \in K$ such that

$$\overline{\mathrm{R}}(K-c_K) - \overline{\mathrm{r}}(K-c_K) = \min\bigl\{\overline{\mathrm{R}}(K-t) - \overline{\mathrm{r}}(K-t) : t \in K\bigr\}$$

(see [2]). The point c_K is the *center of the minimal ring*, i.e., the uniquely determined ring (closed set consisting of all points between two concentric balls) with minimal difference of radii containing $\mathrm{bd}K$. The value $\omega_a(K) := \overline{\mathrm{R}}(K-c_K) - \overline{\mathrm{r}}(K-c_K)$ is called the *width of the minimal ring* of K.

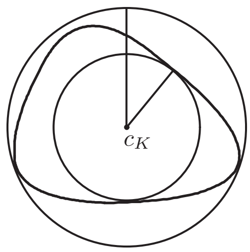

Fig. 1 The minimal ring of a convex body. We observe that the inner and the outer radii of $K - c_K$ do not necessarily coincide with the classical inradius and circumradius of the set, respectively.

Since $\overline{\mathrm{r}}B_n \subseteq K$ and $K \subseteq \overline{\mathrm{R}}B_n$, the in- and outer radii and the quermaßintegrals relate in the following way:

$$\overline{\mathrm{r}}(K)\mathrm{W}_{i+1}(K) \leq \mathrm{W}_i(K) \leq \overline{\mathrm{R}}(K)\mathrm{W}_{i+1}(K), \tag{4}$$

$i = 0, \ldots, n-1$, which is a direct consequence of the monotonicity of the mixed volumes (cf. e.g. [12, p. 282]).

For the statement of the results, and in order to shorten the statements and proofs, we introduce the following notation. For $1 \leq j \leq n-1$ and $0 \leq k \leq n-1$, let

$$\eta_{j,k} = \begin{cases} \dfrac{\mathrm{W}_1(K)\mathrm{W}_j(K) - \mathrm{W}_0(K)\mathrm{W}_{j+1}(K)}{\mathrm{W}_{k+1}(K)^2\omega_a(K)} & \text{if } K \neq rB_n \text{ for all } r>0, \\ 0 & \text{if } K = rB_n. \end{cases}$$

The non-negativity of the values $\eta_{j,k}$ is a direct consequence of the inequalities

$$\mathrm{W}_i(K)\mathrm{W}_j(K) \geq \mathrm{W}_{i-1}(K)\mathrm{W}_{j+1}(K), \quad 1 \leq i \leq j \leq n-1, \tag{5}$$

particular cases of the Aleksandrov-Fenchel inequality (see e.g. [12, Section 7.3]). From now on, for the sake of brevity, we will write $\mathrm{W}_i = \mathrm{W}_i(K)$, $i = 0, \ldots, n$, and analogously for all other functionals, if the distinction of the body is not needed.

2.1 Some previous results

In [1] the above mentioned problem of obtaining lower estimates for the integral of powered *i*-th mean curvatures was studied. Among others, the following more general theorem was proved.

Theorem 1. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. Then, for any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $i = 0, \dots, n-1$,*

$$\int_{\mathrm{bd}K} (F \circ H_i) \, \mathrm{d}\mathscr{H}^{n-1} \geq n\mathrm{W}_1 \frac{F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1} + \eta_{i,0}\right) + F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1} - \eta_{i,0}\right)}{2},$$

$$\int_{\mathrm{bd}K} \left(F \circ \frac{1}{H_i}\right) H_i \, \mathrm{d}\mathscr{H}^{n-1} \geq n\mathrm{W}_{i+1} \frac{F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}} + \eta_{i,i}\right) + F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}} - \eta_{i,i}\right)}{2}.$$

Equality holds in both inequalities if $K = B_n$ (up to dilations).

Indeed, a slightly more general result was obtained (see [1, Theorem 3.2]).

Then, applying Theorem 1 to the convex functions $F(x) = x^{\alpha+1}$ or $F(x) = 1/x^{\alpha}$, $\alpha \geq 0$, two different results can be obtained, providing different bounds for the same integrals. These bounds can be compared, and thus the following theorem is obtained in the spirit of the **main problem**:

Theorem 2. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. Then, for any $\alpha \geq 0$ and all $i = 0, \dots, n-1$,*

$$\int_{\mathrm{bd}K} H_i^{\alpha+1} \mathrm{d}\mathscr{H}^{n-1} \geq \frac{n}{2} \left[\frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(\mathrm{W}_1 + \mathrm{W}_{i+1}\eta_{i,i}\right)^{\alpha}} + \frac{\mathrm{W}_{i+1}^{\alpha+1}}{\left(\mathrm{W}_1 - \mathrm{W}_{i+1}\eta_{i,i}\right)^{\alpha}} \right],$$

$$\int_{\mathrm{bd}K} \frac{1}{H_i^{\alpha}} \mathrm{d}\mathscr{H}^{n-1} \geq \frac{n}{2} \left[\frac{\mathrm{W}_1^{\alpha+1}}{\left(\mathrm{W}_{i+1} + \mathrm{W}_1\eta_{i,0}\right)^{\alpha}} + \frac{\mathrm{W}_1^{\alpha+1}}{\left(\mathrm{W}_{i+1} - \mathrm{W}_1\eta_{i,0}\right)^{\alpha}} \right].$$

Equality holds in both inequalities if $K = B_n$ (up to dilations).

In particular, improvements of Chen's and Ros' estimates for convex hypersurfaces can be obtained by just taking $i = 1$ and, respectively, $\alpha = 1$ or $\alpha = n-2$:

Corollary 1. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. Then,*

$$\int_{\mathrm{bd}K} H^{n-1} \mathrm{d}\mathscr{H}^{n-1} \geq \frac{n}{2} \left[\frac{\mathrm{W}_2^{n-1}}{\left(\mathrm{W}_1 + \mathrm{W}_2\eta_{1,1}\right)^{n-2}} + \frac{\mathrm{W}_2^{n-1}}{\left(\mathrm{W}_1 - \mathrm{W}_2\eta_{1,1}\right)^{n-2}} \right],$$

$$\int_{\mathrm{bd}K} \frac{1}{H} \mathrm{d}\mathscr{H}^{n-1} \geq n \frac{\mathrm{W}_1^2 \mathrm{W}_2}{\mathrm{W}_2^2 - \mathrm{W}_1^2 \eta_{1,0}^2}.$$

Equality holds in all inequalities if $K = B_n$ (up to dilations).

Indeed, on the one hand,

$$n\frac{\mathrm{W}_1^2\mathrm{W}_2}{\mathrm{W}_2^2-\mathrm{W}_1^2\eta_{1,0}^2}\geq n\frac{\mathrm{W}_1^2\mathrm{W}_2}{\mathrm{W}_2^2}=n\frac{\mathrm{W}_1^2}{\mathrm{W}_2}\geq n\mathrm{W}_0=n|K|$$

because of the Aleksandrov-Fenchel inequality (5) for $i=j=1$. On the other hand, since the function $1/x^{n-2}$ is convex, then

$$\frac{n}{2}\left[\frac{\mathrm{W}_2^{n-1}}{\big(\mathrm{W}_1+\mathrm{W}_2\eta_{1,1}\big)^{n-2}}+\frac{\mathrm{W}_2^{n-1}}{\big(\mathrm{W}_1-\mathrm{W}_2\eta_{1,1}\big)^{n-2}}\right]\geq\frac{n}{2}\frac{2\mathrm{W}_2^{n-1}}{\mathrm{W}_1^{n-2}}\geq n|B_n|,$$

where the last inequality follows from the known relations

$$\mathrm{W}_j^{k-i}\geq\mathrm{W}_i^{k-j}\mathrm{W}_k^{j-i}\quad\text{ for }\ 0\leq i<j<k\leq n,\tag{6}$$

which are also consequences of the Aleksandrov-Fenchel inequality (see e.g. [12, (7.66)]). Hence, Corollary 1 improves (2) and (3) in the convex case.

We notice that the Theorem 2 provides lower estimates for the integral of *almost* any power of the *i*-th mean curvatures: bounds for $\int_{\mathrm{bd}K}H_i^{\lambda}\,\mathrm{d}\mathscr{H}^{n-1}$ are given *for any* $\lambda\in(-\infty,0]\cup[1,+\infty)$; the range $(0,1)$ is still an open question.

In this work we consider the opposite case, i.e., we will look for upper bounds for the integrals of powered *i*-th mean curvatures.

3 A probabilistic type result

All the results will be consequences of a very general proposition for functions defined on a general probability space. In [1, Proposition 1.4], this result was proved for a convex function F, whereas now we are interested in the concave case. Although this one can be obtained from the convex case just taking $-F$, for completeness, we include here the proof.

As usual in the literature,

$$\mathbb{E}\rho=\int_{\Omega}\rho(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

will denote the expectation of ρ, $\mathrm{Cov}(\rho,h)=\mathbb{E}h\rho-\mathbb{E}h\mathbb{E}\rho$ the covariance of ρ and h, and $\|\cdot\|_\infty$ the sup-norm, i.e., $\|f\|_\infty=\sup\big\{|f(\omega)|:\omega\in\Omega\big\}$.

Proposition 1. *Let $(\Omega,\mathbb{P})$ be a probability space such that, for any $A\subseteq\Omega$ and any $0\leq p\leq\mathbb{P}(A)$, there exists $B\subseteq A$ with $\mathbb{P}(B)=p$. Let $\rho,h:\Omega\longrightarrow\mathbb{R}$, with $\rho\in L^1(\Omega)$ and $h\in L^\infty(\Omega)$. Then, for any concave function $F:I\longrightarrow\mathbb{R}$, $I\subseteq\mathbb{R}$ where all the expressions below are defined, we have*

$$\mathbb{E}(F\circ\rho)\leq\frac{F\left(\mathbb{E}\rho+\frac{\mathrm{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty}\right)+F\left(\mathbb{E}\rho-\frac{\mathrm{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty}\right)}{2}.$$

Proof. Without loss of generality we assume that $\mathrm{Cov}(\rho,h) \leq 0$; otherwise we just change h by $-h$.

Let m be a median of ρ, i.e., a value for which both

$$\mathbb{P}\big(\{\omega\in\Omega : \rho(\omega)\geq m\}\big)\geq 1/2 \quad \text{and} \quad \mathbb{P}\big(\{\omega\in\Omega : \rho(\omega)\leq m\}\big)\geq 1/2,$$

and let $\Omega_1 \subset \Omega$ and $\Omega_2 = \Omega\setminus\Omega_1$ be such that $\mathbb{P}(\Omega_1) = \mathbb{P}(\Omega_2) = 1/2$ and

$$\begin{aligned}\{\omega\in\Omega : \rho(\omega)>m\} &\subseteq \Omega_1 \subseteq \{\omega\in\Omega : \rho(\omega)\geq m\},\\ \{\omega\in\Omega : \rho(\omega)<m\} &\subseteq \Omega_2 \subseteq \{\omega\in\Omega : \rho(\omega)\leq m\}.\end{aligned}$$

We notice that such Ω_1 always exists. Indeed, by the definition of median,

$$\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)\leq m\}\Big)\geq \frac{1}{2},$$

and so

$$\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)> m\}\Big)\leq \frac{1}{2}.$$

Consequently, since

$$\begin{aligned}&\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)\geq m\}\Big)\\ &\qquad = \mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)> m\}\Big)+\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)= m\}\Big)\geq \frac{1}{2},\end{aligned}$$

we have that

$$\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)= m\}\Big)\geq \frac{1}{2}-\mathbb{P}\Big(\{\omega\in\Omega : \rho(\omega)> m\}\Big)\geq 0.$$

Then, by our assumptions on $(\Omega,\mathbb{P})$, there exists a subset $B\subseteq\{\omega\in\Omega : \rho(\omega)=m\}$ with $\mathbb{P}(B) = 1/2-\mathbb{P}\{\omega\in\Omega : \rho(\omega)>m\}$ and we can take

$$\Omega_1 = \{\omega\in\Omega : \rho(\omega)>m\}\cup B.$$

Now, let

$$\rho_1 = 2\int_{\Omega_1}\rho(\omega)\,\mathrm{d}\mathbb{P}(\omega) \quad \text{and} \quad \rho_2 = 2\int_{\Omega_2}\rho(\omega)\,\mathrm{d}\mathbb{P}(\omega).$$

Since $\rho_1+\rho_2 = 2\,\mathbb{E}\rho$, we can write

$$\rho_1 = \mathbb{E}\rho + b \quad \text{and} \quad \rho_2 = \mathbb{E}\rho - b \tag{7}$$

for some $b\geq 0$. First, we are going to prove that

$$\frac{|\mathrm{Cov}(\rho,h)|}{\|h-\mathbb{E}h\|_\infty}\leq b. \tag{8}$$

Indeed, since

$$-\|h-\mathbb{E}h\|_\infty \le \mathbb{E}h - h(\omega) \le \|h-\mathbb{E}h\|_\infty$$

for every $\omega \in \Omega$ and since $\rho(\omega) \ge m$ if $\omega \in \Omega_1$ and $\rho(\omega) \le m$ if $\omega \in \Omega_2$, then we have that

$$\int_{\Omega_1} \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega)-m\big)\,\mathrm{d}\mathbb{P}(\omega) \le \frac{1}{2}\|h-\mathbb{E}h\|_\infty(\rho_1 - m)$$

and

$$\int_{\Omega_2} \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega)-m\big)\,\mathrm{d}\mathbb{P}(\omega) \le -\frac{1}{2}\|h-\mathbb{E}h\|_\infty(\rho_2 - m).$$

Adding both integrals and using (7) we get

$$\begin{aligned}\mathbb{E}\big((\mathbb{E}h-h)(\rho-m)\big) &= \int_{\Omega} \big(\mathbb{E}h - h(\omega)\big)\big(\rho(\omega)-m\big)\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \frac{1}{2}\|h-\mathbb{E}h\|_\infty(\rho_1-\rho_2) = \|h-\mathbb{E}h\|_\infty b,\end{aligned}$$

and since

$$\mathbb{E}\big((\mathbb{E}h-h)(\rho-m)\big) = \mathbb{E}h\mathbb{E}\rho - \mathbb{E}h\rho = -\operatorname{Cov}(\rho,h),$$

we obtain the required bound (8).

Now, since F is concave, Jensen's inequality (see e.g. [12, p. 20]) yields

$$F(\rho_1) \ge 2\int_{\Omega_1}(F\circ\rho)(\omega)\,\mathrm{d}\mathbb{P}(\omega) \quad \text{and} \quad F(\rho_2) \ge 2\int_{\Omega_2}(F\circ\rho)(\omega)\,\mathrm{d}\mathbb{P}(\omega),$$

which, together with (7) implies that

$$\begin{aligned}\mathbb{E}(F\circ\rho) &= \int_{\Omega_1}(F\circ\rho)(\omega)\,\mathrm{d}\mathbb{P}(\omega) + \int_{\Omega_2}(F\circ\rho)(\omega)\,\mathrm{d}\mathbb{P}(\omega)\\ &\le \frac{F(\rho_1)+F(\rho_2)}{2} = \frac{F(\mathbb{E}\rho+b)+F(\mathbb{E}\rho-b)}{2}.\end{aligned}$$

Finally, since a concave function F satisfies that for any $x\in\mathbb{R}$ and any $0\le a\le b$ the average of the numbers $\big\{F(x+a),F(x-a)\big\}$ is not smaller than the average of $\big\{F(x+b),F(x-b)\big\}$, taking into account (8) we get

$$\mathbb{E}(F\circ\rho) \le \frac{F(\mathbb{E}\rho+b)+F(\mathbb{E}\rho-b)}{2} \le \frac{F\left(\mathbb{E}\rho + \frac{\operatorname{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty}\right) + F\left(\mathbb{E}\rho - \frac{\operatorname{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty}\right)}{2},$$

which conclude the proof. $\square$

If the probability measure can be expressed by means of a density with respect to another (not necessarily a probability) measure μ, we immediately obtain the following result.

Proposition 2. *Let (Ω,μ) be a measure space and let $g:\Omega\longrightarrow\mathbb{R}$ be a positive integrable function with $\int_\Omega g\,\mathrm{d}\mu=1$, and such that for any $A\subseteq\Omega$ and any $0\le p\le\int_A g\,\mathrm{d}\mu$, there exists $B\subseteq A$ with $\int_B g\,\mathrm{d}\mu=p$. Let $\rho,h:\Omega\longrightarrow\mathbb{R}$ be integrable functions with $h\in L^\infty(\Omega)$. Then, for any concave function $F:I\longrightarrow\mathbb{R}$, $I\subseteq\mathbb{R}$ where all the expressions below are defined, we have*

$$\int_\Omega (F\circ\rho)g\,\mathrm{d}\mu\le\frac{F\left(\int_\Omega\rho g\,\mathrm{d}\mu+\eta(\rho,h,g)\right)+F\left(\int_\Omega\rho g\,\mathrm{d}\mu-\eta(\rho,h,g)\right)}{2},$$

where

$$\eta(\rho,h,g)=\frac{\int_\Omega\rho hg\,\mathrm{d}\mu-\left(\int_\Omega\rho g\,\mathrm{d}\mu\right)\left(\int_\Omega hg\,\mathrm{d}\mu\right)}{\|h-\int_\Omega hg\,\mathrm{d}\mu\|_\infty}.$$

4 Upper bounds for integrals of powered i-th mean curvatures

We denote by $h_K(u)=\sup_{x\in K}\langle x,u\rangle$, $u\in\mathbb{R}^n$, the support function of K (see e.g. [12, Section 1.7]), and let

$$q_K(x)=h_K\big(\nu_K(x)\big)=\big\langle x,\nu_K(x)\big\rangle,\quad x\in\mathrm{bd}\,K.$$

Minkowskian integral formulae (see e.g. [12, pp. 296–297]) state that

$$\mathrm{W}_i=\frac{1}{n}\int_{\mathrm{bd}\,K}H_{i-1}\,\mathrm{d}\mathscr{H}^{n-1}=\frac{1}{n}\int_{\mathrm{bd}\,K}q_KH_i\,\mathrm{d}\mathscr{H}^{n-1}\tag{9}$$

for $i=1,\dots,n$. We observe that the volume

$$|K|=\mathrm{W}_0=\frac{1}{n}\int_{\mathrm{bd}\,K}q_KH_0\,\mathrm{d}\mathscr{H}^{n-1}=\frac{1}{n}\int_{\mathrm{bd}\,K}q_K\,\mathrm{d}\mathscr{H}^{n-1}.$$

This section is devoted to look for upper bounds for the integrals of some powers of the i-th mean curvatures. First we show the following general result for an arbitrary concave function.

Theorem 3. *Let $K\in\mathscr{K}_0^n$ be of class C_+^2. For any concave function $F:I\longrightarrow\mathbb{R}$, $I\subseteq\mathbb{R}$ where all the quantities are defined, and all $i=0,\dots,n-1$,*

$$\int_{\mathrm{bd}\,K}(F\circ H_i)\,\mathrm{d}\mathscr{H}^{n-1}\le n\mathrm{W}_1\frac{F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1}+\eta_{i,0}\right)+F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1}-\eta_{i,0}\right)}{2},\tag{10}$$

$$\int_{\mathrm{bd}\,K}\left(F\circ\frac{1}{H_i}\right)H_i\,\mathrm{d}\mathscr{H}^{n-1}\le n\mathrm{W}_{i+1}\frac{F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}}+\eta_{i,i}\right)+F\left(\frac{\mathrm{W}_1}{\mathrm{W}_{i+1}}-\eta_{i,i}\right)}{2}.\tag{11}$$

Equality holds in both inequalities if $K=B_n$ (up to dilations).

Proof. In order to get (10), we consider the probability space $\big(\mathrm{bd}K,\mathscr{H}^{n-1}/(n\mathrm{W}_1)\big)$ and apply Proposition 1 to the functions $\rho=H_i$ and $h=q_K$. Then, using the identities in (9), we get

$$\mathbb{E}\rho=\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1},\qquad \mathbb{E}h=\frac{\mathrm{W}_0}{\mathrm{W}_1}$$

and

$$\mathrm{Cov}(\rho,h)=\mathbb{E}h\rho-\mathbb{E}h\mathbb{E}\rho=\frac{\mathrm{W}_i\mathrm{W}_1-\mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1^2}.$$

Moreover,

$$\|h-\mathbb{E}h\|_\infty=\sup\left\{\left|q_K(x)-\frac{\mathrm{W}_0}{\mathrm{W}_1}\right|:x\in\mathrm{bd}K\right\}=\max\left\{\overline{\mathrm{R}}-\frac{\mathrm{W}_0}{\mathrm{W}_1},\frac{\mathrm{W}_0}{\mathrm{W}_1}-\overline{\mathrm{r}}\right\},$$

and since the functionals H_j,W_j are translation invariant, the smallest possible upper bound for $\int_{\mathrm{bd}K}(F\circ H_i)\,\mathrm{d}\mathscr{H}^{n-1}$ will be obtained for the translation of K such that the above maximum is minimal. Therefore, we can write

$$\int_{\mathrm{bd}K}(F\circ H_i)\,\mathrm{d}\mathscr{H}^{n-1}\le n\mathrm{W}_1\frac{F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1}+\eta\right)+F\left(\frac{\mathrm{W}_{i+1}}{\mathrm{W}_1}-\eta\right)}{2},$$

with

$$\delta=\frac{\mathrm{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty}=\frac{\mathrm{W}_i\mathrm{W}_1-\mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1^2\min\limits_{x\in K}\max\left\{\overline{\mathrm{R}}(K-x)-\frac{\mathrm{W}_0}{\mathrm{W}_1},\frac{\mathrm{W}_0}{\mathrm{W}_1}-\overline{\mathrm{r}}(K-x)\right\}}.$$

Now we observe that, by (4),

$$\begin{aligned}\overline{\mathrm{R}}(K-x)-\frac{\mathrm{W}_0}{\mathrm{W}_1}&\le\overline{\mathrm{R}}(K-x)-\overline{\mathrm{r}}(K-x),\qquad\text{and}\\ \frac{\mathrm{W}_0}{\mathrm{W}_1}-\overline{\mathrm{r}}(K-x)&\le\overline{\mathrm{R}}(K-x)-\overline{\mathrm{r}}(K-x),\end{aligned}$$

and since F is a concave function, we can replace δ by a smaller number, namely,

$$\delta\ge\frac{\mathrm{W}_i\mathrm{W}_1-\mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1^2\min\limits_{x\in K}\left\{\overline{\mathrm{R}}(K-x)-\overline{\mathrm{r}}(K-x)\right\}}=\frac{\mathrm{W}_i\mathrm{W}_1-\mathrm{W}_0\mathrm{W}_{i+1}}{\mathrm{W}_1^2\omega_a}=\eta_{i,0}.$$

Altogether we get (10).

Inequality (11) is obtained analogously, but now as a consequence of Proposition 2 for $\rho=1/H_i$, $h=q_K$ and $g=H_i/(n\mathrm{W}_{i+1})$; we notice that, by (9), $\int_{\mathrm{bd}K}g\,\mathrm{d}\mathscr{H}^{n-1}=1$.

Finally, equality trivially holds for $K=B_n$ (up to dilations) just noticing that $\mathrm{W}_i(B_n)=|B_n|$ for all $i=0,\dots,n$. $\square$

In order to get bounds for the integral of some powers of the mean curvatures, we may apply Theorem 3 to the concave function $F(x)=x^\alpha$ for $0\le\alpha\le1$.

Theorem 4. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. Then, for all $i = 0, \dots, n-1$, the following inequalities hold:*

- *If* $0 \leq \alpha \leq 1/2$,

$$\int_{\mathrm{bd}K} H_i^{\alpha} \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_1^{1-\alpha} \Big[\big(\mathrm{W}_{i+1} + \mathrm{W}_1\, \eta_{i,0} \big)^{\alpha} + \big(\mathrm{W}_{i+1} - \mathrm{W}_1\, \eta_{i,0} \big)^{\alpha} \Big]. \quad (12)$$

- *If* $1/2 \leq \alpha \leq 1$,

$$\int_{\mathrm{bd}K} H_i^{\alpha} \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_{i+1}^{\alpha} \Big[\big(\mathrm{W}_1 + \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{1-\alpha} + \big(\mathrm{W}_1 - \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{1-\alpha} \Big]. \quad (13)$$

Equality holds in both inequalities if $K = B_n$ (up to dilations).

Proof. Let $0 \leq \alpha \leq 1$. On the one hand, taking $F(x) = x^{\alpha}$ in (10) we directly get

$$\int_{\mathrm{bd}K} H_i^{\alpha} \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_1^{1-\alpha} \Big[\big(\mathrm{W}_{i+1} + \mathrm{W}_1\, \eta_{i,0} \big)^{\alpha} + \big(\mathrm{W}_{i+1} - \mathrm{W}_1\, \eta_{i,0} \big)^{\alpha} \Big]. \quad (14)$$

On the other hand, (11) applied to $F(x) = x^{\alpha}$ yields

$$\int_{\mathrm{bd}K} H_i^{1-\alpha} \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_{i+1}^{1-\alpha} \Big[\big(\mathrm{W}_1 + \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{\alpha} + \big(\mathrm{W}_1 - \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{\alpha} \Big]$$

or, equivalently,

$$\int_{\mathrm{bd}K} H_i^{\alpha} \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_{i+1}^{\alpha} \Big[\big(\mathrm{W}_1 + \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{1-\alpha} + \big(\mathrm{W}_1 - \mathrm{W}_{i+1}\, \eta_{i,i} \big)^{1-\alpha} \Big]. \quad (15)$$

Therefore, we just have to compare both bounds, depending on the value of α. In order to do it, we denote by

$$x = \frac{\mathrm{W}_1}{\mathrm{W}_{i+1}} \eta_{i,0} = \frac{\mathrm{W}_{i+1}}{\mathrm{W}_1} \eta_{i,i} = \frac{\mathrm{W}_1 \mathrm{W}_i - \mathrm{W}_0 \mathrm{W}_{i+1}}{\mathrm{W}_1 \mathrm{W}_{i+1} \omega_a}.$$

Using (4) and (5) we get that

$$0 \leq \mathrm{W}_1 \mathrm{W}_i - \mathrm{W}_0 \mathrm{W}_{i+1} \leq \mathrm{W}_1 \mathrm{W}_{i+1} \big(\overline{\mathrm{R}}(K - c_K) - \overline{\mathrm{r}}(K - c_K) \big) = \mathrm{W}_1 \mathrm{W}_{i+1} \omega_a,$$

and therefore, $0 \leq x \leq 1$. Using this notation, the upper bounds in (14) and (15) can be written, respectively, as

$$\mathrm{W}_1^{1-\alpha} \mathrm{W}_{i+1}^{\alpha} \big[(1+x)^{\alpha} + (1-x)^{\alpha} \big] =: (\mathrm{b1}),$$
$$\mathrm{W}_1^{1-\alpha} \mathrm{W}_{i+1}^{\alpha} \big[(1+x)^{1-\alpha} + (1-x)^{1-\alpha} \big] =: (\mathrm{b2}).$$

Then (b1) is, say, smaller than (b2), if and only if

$$(1+x)^{\alpha} + (1-x)^{\alpha} \leq (1+x)^{1-\alpha} + (1-x)^{1-\alpha}, \quad (16)$$

and it clearly holds when $\alpha \leq 1/2$. This shows (12). Finally, (b1) $\geq$ (b2) is equivalent to have the reverse inequality in (16), which holds if $\alpha \geq 1/2$. It states (13) concludes the proof of the theorem. $\square$

It may also have interest to obtain an estimate for the entropy of the i-th mean curvatures, which is defined by

$$-\int_{\mathrm{bd}K} H_i \log H_i \, \mathrm{d}\mathscr{H}^{n-1}.$$

We do it in the following result.

Corollary 2. *Let $K \in \mathscr{K}_0^n$ be of class C_+^2. For all $i = 0, \dots, n-1$,*

$$-\int_{\mathrm{bd}K} H_i \log H_i \, \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_{i+1} \log \left(\frac{\mathrm{W}_1^2}{\mathrm{W}_{i+1}^2} - \eta_{i,i}^2 \right).$$

Equality holds if $K = B_n$ (up to dilations).

Proof. It is a direct consequence of inequality (11), just considering the concave function $F(x) = \log x$:

$$-\int_{\mathrm{bd}K} H_i \log H_i \, \mathrm{d}\mathscr{H}^{n-1} = \int_{\mathrm{bd}K} H_i \log \frac{1}{H_i} \, \mathrm{d}\mathscr{H}^{n-1} \leq \frac{n}{2} \mathrm{W}_{i+1} \log \left(\frac{\mathrm{W}_1^2}{\mathrm{W}_{i+1}^2} - \eta_{i,i}^2 \right). \square$$

4.1 On the radii of curvature of convex bodies

If $K \in \mathscr{K}_0^n$ is of class C_+^2, we can consider the $n-1$ principal radii of curvature $r_1, \dots, r_{n-1}$ of K at $u \in \mathbb{S}^{n-1}$, i.e., the eigenvalues of the reverse Weingarten map (see e.g. [12, p. 116] for a detailed explanation). Then, for $i = 1, \dots, n-1$,

$$s_i = \frac{1}{\binom{n-1}{i}} \sum_{1 \leq j_1 < \cdots < j_i \leq n-1} r_{j_1} \cdots r_{j_i}$$

is the i-th normalized elementary symmetric function of the principal radii of curvature, with $s_0 = 1$. We observe that, properly ordering the indices,

$$r_i(u) = \frac{1}{k_i\big(x_K(u)\big)}, \quad i = 1, \dots, n-1,$$

where $x_K(u) \in \mathrm{bd}K$ is the unique point of the boundary at which u is the outer normal vector. Moreover, for all $u \in \mathbb{S}^{n-1}$ and $x \in \mathrm{bd}K$ we have the relations

$$s_i(u) = \frac{H_{n-i-1}}{H_{n-1}}\big(x_K(u)\big) \quad \text{and} \quad H_i(x) = \frac{s_{n-i-1}}{s_{n-1}}\big(\nu_K(x)\big),$$

and so there exist also Minkowskian integral formulae for the s_i's (see e.g. [12, pp. 296–297]): for all $i=0,\dots,n-1$,

$$\mathrm{W}_i=\frac{1}{n}\int_{\mathbb{S}^{n-1}} s_{n-i}\,\mathrm{d}\mathscr{H}^{n-1}=\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K s_{n-i-1}\,\mathrm{d}\mathscr{H}^{n-1}. \tag{17}$$

In [1] the analogous result to Theorem 1 for the so-called i-th (normalized) elementary symmetric function of the principal radii of curvature was obtained. In a similar way we can also get the corresponding result for a concave function, which will provide upper bounds for the integrals of some powers of the s_i's.

Theorem 5. *Let $K\in\mathscr{K}_0^n$ be of class C_+^2. For any concave function $F:I\longrightarrow\mathbb{R}$, $I\subseteq\mathbb{R}$ where all the quantities are defined, and all $i=0,\dots,n-1$,*

$$\int_{\mathbb{S}^{n-1}}(F\circ s_i)\,\mathrm{d}\mathscr{H}^{n-1}\le n|B_n|\frac{F\left(\frac{\mathrm{W}_{n-i}}{|B_n|}+\overline{\eta}_{i,1}\right)+F\left(\frac{\mathrm{W}_{n-i}}{|B_n|}-\overline{\eta}_{i,1}\right)}{2},$$
$$\int_{\mathbb{S}^{n-1}}\left(F\circ\frac{1}{s_i}\right)s_i\,\mathrm{d}\mathscr{H}^{n-1}\le n\mathrm{W}_{n-i}\frac{F\left(\frac{|B_n|}{\mathrm{W}_{n-i}}+\overline{\eta}_{i,i+1}\right)+F\left(\frac{|B_n|}{\mathrm{W}_{n-i}}-\overline{\eta}_{i,i+1}\right)}{2},$$

where now, for any $0\le j,k\le n$,

$$\overline{\eta}_{j,k}=\begin{cases}\dfrac{\mathrm{W}_{n-j}\mathrm{W}_{n-1}-\mathrm{W}_{n-j-1}|B_n|}{\mathrm{W}_{n-k+1}^2\,\omega_a} & \text{if } K\neq rB_n \text{ for all } r>0,\\ 0 & \text{if } K=rB_n.\end{cases}$$

Equality holds in both inequalities if $K=B_n$ (up to dilations).

Proof. In order to get the first inequality we apply Proposition 1 to the probability space $\left(\mathbb{S}^{n-1},\mathscr{H}^{n-1}/(n|B_n|)\right)$ and to the functions $\rho=s_i$ and $h=h_K$. Then, using the Minkowski integral formula (17), we get

$$\mathbb{E}\rho=\frac{\mathrm{W}_{n-i}}{|B_n|},\qquad \mathbb{E}h=\frac{\mathrm{W}_{n-1}}{|B_n|}$$

and

$$\mathrm{Cov}(\rho,h)=\frac{\mathrm{W}_{n-i-1}|B_n|-\mathrm{W}_{n-i}\mathrm{W}_{n-1}}{|B_n|^2}.$$

In addition,

$$\begin{aligned}\|h-\mathbb{E}h\|_\infty&=\sup\left\{\left|h_K(u)-\frac{\mathrm{W}_{n-1}}{|B_n|}\right|:u\in\mathbb{S}^{n-1}\right\}\\&=\max\left\{\overline{\mathrm{R}}-\frac{\mathrm{W}_{n-1}}{|B_n|},\frac{\mathrm{W}_{n-1}}{|B_n|}-\overline{\mathrm{r}}\right\},\end{aligned}$$

and since the functionals s_j,W_j are translation invariant, the smallest possible upper bound for $\int_{\mathrm{bd}K}(F\circ s_i)\,\mathrm{d}\mathscr{H}^{n-1}$ will be obtained for the translation of K such that the above maximum is minimal.

Moreover, using (4) we have

$$\begin{aligned}\frac{\operatorname{Cov}(\rho,h)}{\|h-\mathbb{E}h\|_\infty} &= \frac{\mathrm{W}_{n-i-1}|B_n|-\mathrm{W}_{n-i}\mathrm{W}_{n-1}}{|B_n|^2\min\limits_{x\in K}\max\left\{\overline{\mathrm{R}}(K-x)-\dfrac{\mathrm{W}_{n-1}}{|B_n|},\dfrac{\mathrm{W}_{n-1}}{|B_n|}-\overline{\mathrm{r}}(K-x)\right\}}\\ &\geq \frac{\mathrm{W}_{n-i-1}|B_n|-\mathrm{W}_{n-i}\mathrm{W}_{n-1}}{|B_n|^2\min\limits_{x\in K}\left\{\overline{\mathrm{R}}(K-x)-\overline{\mathrm{r}}(K-x)\right\}}=\frac{\mathrm{W}_{n-i-1}|B_n|-\mathrm{W}_{n-i}\mathrm{W}_{n-1}}{|B_n|^2\omega_a}=-\overline{\eta}_{i,1}.\end{aligned}$$

Altogether and the concavity of F show the first inequality.

Second inequality is obtained analogously, but now as a consequence of Proposition 2 for $\rho=1/s_i$, $h=h_K$ and $g=s_i/(n\mathrm{W}_{n-i})$; we notice that, by (17), $\int_{\mathbb{S}^{n-1}} g\,\mathrm{d}\mathscr{H}^{n-1}=1$. The equality case is trivial. □

If we replace $F(x)$ by the concave function x^α, $0\leq\alpha\leq 1$, we get the corresponding result to Theorem 4 for the s_i's.

Theorem 6. *Let $K\in\mathscr{K}_0^n$ be of class C_+^2. Then, for all $i=0,\dots,n-1$, the following inequalities hold:*

- *If $0\leq\alpha\leq 1/2$,*

$$\int_{\mathbb{S}^{n-1}} s_i^\alpha\,\mathrm{d}\mathscr{H}^{n-1}\leq\frac{n}{2}|B_n|^{1-\alpha}\Big(\mathrm{W}_{n-i}+|B_n|\overline{\eta}_{i,1}\Big)^\alpha+\Big(\mathrm{W}_{n-i}-|B_n|\overline{\eta}_{i,1}\Big)^\alpha.$$

- *If $1/2\leq\alpha\leq 1$,*

$$\int_{\mathbb{S}^{n-1}} s_i^\alpha\,\mathrm{d}\mathscr{H}^{n-1}\leq\frac{n}{2}\mathrm{W}_{n-i}^\alpha\Big[\big(|B_n|+\mathrm{W}_{n-i}\,\overline{\eta}_{i,i+1}\big)^{1-\alpha}+\big(|B_n|-\mathrm{W}_{n-i}\,\eta_{i,i+1}\big)^{1-\alpha}\Big].$$

Equality holds in both inequalities if $K=B_n$ (up to dilations).

5 Another consequence: the radial function and the dual quermaßintegrals

In this section we will apply Proposition 1 in a difference setting: instead of working with convex bodies we will consider the so-called starshaped sets. A non-empty set $S\subset\mathbb{R}^n$ is called *starshaped* (with respect to the origin) if the line segment $[0,x]\subseteq S$ for all $x\in S$. For a compact starshaped set K, the radial function is defined as

$$\rho_K(u)=\max\big\{\lambda\geq 0:\lambda u\in K\big\},\quad u\in\mathbb{R}^n\backslash\{0\}.$$

Clearly, $\rho_K(u)u\in\operatorname{bd}K$. We will denote by $\mathscr{S}_0^n$ the family of all compact starshaped sets in $\mathbb{R}^n$ having the origin as an interior point.

Closely related to the radial function are dual quermaßintegrals (and dual mixed volumes), which were introduced by Lutwak in [9]; they were the starting point for the development of the nowadays known as dual Brunn-Minkowski theory (see e.g. [12, Section 9.3]). For $K \in \mathscr{S}_0^n$ and $i = 0, \dots, n$, the *dual quermaßintegral of order* $n-i$, $\widetilde{\mathrm{W}}_{n-i}(K)$, is defined by

$$\widetilde{\mathrm{W}}_{n-i}(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \rho_K^i \,\mathrm{d}\mathscr{H}^{n-1}. \tag{18}$$

The functional $\widetilde{\mathrm{W}}_{n-i}$ is non-negative, monotonous and homogeneous of degree i (see e.g. [7, Section A.7]), although it is not translation invariant. In particular, the use of spherical coordinates immediately yields $\widetilde{\mathrm{W}}_0(K) = |K|$, whereas $\widetilde{\mathrm{W}}_n(K) = |B_n|$ and $2\widetilde{\mathrm{W}}_{n-1}(K)/|B_n|$ is the average length of chords of K through the origin. Moreover, if $K \in \mathscr{K}_0^n$ then $\widetilde{\mathrm{W}}_i(K) \leq \mathrm{W}_i(K)$ for all $i = 0, \dots, n$ (see [9]).

For $K \in \mathscr{S}_0^n$, its in- and outer radii, $\overline{\mathrm{r}}(K)$, $\overline{\mathrm{R}}(K)$, are defined analogously to the convex case, and from the already mentioned monotonicity of the dual quermaßintegrals we get (cf. (4))

$$\overline{\mathrm{r}}(K)^{n-j}\, \widetilde{\mathrm{W}}_{n-j+k}(K) \leq \widetilde{\mathrm{W}}_k(K) \leq \overline{\mathrm{R}}^{n-j}(K)\, \widetilde{\mathrm{W}}_{n-j+k}(K). \tag{19}$$

We observe that definition (18) can be extended to any real number, and thus, in contrast to the case of the classical quermaßintegrals, dual quermaßintegrals can be defined for any $i \in \mathbb{R}$. Now, the *dual Aleksandrov-Fenchel inequalities* (see e.g. [9, Theorem 2]) read (cf. (6))

$$\widetilde{\mathrm{W}}_j(K)^{k-i} \leq \widetilde{\mathrm{W}}_i(K)^{k-j} \widetilde{\mathrm{W}}_k(K)^{j-i}, \quad \text{for } \ i \leq j \leq k. \tag{20}$$

In [1], a slightly stronger version of the following result was obtained. Again, for the sake of brevity, we will write $\widetilde{\mathrm{W}}_i = \widetilde{\mathrm{W}}_i(K)$.

Theorem 7. *Let $K \in \mathscr{S}_0^n$. For any convex function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $i = 0, \dots, n$,*

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \rho_K^i\right) \mathrm{d}\mathscr{H}^{n-1} \geq n|B_n| \frac{F\left(\frac{\widetilde{\mathrm{W}}_{n-i}}{|B_n|} + \widetilde{\eta}_{i,0}\right) + F\left(\frac{\widetilde{\mathrm{W}}_{n-i}}{|B_n|} - \widetilde{\eta}_{i,0}\right)}{2}, \tag{21}$$

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \frac{1}{\rho_K^i}\right) \rho_K^i \,\mathrm{d}\mathscr{H}^{n-1} \geq n\widetilde{\mathrm{W}}_{n-i} \frac{F\left(\frac{|B_n|}{\widetilde{\mathrm{W}}_{n-i}} + \widetilde{\eta}_{i,i}\right) + F\left(\frac{|B_n|}{\widetilde{\mathrm{W}}_{n-i}} - \widetilde{\eta}_{i,i}\right)}{2},$$

where now, for any $0 \leq j,k \leq n$,

$$\widetilde{\eta}_{j,k} = \frac{|K||B_n| - \widetilde{\mathrm{W}}_{n-j}\widetilde{\mathrm{W}}_j}{\widetilde{\mathrm{W}}_{n-k}^2 \left(\overline{\mathrm{R}}^{n-j} - \overline{\mathrm{r}}^{n-j}\right)} \quad \text{if } K \neq rB_n,\ r > 0,$$
$$\widetilde{\eta}_{j,k} = 0 \qquad \text{if } K = rB_n \text{ for some } r > 0.$$

Equality holds in both inequalities if $K = B_n$ (up to dilations).

We observe that the relation $|K||B_2^n| \geq \widetilde{\mathrm{W}}_{n-j}\widetilde{\mathrm{W}}_j$, a consequence of the dual Aleksandrov-Fenchel inequality (20), ensures that $\widetilde{\eta}_{j,k} \geq 0$.

Following the same argument as in the proof of the above theorem, we obtain the corresponding result for the case of a concave function.

Theorem 8. *Let $K \in \mathscr{S}_0^n$. For any concave function $F : I \longrightarrow \mathbb{R}$, $I \subseteq \mathbb{R}$ where all the quantities are defined, and all $i = 0,\dots,n$,*

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \rho_K^i\right) \mathrm{d}\mathscr{H}^{n-1} \leq n|B_n| \frac{F\left(\frac{\widetilde{\mathrm{W}}_{n-i}}{|B_n|} + \widetilde{\eta}_{i,0}\right) + F\left(\frac{\widetilde{\mathrm{W}}_{n-i}}{|B_n|} - \widetilde{\eta}_{i,0}\right)}{2}, \tag{22}$$

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \frac{1}{\rho_K^i}\right) \rho_K^i \,\mathrm{d}\mathscr{H}^{n-1} \leq n\widetilde{\mathrm{W}}_{n-i} \frac{F\left(\frac{|B_n|}{\widetilde{\mathrm{W}}_{n-i}} + \widetilde{\eta}_{i,i}\right) + F\left(\frac{|B_n|}{\widetilde{\mathrm{W}}_{n-i}} - \widetilde{\eta}_{i,i}\right)}{2}.$$

Equality holds in both inequalities if $K = B_n$ (up to dilations).

Proof. In order to prove (22) we apply Proposition 1 to the probability space $\left(\mathbb{S}^{n-1}, \mathscr{H}^{n-1}/(n|B_n|)\right)$ and the functions $\rho = \rho_K^i$ and $h = \rho_K^{n-i}$. Then, using (18), $\mathbb{E}\rho = \widetilde{\mathrm{W}}_{n-i}/|B_n|$, $\mathbb{E}h = \widetilde{\mathrm{W}}_i/|B_n|$ and

$$\mathrm{Cov}(\rho, h) = \frac{|K||B_n| - \widetilde{\mathrm{W}}_{n-i}\widetilde{\mathrm{W}}_i}{|B_n|^2}.$$

Moreover, since $\rho_K(u)u \in \mathrm{bd}\,K$, the relations (19) yield

$$\begin{aligned} \|h - \mathbb{E}h\|_\infty &= \sup\left\{\left|\rho_K(u)^{n-i} - \frac{\widetilde{\mathrm{W}}_i}{|B_n|}\right| : u \in \mathbb{S}^{n-1}\right\} \\ &= \max\left\{\overline{\mathrm{R}}^{n-i} - \frac{\widetilde{\mathrm{W}}_i}{|B_n|}, \frac{\widetilde{\mathrm{W}}_i}{|B_n|} - \overline{\mathrm{r}}^{n-i}\right\} \geq \overline{\mathrm{R}}^{n-i} - \overline{\mathrm{r}}^{n-i}. \end{aligned}$$

Altogether and the concavity of F shows the first inequality.

Second inequality is obtained analogously, but now as a consequence of Proposition 2 for $\rho = 1/\rho_K^i$, $h = \rho_K^{n-i}$ and $g = \rho_K^i/(n\widetilde{\mathrm{W}}_{n-i})$. The equality case is trivial. □

We observe that since (18) can be defined for any $i \in \mathbb{R}$, taking $F(x) = x^\alpha$ or $F(x) = 1/x^\alpha$ for suitable powers $\alpha \geq 0$, new inequalities relating the dual quermaßintegrals with the in- and outer radii can be obtained. Indeed, even Theorems 7 and 8 hold true for all $i \in \mathbb{R}$, just properly defining the values $\widetilde{\eta}_{j,k}$.

For instance, taking $F(x) = x^2$ in (21), then

$$\int_{\mathbb{S}^{n-1}} \left(F \circ \rho_K^i\right) \mathrm{d}\mathscr{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \rho_K^{2i} \,\mathrm{d}\mathscr{H}^{n-1} = n\widetilde{\mathrm{W}}_{n-2i}$$

for any $i = 0,\dots,n$, and hence we get

$$|B_n|^2 \left(|B_n|\widetilde{\mathrm{W}}_{n-2i} - \widetilde{\mathrm{W}}_{n-i}^2\right) \left(\overline{\mathrm{R}}^{n-i} - \overline{\mathrm{r}}^{n-i}\right)^2 \geq \left(|K||B_n| - \widetilde{\mathrm{W}}_{n-i}\widetilde{\mathrm{W}}_i\right)^2,$$

with equality for the ball.

If we consider now the concave function $F(x)=\sqrt{x}$ and apply (22), we obtain

$$2\widetilde{\mathrm{W}}_{n-i/2}^2 \le |B_n|\widetilde{\mathrm{W}}_{n-i} + \sqrt{|B_n|^2\widetilde{\mathrm{W}}_{n-i}^2 - \frac{\left(|K||B_n| - \widetilde{\mathrm{W}}_{n-i}\widetilde{\mathrm{W}_i}\right)^2}{\left(\overline{\mathrm{R}}^{n-i} - \overline{\mathrm{r}}^{n-i}\right)}};$$

here we are assuming that $K \neq rB_n$, otherwise we get a trivial identity.

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