
On successive minima and successive radii

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Abstract

We establish new inequalities relating products of the successive minima of a 0-symmetric convex body to the product of its successive outer and inner radii. These inequalities are motivated by results of Henk and Hernández Cifre [11], but relate only the sequences of successive minima and radii, without involving the volume.

1 Introduction

We work in the n -dimensional Euclidean space $\mathbb{R}^n$ endowed with the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\| \cdot \|$. Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$, i.e., compact convex sets with nonempty interior, and we write $\mathcal{K}_0^n$ to represent the subfamily of $\mathcal{K}^n$ of all 0-symmetric convex bodies, that is, those $K \in \mathcal{K}^n$ with $K = -K$. We denote the n -dimensional unit ball by B_n . The volume of a measurable set $A \subset \mathbb{R}^n$, i.e., its n -dimensional Lebesgue measure, is denoted by $\text{vol}(A)$ (or $\text{vol}_n(A)$ if the dimension needs to be specified) and, in particular, we write $\kappa_n := \text{vol}(B_n)$. Furthermore, the dimension of A , i.e., the dimension of the smallest affine subspace containing A , is denoted by $\dim A$.

On the other hand, we represent by $\mathbb{Z}^n$ the integer lattice, i.e., the set of all points with integral coordinates in $\mathbb{R}^n$. Then, any lattice Λ in $\mathbb{R}^n$ can be obtained as $\Lambda = B\mathbb{Z}^n$ for some $B \in \text{GL}_n(\mathbb{R})$. The determinant of Λ is defined as $\det \Lambda = |\det B|$, and it can be proved that it does not depend on the matrix B generating the lattice. As general references for lattices we refer to [6, 7].

A key notion in the Geometry of Numbers is that of the successive minima. Given a 0-symmetric convex body $K \in \mathcal{K}_0^n$, the i -th successive minimum $\lambda_i(K, \Lambda)$ of K with respect to a lattice Λ is defined as

$$\lambda_i(K, \Lambda) = \min\{\lambda \geq 0 : \dim(\lambda K \cap \Lambda) \geq i\}, \quad i = 1, \dots, n.$$

Obviously, $\lambda_1(K, \Lambda) \leq \dots \leq \lambda_n(K, \Lambda)$ and the first successive minimum $\lambda_1(K, \Lambda)$ is the smallest dilation factor such that $\lambda_1(K, \Lambda)K$ contains a nonzero lattice point. With this notation, Minkowski's second theorem on successive minima (see e.g. [7, Sections 9.1 and 9.4], [9]) states that if $K \in \mathcal{K}_0^n$ then

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1(K, \Lambda) \cdots \lambda_n(K, \Lambda) \text{vol}(K) \leq 2^n \det \Lambda. \quad (1)$$

In the case of the integer lattice $\mathbb{Z}^n$ we will just write $\lambda_i(K)$ instead of $\lambda_i(K, \mathbb{Z}^n)$.

In this work we are mainly interested in studying the relation between the successive minima and particular sequences of functionals associated to a convex body, the so-called successive radii. Next we introduce them.

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The set of all i -dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$, the orthogonal projection of K onto L is denoted by $K|L$ and we represent by $L^\perp$ its orthogonal complement. Furthermore, we denote the linear hull of the vectors $u_1, \dots, u_m$ by $\text{lin}\{u_1, \dots, u_m\}$.

The diameter, the minimal width, the circumradius and the inradius of a convex body K are denoted by $D(K)$, $\omega(K)$, $R(K)$, and $r(K)$, respectively. For more information on these functionals and their properties we refer to [3, pp. 56-59]. If f is a functional on $\mathcal{K}^n$ and K is contained in an affine subspace H of dimension k , we write $f(K; H)$ to express that f has to be evaluated in the corresponding ambient space $\mathbb{R}^k$ (where we identify H with $\mathbb{R}^k$). With this notation we define the following successive outer and inner radii.

Definition 1. For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let

$$R_i(K) = \min_{L \in \mathcal{L}_i^n} R(K|L) \quad \text{and} \quad r_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} r(K \cap (x + L); x + L).$$

Geometrically, $R_i(K)$ is the smallest radius of a solid cylinder with i -dimensional spherical cross section containing K , whereas $r_i(K)$ is the radius of the greatest i -dimensional ball contained in K .

If we replace in the definition of R_i the min-condition by a max-condition and in the definition of r_i the first max-condition by a min-condition, we obtain other families of successive radii:

Definition 2. For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let

$$\bar{R}_i(K) = \max_{L \in \mathcal{L}_i^n} R(K|L) \quad \text{and} \quad \bar{r}_i(K) = \min_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} r(K \cap (x + L); x + L).$$

From the definition it follows trivially that the first and last radii of each family coincide either with the circumradius, the inradius, the minimal width or the diameter, namely,

$$\begin{aligned} R(K) &= R_n(K) = \bar{R}_n(K), & r(K) &= r_n(K) = \bar{r}_n(K), \\ \frac{D(K)}{2} &= \bar{R}_1(K) = r_1(K), & \frac{\omega(K)}{2} &= R_1(K) = \bar{r}_1(K). \end{aligned}$$

Moreover, it is clear that both types of outer successive radii are increasing in i , whereas inner successive radii are decreasing in i for $1 \leq i \leq n$.

The first systematic study of all these families of successive radii was developed in [2]. In particular, the authors proved several relations between the different series of inner and outer radii and the volume of an arbitrary convex body $K \in \mathcal{K}^n$, namely:

$$\frac{2^n}{n!} r_1(K) \cdots r_n(K) \leq \text{vol}(K) \leq 2^n R_1(K) \cdots R_n(K), \quad (2)$$

$$\kappa_n \bar{r}_1(K) \cdots \bar{r}_n(K) \leq \text{vol}(K) \leq \kappa_n \bar{R}_1(K) \cdots \bar{R}_n(K). \quad (3)$$

For more information on successive radii, their size for special bodies, as well as their relation with other measures, we refer, among others, to [1, 2, 4, 5, 11, 10] and the references therein.

In particular, in [11], Henk and Hernández Cifre studied the relations between the successive radii given in Definition 1 and the successive minima of a 0-symmetric convex body with respect to the integer lattice, establishing inequalities connecting products of the successive minima with products of the radii of the set: for $K \in \mathcal{K}_0^n$ and $i = 1, \dots, n-1$ one has

$$\begin{aligned} \lambda_{i+1}(K) \cdots \lambda_n(K) \text{vol}(K) &\leq 2^n R_{n-i+1}(K) \cdots R_n(K), \\ \lambda_1(K) \cdots \lambda_i(K) \text{vol}(K) &\geq \frac{2^n}{n!} r_{i+1}(K) \cdots r_n(K). \end{aligned} \quad (4)$$

These results build a bridge between Minkowski's second fundamental theorem and classical volume bounds involving inner and outer radii: the "limit" cases $i = 0$ and $i = n$ in the above inequalities (i.e.,

when either no successive radii or no successive minima appear) recover Minkowski's second theorem for successive minima and inequality (2).

Following the same argument as in [11], but now using the relations in (3), analogous inequalities can be obtained involving the successive radii stated in Definition 2: for $K \in \mathcal{K}_0^n$ and $i = 1, \dots, n-1$ one has

$$\begin{aligned}\lambda_{i+1}(K) \cdots \lambda_n(K) \operatorname{vol}(K) &\leq 2^{n-i} \kappa_i \bar{R}_1(K) \cdots \bar{R}_i(K), \\ \lambda_1(K) \cdots \lambda_i(K) \operatorname{vol}(K) &\geq \frac{2^i}{i! \binom{n}{i}} \kappa_{n-i} \bar{r}_1(K) \cdots \bar{r}_{n-i}(K).\end{aligned}$$

We observe that these bounds improve the ones which can be obtained directly from (4) using that the radii defined via the minimum are trivially smaller than the ones defined via the maximum, because $2^{n-i} \kappa_i \leq 2^n$ and $2^n/n! \leq (2^i \kappa_{n-i})/(i! \binom{n}{i})$.

In the present work, we investigate new relations between the successive minima and the successive radii of a 0-symmetric convex body that do not involve the volume. Our main results are the following:

Theorem 3. *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdots \lambda_n(K) R_1(K) \cdots R_{n-i}(K) \leq n!.$$

Furthermore,

$$\lambda_{i+1}(K) \cdots \lambda_n(K) r_{i+1}(K) \cdots r_n(K) \leq n!.$$

Theorem 4. *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\frac{1}{i!} \leq \lambda_1(K) \cdots \lambda_i(K) r_1(K) \cdots r_i(K).$$

Furthermore,

$$\frac{1}{i!} \leq \lambda_1(K) \cdots \lambda_i(K) R_{n-i+1}(K) \cdots R_n(K).$$

Unfortunately, the above inequalities are not sharp, and it remains an open question to determine the optimal constants.

In the next section, we present the proofs of these results, as well as some consequences for general (not necessarily 0-symmetric) convex bodies.

2 Proofs of the main results

The general proof method is inspired by the approach developed by Henk and Hernández Cifre in [11].

Proof of Theorem 3. Let $z_1, \dots, z_i \in K$ be i linearly independent points with $\lambda_j(K)z_j \in \lambda_j(K)K \cap \mathbb{Z}^n$. We consider a suitable $(n-i)$ -dimensional coordinate plane

$$L_{n-i} = \{x \in \mathbb{R}^n : x_{j_1} = \cdots = x_{j_i} = 0, j_k \in \{1, \dots, n\}\}$$

such that

$$\operatorname{lin}\{z_1, \dots, z_i\} \cap L_{n-i} = \{0\}. \quad (5)$$

Now we denote by $\Lambda_{n-i} = \mathbb{Z}^n \cap L_{n-i}$, which is the sublattice of all points in L_{n-i} with integer coordinates. Then, Minkowski's second theorem applied to $K \cap L_{n-i}$ and Λ_{n-i} ensures that

$$\lambda_1(K \cap L_{n-i}, \Lambda_{n-i}) \cdots \lambda_{n-i}(K \cap L_{n-i}, \Lambda_{n-i}) \operatorname{vol}_{n-i}(K \cap L_{n-i}) \leq 2^{n-i}.$$

From (5) we get that, for any $j = 1, \dots, n-i$, the set $\lambda_j(K \cap L_{n-i}, \Lambda_{n-i})K$ contains $i+j$ linearly independent points of $\mathbb{Z}^n$, and therefore,

$$\lambda_{i+j}(K) \leq \lambda_j(K \cap L_{n-i}, \Lambda_{n-i}), \quad j = 1, \dots, n-i.$$

Thus, we obtain that

$$\lambda_{i+1}(K) \cdots \lambda_n(K) \operatorname{vol}_{n-i}(K \cap L_{n-i}) \leq 2^{n-i}. \quad (6)$$

We write, for the sake of brevity, $L_i := L_{n-i}^\perp$. In [8] it was proved that, if K is a 0-symmetry convex body, then

$$\operatorname{vol}_{n-i}(K \cap L_{n-i}) \geq \frac{\operatorname{vol}(K)}{\operatorname{vol}_i(K|L_i)}. \quad (7)$$

Now, denoting by $\omega_i(K) := \max_{L \in \mathcal{L}_i^n} \omega(K|L)$ and $D_i(K) := \min_{L \in \mathcal{L}_i^n} D(K|L)$, Theorem 2.1 in [2] applied to the i -dimensional convex body $K|L_i$ ensures that $\operatorname{vol}(K) \geq (1/n!) D_1(K) \cdots D_n(K)$ and $\operatorname{vol}_i(K|L_i) \leq \omega_1(K|L_i; L_i) \cdots \omega_i(K|L_i; L_i)$. Then, since $\omega_j(K|L_i; L_i) \leq \omega_j(K)$ for all $j = 1, \dots, i$ (see [2, Lemma 2.1]) and using that fact that K is 0-symmetric, we get

$$\begin{aligned} \operatorname{vol}(K) &\geq \frac{2^n}{n!} R_1(K) \cdots R_n(K) \quad \text{and} \\ \operatorname{vol}_i(K|L_i) &\leq \omega_1(K) \cdots \omega_i(K) = 2^i \max_{L \in \mathcal{L}_1^n} r(K|L) \cdots \max_{L \in \mathcal{L}_i^n} r(K|L). \end{aligned}$$

Moreover, it is known that $\max_{L \in \mathcal{L}_j^n} r(K|L) \leq R_{n-j+1}(K)$ for all $j = 1, \dots, n$ (see [2, Lemma 2.1]), and so we may conclude that

$$\operatorname{vol}_i(K|L_i) \leq 2^i R_{n-i+1}(K) \cdots R_n(K).$$

These bounds for $\operatorname{vol}_i(K|L_i)$ and $\operatorname{vol}(K)$ yield, together with (7), that

$$\operatorname{vol}_{n-i}(K \cap L_{n-i}) \geq \frac{2^n}{n!} \frac{R_1(K) \cdots R_n(K)}{2^i R_{n-i+1}(K) \cdots R_n(K)} = \frac{2^{n-i}}{n!} R_1(K) \cdots R_{n-i}(K),$$

and using (6) we finally obtain

$$\lambda_{i+1}(K) \cdots \lambda_n(K) R_1(K) \cdots R_{n-i}(K) \leq n!.$$

The relation $r_j(K) \leq \max_{L \in \mathcal{L}_j^n} r(K|L) \leq R_{n-j+1}(K)$ gives the second bound in the theorem. $\square$

Proof of Theorem 4. Let $z_1, \dots, z_i \in K$ be i linearly independent points with $\lambda_j(K) z_j \in \lambda_j(K) K \cap \mathbb{Z}^n$. Denoting by $u_j := \lambda_j(K) z_j$, let Λ_i the i -dimensional sublattice of $\mathbb{Z}^n$ determined by $\{u_1, \dots, u_i\}$. Since $\Lambda_i \subset \mathbb{Z}^n$, it is clear that $\det \Lambda_i \geq \det \mathbb{Z}^n = 1$. Then, the lower bound in Minkowski's second theorem (1) yields

$$\frac{2^i}{i!} \leq \frac{2^i}{i!} \det \Lambda_i \leq \lambda_1(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \cdots \lambda_i(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \operatorname{vol}_i(K \cap \operatorname{lin} \Lambda_i).$$

Moreover, $\lambda_j(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) = \lambda_j(K)$ for all $j = 1, \dots, i$, and so we can write

$$\frac{2^i}{i!} \leq \lambda_1(K) \cdots \lambda_i(K) \operatorname{vol}_i(K \cap \operatorname{lin} \Lambda_i). \quad (8)$$

Denoting by $\tilde{\omega}_i(K) := \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \omega(K \cap (x + L); x + L)$ and writing $K_i := K \cap \operatorname{lin} \Lambda_i$ for the sake of brevity, Theorem 2.1 in [2] applied to the i -dimensional convex body K_i ensures that

$$\operatorname{vol}_i(K_i) \leq \tilde{\omega}_1(K_i; \operatorname{lin} \Lambda_i) \cdots \tilde{\omega}_i(K_i; \operatorname{lin} \Lambda_i).$$

Then, since $\tilde{\omega}_j(K_i; \operatorname{lin} \Lambda_i) \leq \tilde{\omega}_j(K)$ for all $j = 1, \dots, i$ (see [2, Lemma 2.1]) and due to the fact that K is 0-symmetric, we get

$$\operatorname{vol}_i(K_i) \leq \tilde{\omega}_1(K) \cdots \tilde{\omega}_i(K) = 2^i r_1(K) \cdots r_i(K).$$

This, together with (8), concludes the proof of the first inequality:

$$\frac{1}{i!} \leq \frac{1}{2^i} \lambda_1(K) \cdots \lambda_i(K) \operatorname{vol}_i(K_i) \leq \lambda_1(K) \cdots \lambda_i(K) r_1(K) \cdots r_i(K).$$

Again, the relation $r_j(K) \leq \max_{L \in \mathcal{L}_j^n} r(K|L) \leq R_{n-j+1}(K)$ gives the second bound in the theorem. $\square$

We now extend Theorems 3 and 4 to arbitrary convex bodies. To this end we consider the difference body $DK := K + (-K)$ of a convex body $K \in \mathcal{K}^n$, which is clearly 0-symmetric. For a deep study of the properties of this set we refer, for instance, to [6, Section 9.5]. In [4, Proposition 4.2] it is proved that the inner and outer radii of the difference body satisfy that $r_j(DK) \geq 2r_j(K)$ and $R_j(DK) \leq 2R_j(K)$, for $j = 1, \dots, n$. Combining these properties with Theorems 3 and 4 we immediately obtain the following results for general convex bodies.

Corollary 5. *Let $K \in \mathcal{K}^n$. For $i = 1, \dots, n - 1$ we have*

$$\lambda_{i+1}(DK) \cdots \lambda_n(DK) r_{i+1}(K) \cdots r_n(K) \leq \frac{n!}{2^{n-i}}.$$

Corollary 6. *Let $K \in \mathcal{K}^n$. For $i = 1, \dots, n - 1$ we have*

$$\frac{1}{2^i i!} \leq \lambda_1(DK) \cdots \lambda_i(DK) R_{n-i+1}(K) \cdots R_n(K).$$

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