STEINER POLYNOMIALS VIA ULTRA-LOGCONCAVE SEQUENCES

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Abstract. We investigate structural properties of the cone of roots of relative Steiner polynomials of convex bodies. We prove that they are closed, monotonous with respect to the dimension, and that they cover the whole upper half-plane, except the positive real axis, when the dimension tends to infinity. In particular, it turns out that relative Steiner polynomials are stable polynomials if and only if the dimension is ≤ 9. Moreover, pairs of convex bodies whose relative Steiner polynomial has a complex root on the boundary of such a cone have to satisfy some Aleksandrov-Fenchel inequality with equality. An essential tool for the proofs of the results is the characterization of Steiner polynomials via ultra-logconcave sequences.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let B_n be the n-dimensional unit ball. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$. The volume of a set $M \subsetneq \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{vol}(M)$, its boundary by $\mathrm{bd}\, M$ and its convex, affine and linear hulls by $\mathrm{conv}\, M$, $\mathrm{aff}\, M$ and $\mathrm{lin}\, M$, respectively. For two convex bodies $K, E \in \mathcal{K}^n$ and a non-negative real number λ, the volume of the Minkowski sum $K + \lambda\, E$ is expressed as a polynomial of degree at most n in λ, and it is written as

$$\mathrm{vol}(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K; E)\lambda^i. \tag{1.1}$$

This expression is called *Minkowski-Steiner formula* or *relative Steiner formula* of K. The coefficients $\mathrm{W}_i(K; E)$ are the *relative quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes* for which we refer to [18, s. 5.1]. In particular, we have $\mathrm{W}_0(K; E) = \mathrm{vol}(K)$,

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$\mathrm{W}_n(K;E)=\mathrm{vol}(E)$, $\mathrm{W}_i(\mu_1 K;\mu_2 E)=\mu_1^{n-i}\mu_2^i\mathrm{W}_i(K;E)$ for $\mu_1,\mu_2\geq 0$ and $\mathrm{W}_i(K;E)=\mathrm{W}_{n-i}(E;K)$.

In the following we regard the right hand side in (1.1) as a formal polynomial in a complex variable $z\in\mathbb{C}$, which we will denote by

$$f_{K;E}(z)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K;E)z^i.$$

It is known that $\mathrm{W}_i(K;E)\geq 0$, with equality if and only if $\dim K<n-i$ or $\dim E<i$ (see e.g. [18, Theorem 5.1.7]). Hence, with respect to the dimensions of the bodies K,E we may write

$$f_{K;E}(z)=\sum_{i=n-\dim K}^{\dim E}\binom{n}{i}\mathrm{W}_i(K;E)z^i.$$

Moreover, since $\mathrm{W}_i(K;E)=\mathrm{W}_{n-i}(E;K)$ we have $f_{K;E}(z)=z^n\,f_{E;K}(1/z)$, and thus, up to multiplication by real constants,

$$f_{K;E}(z)\text{ and }f_{E;K}(z)\text{ have the same non-trivial roots.}\tag{1.2}$$

Here we are interested in the location of the roots of $f_{K;E}(z)$. To this end, let $\mathbb{C}^+=\{z\in\mathbb{C}:\mathrm{Im}(z)\geq 0\}$ be the set of complex numbers with non-negative imaginary part, and we denote by $\mathbb{R}_{\leq 0}$ and $\mathbb{R}_{>0}$ the non-positive and positive real axes, respectively. For any dimension $n\geq 2$, let

$$\mathcal{R}(n)=\bigl\{z\in\mathbb{C}^+:f_{K;E}(z)=0\text{ for }K,E\in\mathcal{K}^n,\ \dim(K+E)=n\bigr\}\tag{1.3}$$

be the set of all roots of all non-trivial Steiner polynomials in the upper half-plane. Note, that if $\dim(K+E)<n$ then all relative quermassintegrals vanish and so $f_{K;E}(z)\equiv 0$.

By the isoperimetric inequality for arbitrary gauge bodies E (cf. e.g., [18, p. 317-318]), it is easy to see that $\mathcal{R}(2)=\mathbb{R}_{\leq 0}$ is exactly the non-positive real axis and, in particular, it is a convex cone. For arbitrary dimensions this was verified in [8]. More precisely, the following result was shown.

Theorem 1.1 ([8, Theorem 1.1])**.** *$\mathcal{R}(n)$ is a convex cone containing $\mathbb{R}_{\leq 0}$.*

Hence one ray of the boundary of $\mathcal{R}(n)$ consists of the non-positive real axis $\mathbb{R}_{\leq 0}$, and, of course, any odd-degree Steiner polynomial has a root on this boundary. The "other ray" of the boundary of $\mathcal{R}(n)$ seems to have more geometric structure. We call a pair of convex bodies $(K,E)\in\mathcal{K}^n\times\mathcal{K}^n$ a *boundary-pair* if the Steiner polynomial $f_{K;E}(z)$ has a root on the boundary $\mathrm{bd}\,\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$, and in view of (1.2) we may additionally assume $\dim K\leq\dim E$.

Regarding the 3-dimensional case, in [8] the following characterization was given.

Proposition 1.1 ([8, Theorem 1.2])**.** $\mathcal{R}(3)=\bigl\{x+y\mathrm{i}\in\mathbb{C}^+:x+\sqrt{3}\,y\leq 0\bigr\}$. *Moreover, a pair (K,E) is a boundary-pair if and only if* $\dim K=2$, $\dim E=3$ *and* $\mathrm{W}_2(K;E)^2=\mathrm{W}_1(K;E)\mathrm{W}_3(K;E)$.

We notice that two convex bodies $K,E\in\mathcal{K}^n$ satisfy the above conditions if and only if $E\in\mathcal{K}^3_0$ is a cap-body of (an homothet of) a planar convex body K (see [1]). A convex body L is called a cap-body of $M\in\mathcal{K}^n$ if L is the convex hull of M and countably many points such that the line segment joining any pair of these points intersects M.

Here we will also extend the exact description of the cones $\mathcal{R}(n)$ to the case $n=4$, and get similarly to $n=3$ the following characterization.

Proposition 1.2. $\mathcal{R}(4)=\{x+y\mathrm{i}\in\mathbb{C}^+ : x+y\leq 0\}$. *Moreover, a pair* (K,E) *is a boundary-pair if and only if* $\dim K=3$, $\dim E=4$ *and, for* $i=2,3$, $\mathrm{W}_i(K;E)^2=\mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E)$.

However, in contrast to the case $n=3$ we are not aware of an equivalent geometric description of the boundary pairs (K,E) in dimension 4.

The cones $\mathcal{R}(2),\mathcal{R}(3),\mathcal{R}(4)$ are in particular closed, and our first main result verifies this in any dimension.

Theorem 1.2. *The cone* $\mathcal{R}(n)$ *is closed.*

The low dimensional cones are also strictly nested, i.e., $\mathcal{R}(2)\subsetneq\mathcal{R}(3)\subsetneq\mathcal{R}(4)$. Our second theorem shows that this is also true in general.

Theorem 1.3. $\mathcal{R}(n)\subsetneq\mathcal{R}(n+1)$.

So, the following question arises in a natural way: does $\mathcal{R}(n)$ cover the whole upper half-plane $\mathbb{C}^+$, except $\mathbb{R}_{>0}$, when n tends to infinity? Next theorem gives an affirmative answer to it.

Theorem 1.4. *Let* $\gamma\in\mathbb{C}^+\setminus\mathbb{R}_{>0}$. *Then there exists* $n_\gamma\in\mathbb{N}$ *with* $\gamma\in\mathcal{R}(n)$ *for all* $n\geq n_\gamma$.

It is well-known that the relative quermassintegrals of two convex bodies satisfy the inequalities

$$\mathrm{W}_i(K;E)^2\geq\mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E),\quad 1\leq i\leq n-1, \tag{1.4}$$

which are particular cases of the Aleksandrov-Fenchel inequality; we notice that the complete classification of the equality cases is an unsolved problem (see e.g. [18, ss. 6.3, 6.6]). By the proof of Theorem 1.3 the following corollary is obtained, which also shows that boundary-pairs have a special geometric meaning (cf. Propositions 1.1 and 1.2).

Corollary 1.1. *For* $n\geq 3$, *let* (K,E) *be a boundary-pair. Then there exists* $i\in\{1,\dots,n-1\}$ *such that*

$$\mathrm{W}_i(K;E)^2=\mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E), \tag{1.5}$$

i.e., K,E *are extremal sets for at least one Aleksandrov-Fenchel inequality.*

According to Propositions 1.1 and 1.2 all Steiner polynomials for $n=3,4$ of boundary-pairs are (up to multiplication by a constant) of the type

$$\sum_{i=1}^{3}\binom{3}{i}\lambda^{3-i}z^i \quad\text{and}\quad \sum_{i=1}^{4}\binom{4}{i}\lambda^{4-i}z^i,$$

for all real $\lambda \geq 0$. Since the parameter λ implies just a multiplication of the roots and $\mathcal{R}(n)$ is a convex cone, we can say that a representative of the Steiner polynomials of boundary-pairs is given by a truncated binomial polynomial (setting $\lambda = 1$) for $n = 3, 4$. We believe that this is true in general, and so for $0 \leq j < k \leq n$ we define

$$P_{j,k}^n(z) := \sum_{i=j}^{k} \binom{n}{i} z^i,$$

the truncation of the binomial polynomial $(z+1)^n$ with indices $j < k$.

Conjecture 1.1. *Let $n \geq 5$ and let $\gamma \in \operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$. Then there exist a truncated binomial polynomial $P_{j,k}^n(z)$, $0 < j < k < n$, and $\lambda > 0$, such that $P_{j,k}^n(\lambda\gamma) = 0$.*

Notice that the conjecture would directly imply that if (K, E) is a boundary-pair, then K, E are extremal sets for exactly $n-3$ Aleksandrov-Fenchel inequalities (cf. Corollary 1.1).

The property that all roots of 3-dimensional Steiner polynomials lie in the left half-plane was part of a conjecture posed by Sangwine-Yager [16] (cf. e.g., [17, p. 65]), motivated by a problem of Teissier [21]. There it was claimed that Steiner polynomials satisfy $\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \operatorname{Re}(z) \leq 0\}$. This inclusion is known to be true for dimensions ≤ 9. In fact, in [8, Proposition 1.1] it was shown that

$$\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \operatorname{Re}(z) < 0\} \cup \{0\} \quad \text{for } n \leq 9, \tag{1.6}$$

i.e., all non-trivial roots are in the open left half-plane. We will call this property "weak" stability. In [7] the conjecture was shown to be false in dimensions ≥ 12 for a special family of bodies (see also [13] for another family of high dimensional convex bodies with this property). By looking at the roots of particular truncated polynomials, we get rid of the gap, showing that for $n = 10, 11$ Steiner polynomials are also not weakly stable.

Proposition 1.3. *Steiner polynomials are weakly stable polynomials, i.e., $\mathcal{R}(n) \subseteq \{z \in \mathbb{C}^+ : \operatorname{Re}(z) < 0\} \cup \{0\}$, if and only if $n \leq 9$.*

Figure 1 depicts the above results, and for further information on the roots of Steiner polynomials in the context of Teissier's problem we refer to [7, 9, 11, 12, 13].

An essential tool for the proofs of the above results is a characterization of Steiner polynomials via ultra-logconcave sequences (Lemma 2.1), which we show in Section 2. There we also discuss some additional applications of this characterization. In Sections 3 and 4 we present the proofs of the main results, namely Theorems 1.2, 1.3 and 1.4, as well as some consequences. Finally, in Section 5 we characterize the 4-dimensional cone $\mathcal{R}(4)$.

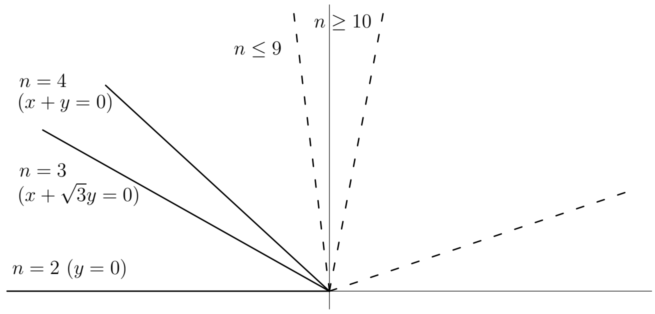

FIGURE 1. Structure of the cones $\mathcal{R}(n)$.

2. Ultra-logconcave sequences

A sequence of non-negative real numbers $a_0, \dots, a_n$ is said to be *ultra-logconcave* if

$$c_{i,n}\, a_i^2 \geq a_{i-1}\, a_{i+1} \quad \text{with } c_{i,n} = \frac{\binom{n}{i-1}\binom{n}{i+1}}{\binom{n}{i}^2} = \frac{i}{i+1}\frac{n-i}{n-i+1}, \tag{2.1}$$

$1 \leq i \leq n-1$. For further information on ultra-logconcave sequences we refer to [4, 14] and the references inside. This property for real numbers allows to characterize Steiner polynomials.

Lemma 2.1. *A real polynomial $\sum_{i=0}^{n} a_i z^i$, $a_i \geq 0$, is a Steiner polynomial $f_{K;E}(z)$ for a pair of convex bodies $K, E \in \mathcal{K}^n$, with $\dim E = r$, $\dim K = s$, $\dim(K+E) = n$, if and only if*

i) *$a_i > 0$ for all $n - s \leq i \leq r$, and $a_i = 0$ otherwise, and*
ii) *the sequence $a_0, \dots, a_n$ is ultra-logconcave, i.e.,*

$$c_{i,n}\, a_i^2 \geq a_{i-1}\, a_{i+1} \quad \text{for } 1 \leq i \leq n-1.$$

This result essentially follows from a theorem of Shephard ([19, Theorem 4], see also [18, p. 333], and for the 2-dimensional case see [6]), which states that any given set of $n+1$ non-negative real numbers $\mathrm{W}_0, \dots, \mathrm{W}_n \geq 0$ satisfying the inequalities $\mathrm{W}_i \mathrm{W}_j \geq \mathrm{W}_{i-1}\mathrm{W}_{j+1}$, $1 \leq i \leq j \leq n-1$, arises as the set of relative quermassintegrals of two convex bodies. There, an explicit construction of the two convex bodies is given in the case when all $\mathrm{W}_i > 0$, whereas the general case is obtained by a rather non-constructive topological argument. Here we reduce the number of involved inequalities, and extend the construction of the two convex bodies to $\mathrm{W}_i \geq 0$.

Proof of Lemma 2.1. If $\sum_{i=0}^{n} a_i z^i$ is the Steiner polynomial of two convex bodies $K, E \in \mathcal{K}^n$, then $a_i = \binom{n}{i}\mathrm{W}_i(K;E)$, and i), ii) are well-known properties of quermassintegrals. For i) see [18, Theorem 5.1.7] and ii) is (1.4).

Now we assume i) and ii). If $s = 0$ or $r = 0$ then both $a_n z^n$ and a_0 are obviously Steiner polynomials, and so we may assume $r, s \geq 1$. Setting

$\mathrm{W}_i = a_i/\binom{n}{i}$, we directly get that

$$\begin{aligned} &\mathrm{W}_i > 0 \text{ for all } n-s \le i \le r \text{ and } \mathrm{W}_i = 0 \text{ otherwise, and} \\ &\mathrm{W}_i^2 \ge \mathrm{W}_{i-1}\mathrm{W}_{i+1},\ 1 \le i \le n-1. \end{aligned} \tag{2.2}$$

In the rest of the proof we will construct two convex bodies K, E, with $\dim K = s$, $\dim E = r$, $\dim(K+E) = n$ and $\mathrm{W}_i = \mathrm{W}_i(K;E)$, and so $\sum_{i=0}^n a_i z^i = f_{K;E}(z)$. To this end we extend the construction in [19] to handle lower dimensional bodies as well and, as in [19], the sets K, E will be simplices.

Let e_i, $1 \le i \le n$, denote the i-th canonical unit vector, and let $q_i = \alpha_i \mathrm{e}_i$, where $\alpha_i > 0$ for $i = 1, \dots, r$, $\alpha_i = 0$ for $i = r+1, \dots, n$, and $\alpha_i \ge \alpha_{i+1}$ if $i = n-s+1, \dots, r-1$. These numbers α_i's will be fixed at the end of the proof. Let K, E be the, respectively, s- and r-dimensional simplices

$$K = \operatorname{conv}\{0, \mathrm{e}_{n-s+1}, \dots, \mathrm{e}_n\}, \quad E = \operatorname{conv}\{0, q_1, \dots, q_r\}. \tag{2.3}$$

Then $K+E = \operatorname{conv}\{0, \mathrm{e}_i, q_j, \mathrm{e}_i + q_j : n-s+1 \le i \le n, 1 \le j \le r\}$, but since $\alpha_j \ge \alpha_{j+1}$ for $j = n-s+1, \dots, r-1$, the points $\mathrm{e}_i + q_j \in \operatorname{conv}\{0, \mathrm{e}_i + q_i, \mathrm{e}_j + q_j\}$ if $i < j$, and thus

$$\begin{aligned} K+E = \operatorname{conv}\{0, \mathrm{e}_i + q_j : j \le i, n-s+1 \le i \le n, 1 \le j \le r, \\ \mathrm{e}_i, q_j : r+1 \le i \le n, 1 \le j \le n-s\}. \end{aligned} \tag{2.4}$$

Now for $n-s+1 \le m \le r+1$, let

$$K_m = \operatorname{conv}\{0, \mathrm{e}_m, \dots, \mathrm{e}_n\}, \quad E_m = \operatorname{conv}\{q_1, \dots, q_m\},$$

with $K_{n+1} = \{0\}$, $E_{n+1} = \operatorname{conv}\{0, q_1, \dots, q_n\}$; notice that $q_{r+1} = 0$. In the following we will show by induction on the dimension that $K+E$ is the interior-disjoint union of the sets $K_m + E_m$, i.e.,

$$K+E = \bigcupdot_{m=n-s+1}^{r+1} (K_m + E_m), \tag{2.5}$$

where $\cupdot$ denotes interior-disjoint union.

For $n = 1$ the assertion is trivial. So let $n \ge 2$, and let

$$\overline{K} = \operatorname{conv}\{0, \mathrm{e}_{n-s+1}, \dots, \mathrm{e}_{n-1}\}, \quad \overline{E} = \begin{cases} \operatorname{conv}\{0, q_1, \dots, q_r\} & \text{if } r < n, \\ \operatorname{conv}\{0, q_1, \dots, q_{n-1}\} & \text{if } r = n, \end{cases}$$

with $\overline{K} = \{0\}$ if $s = 1$. Notice that in both cases, $\dim \overline{K} + \dim \overline{E} = s-1+r \ge n-1$. Similarly as before we consider, for $n-s+1 \le m \le r+1$ (if $r < n$) or $n-s+1 \le m < n$ (if $r = n$),

$$\overline{K}_m = \operatorname{conv}\{0, \mathrm{e}_m, \dots, \mathrm{e}_{n-1}\}, \quad \overline{E}_m = \operatorname{conv}\{q_1, \dots, q_m\},$$

where $\overline{K}_n = \{0\}$ and $\overline{E}_n = \operatorname{conv}\{0, q_1, \dots, q_{n-1}\}$ (also for $m = r = n$). By induction hypothesis,

$$\overline{K} + \overline{E} = \left\{ \begin{matrix} \bigcupdot_{m=n-s+1}^{r+1} (\overline{K}_m + \overline{E}_m) & \text{if } r < n \\ \bigcupdot_{m=n-s+1}^{n} (\overline{K}_m + \overline{E}_m) & \text{if } r = n \end{matrix} \right\} =: \bigcupdot_{m=n-s+1}^{r+1,n} (\overline{K}_m + \overline{E}_m),$$

and taking the orthogonal projection π_n onto the coordinate hyperplane $\mathrm{e}_n = 0$ and the restriction $\pi := (\pi_n)|_{K+E}$, we get

$$K + E = \pi^{-1}\big(\overline{K} + \overline{E}\big) = \bigcup_{m=n-s+1}^{r+1,n} \cdot \, \pi^{-1}\big(\overline{K}_m + \overline{E}_m\big).$$

It is easy to see that $\pi^{-1}\big(\overline{K}_m+\overline{E}_m\big) = K_m+E_m$ for $m = n-s+1,\dots,r+1$ when $r < n$ and $m = n-s+1,\dots,n-1$ when $r = n$. So we get the required union for $K+E$ in $r+s-n+1$ interior-disjoint parts (cf. (2.5)) when $r < n$. Finally, if $r = n$,

$$\pi^{-1}\big(\overline{K}_n+\overline{E}_n\big) = \mathrm{conv}\{0, q_j, q_j+\mathrm{e}_n : 1 \le j \le n\} = (K_{n+1}+E_{n+1}) \cup (K_n+E_n),$$

providing the $s+1$ interior-disjoint parts in (2.5) when $r = n$.

Next, based on relation (2.5), we can compute the volume of the polytope $K+E$. Since $(\mathrm{aff}\, K_m) \cap (\mathrm{aff}\, E_m) = \{q_m\}$ we get, for all $n-s+1 \le m \le r+1$ $(m \ne n+1)$, that

$$\begin{aligned}
\mathrm{vol}(K_m + E_m) &= \mathrm{vol}\Big(K_m + \big(E_m|(\mathrm{lin}\, K_m)^{\perp}\big)\Big)\\
&= \mathrm{vol}_{n-m+1}(K_m)\mathrm{vol}_{m-1}\big(\mathrm{conv}\{0, q_1, \dots, q_{m-1}\}\big)\\
&= \frac{1}{(n-m+1)!}\frac{\alpha_1 \dots \alpha_{m-1}}{(m-1)!} = \frac{1}{n!}\binom{n}{m-1}\alpha_1 \dots \alpha_{m-1};
\end{aligned}$$

here we use vol_i to denote the i-dimensional volume in $\mathbb{R}^i$, $L^{\perp}$ for the orthogonal complement of a linear subspace L and $M|L$ for the orthogonal projection of $M \subsetneq \mathbb{R}^n$ onto L. Observe that if $r = n$ then $\mathrm{vol}(K_{n+1} + E_{n+1}) = \mathrm{vol}(E_{n+1}) = (1/n!)\,\alpha_1 \dots \alpha_n$. Thus, by (2.5),

$$\mathrm{vol}(K+E) = \sum_{m=n-s+1}^{r+1} \mathrm{vol}(K_m + E_m) = \sum_{i=n-s}^{r} \binom{n}{i}\frac{1}{n!}\,\alpha_1 \dots \alpha_i,$$

where, if $s = n$, the first summand $(i = 0)$ is just $1/n!$. This says that $\mathrm{W}_i(K;E) = (1/n!)\,\alpha_1 \dots \alpha_i$ for $n-s \le i \le r$, and $\mathrm{W}_i(K;E) = 0$ otherwise.

Now we go back to our given sequence of real numbers $\mathrm{W}_0, \dots, \mathrm{W}_n \ge 0$ satisfying (2.2). Let

$$\alpha_i = \begin{cases} (n!\mathrm{W}_{n-s})^{1/(n-s)} & \text{for } i = 1, \dots, n-s,\\ \mathrm{W}_i/\mathrm{W}_{i-1} & \text{for } i = n-s+1, \dots, r,\\ 0 & \text{for } i = r+1, \dots, n. \end{cases}$$

Since $\mathrm{W}_i^2 \ge \mathrm{W}_{i-1}\mathrm{W}_{i+1}$ we have $\alpha_i \ge \alpha_{i+1}$ for $n-s+1 \le i \le r$, and taking K, E as defined in (2.3) we get, for all $i = n-s, \dots, r$,

$$\mathrm{W}_i(K;E) = \frac{1}{n!}\,\alpha_1 \dots \alpha_{n-s}\alpha_{n-s+1} \dots \alpha_i = \frac{1}{n!}(n!\mathrm{W}_{n-s})\frac{\mathrm{W}_i}{\mathrm{W}_{n-s}} = \mathrm{W}_i,$$

and $\mathrm{W}_i = 0$ otherwise. □

For complex numbers $z_1, \ldots, z_r \in \mathbb{C}$ let

$$\sigma_i(z_1, \ldots, z_r) = \sum_{\substack{J \subseteq \{1,\ldots,r\} \\ \#J=i}} \prod_{j \in J} z_j$$

denote the i-th elementary symmetric function of $z_1, \ldots, z_r$, $1 \leq i \leq r$. In addition we set $\sigma_0(z_1, \ldots, z_r) = 1$. Using this notation the following corollary is an immediate consequence of Lemma 2.1.

Corollary 2.1. *The complex numbers $\gamma_1, \ldots, \gamma_r \in \mathbb{C}$ are the roots of a Steiner polynomial $f_{K;E}(z)$ of degree $r \leq n$, with $K, E \in \mathcal{K}^n$, $\dim E = r$, $\dim K = s$, $\dim(K + E) = n$, if and only if*

(2.6)

i) $(-1)^i \sigma_i(\gamma_1, \ldots, \gamma_r) > 0, \quad 0 \leq i \leq r + s - n,$

$\sigma_i(\gamma_1, \ldots, \gamma_r) = 0, \quad r + s - n + 1 \leq i \leq r,$

ii) $c_{r-i,n}\, \sigma_i(\gamma_1, \ldots, \gamma_r)^2 \geq \sigma_{i-1}(\gamma_1, \ldots, \gamma_r)\, \sigma_{i+1}(\gamma_1, \ldots, \gamma_r), \ 1 \leq i \leq r - 1.$

We conclude this section by three immediate applications of Lemma 2.1.

Proposition 2.1. *All truncated binomial polynomials $P^n_{j,k}(z) = \sum_{i=j}^{k} \binom{n}{i} z^i$, $0 \leq j < k \leq n$, are Steiner polynomials of convex bodies $K, E \in \mathcal{K}^n$ with $\dim K = n - j$, $\dim E = k$ and $\dim(K + E) = n$.*

Hence in the following we consider $P^n_{j,k}(z)$ as Steiner polynomials. In fact, by the proof of Lemma 2.1, $P^n_{j,k}(z)$ can be realized as the Steiner polynomial $f_{K;E}(z)$ of the bodies $K = \operatorname{conv}\{0, \mathrm{e}_{j+1}, \ldots, \mathrm{e}_n\}$ and $E = \operatorname{conv}\{0, c\,\mathrm{e}_1, \ldots, c\,\mathrm{e}_j, \mathrm{e}_{j+1}, \ldots, \mathrm{e}_k\}$ with $c = (n!)^{1/j}$.

Second consequence deals with the derivative and antiderivative of Steiner polynomials.

Proposition 2.2. *Let $f_{K;E}(z) = \sum_{i=0}^{n} a_i\, z^i$ be the Steiner polynomial of two convex bodies $K, E \in \mathcal{K}^n$, $\dim(K + E) = n$. Then both, its derivative as well as its antiderivative*

$$f'_{K;E}(z) = \sum_{i=0}^{n-1} (i+1)\, a_{i+1}\, z^i \quad \text{and} \quad \int f_{K;E}(z)\, \mathrm{d}z = \sum_{i=1}^{n+1} \frac{a_{i-1}}{i}\, z^i$$

are Steiner polynomials of appropriate convex bodies in $\mathcal{K}^{n-1}$ and $\mathcal{K}^{n+1}$, respectively.

If $\dim K = n$, we may also add any constant term c to the antiderivative as long as $c \leq n a_0^2 / \big((n+1)a_1\big)$.

The last consequence regards Steiner polynomials with only real roots.

Proposition 2.3. *For any given n real numbers $\gamma_i \leq 0$, $i = 1, \ldots, n$, there exist $K, E \in \mathcal{K}^n$ such that $f_{K;E}(\gamma_i) = 0$ for all $i = 1, \ldots, n$.*

This is, for instance, due to the fact that the elementary symmetric functions form an ultra-logconcave sequence (Newton inequalities, see e.g. [5]),

$$\left(\frac{\sigma_i\left(\gamma_1,\ldots,\gamma_n\right)}{\binom{n}{i}}\right)^2 \geq \frac{\sigma_{i-1}\left(\gamma_1,\ldots,\gamma_n\right)}{\binom{n}{i-1}}\frac{\sigma_{i+1}\left(\gamma_1,\ldots,\gamma_n\right)}{\binom{n}{i+1}},$$

and so Lemma 2.1 gives the result. In the case $n=2$ this means that given any pair $\gamma,\gamma'\in\mathcal{R}(2)$, we can find a Steiner polynomial having these two roots. This property is, however, not true in higher dimension if we also allow complex (non-real) numbers to be involved. Indeed, in [8, pp. 160-161] it is shown that if $-a+b\mathrm{i}\in\mathcal{R}(3)$, then $-a+b\mathrm{i},-a-b\mathrm{i},-c$ are the roots of a Steiner polynomial if and only if either $c\leq a-\sqrt{3}\,b$ or $c\geq(a^2+b^2)/(a-\sqrt{3}\,b)$.

3. On the boundary of the cones $\mathcal{R}(n)$

We start showing that all cones $\mathcal{R}(n)$ are closed.

Proof of Theorem 1.2. Let $\gamma\in\operatorname{bd}\mathcal{R}(n)$. Since we already know that the non-positive real axis is always contained in $\mathcal{R}(n)$, we assume that $\gamma\notin\mathbb{R}$. Let $(\gamma_j)_{j\in\mathbb{N}}\subsetneq\operatorname{int}\mathcal{R}(n)$ be a sequence of complex numbers converging to γ. For each $j\in\mathbb{N}$, since $\gamma_j\in\operatorname{int}\mathcal{R}(n)$, there exists a pair of convex bodies $(K_j,E_j)\in\mathcal{K}^n\times\mathcal{K}^n$, $\dim(K_j+E_j)=n$, such that $f_{K_j;E_j}(\gamma_j)=0$.

Notice that we can always choose K_j,E_j such that $\operatorname{vol}(K_j+E_j)=1$. Otherwise, since $\operatorname{vol}(K_j+E_j)>0$, it suffices to consider the new convex bodies $K_j'=1/\operatorname{vol}(K_j+E_j)^{1/n}K_j$ and $E_j'=1/\operatorname{vol}(K_j+E_j)^{1/n}E_j$, for which it clearly holds $f_{K_j';E_j'}(\gamma_j)=\bigl(1/\operatorname{vol}(K_j+E_j)\bigr)f_{K_j;E_j}(\gamma_j)=0$, and moreover,

$$\operatorname{vol}(K_j'+E_j')=f_{K_j';E_j'}(1)=\frac{1}{\operatorname{vol}(K_j+E_j)}f_{K_j;E_j}(1)=1.$$

Observe that since $\operatorname{vol}(K_j+E_j)=\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i(K_j;E_j)=1$, all quermassintegrals $\mathrm{W}_i(K_j;E_j)\in[0,1]$, $i=0,\ldots,n$, and not all of them are zero. Then, denoting by $\mathrm{W}_{i,j}=\mathrm{W}_i(K_j;E_j)$, we can assure that the bounded sequence of $(n+1)$-tuples of numbers $(\mathrm{W}_{0,j},\ldots,\mathrm{W}_{n,j})_{j\in\mathbb{N}}$ has a convergent subsequence to an $(n+1)$-tuple $(\mathrm{W}_0,\ldots,\mathrm{W}_n)$, and without loss of generality we assume that $(\mathrm{W}_{0,j},\ldots,\mathrm{W}_{n,j})_{j\in\mathbb{N}}$ is the convergent subsequence.

By continuity, the numbers $\mathrm{W}_0,\ldots,\mathrm{W}_n$ also satisfy inequalities (1.4), and thus the sequence $\bigl\{a_i=\binom{n}{i}\mathrm{W}_i : i=0,\ldots,n\bigr\}$ is ultra-logconcave. Moreover,

$$\sum_{i=0}^n\binom{n}{i}\mathrm{W}_i=\lim_{j\to\infty}\sum_{i=0}^n\binom{n}{i}\mathrm{W}_{i,j}=\lim_{j\to\infty}\operatorname{vol}(K_j+E_j)=1,$$

i.e., the polynomial $\sum_{i=0}^n\binom{n}{i}\mathrm{W}_iz^i=\sum_{i=0}^n a_iz^i\neq0$. Therefore, the property $a_i>0$ for all $n-s\leq i\leq r$ and $a_i=0$ otherwise, holds for suitable $r,s\in\{1,\ldots,n\}$. Then Lemma 2.1 ensures that $\sum_{i=0}^n\binom{n}{i}\mathrm{W}_iz^i$ is a Steiner polynomial of two convex bodies $K,E\in\mathcal{K}^n$ with $\dim K=s$, $\dim E=r$. By continuity, since $f_{K_j;E_j}(\gamma_j)=0$ for all $j\in\mathbb{N}$ and the sequence of complex numbers $(\gamma_j)_{j\in\mathbb{N}}$ converges to γ, we have $f_{K;E}(\gamma)=0$, i.e., $\gamma\in\mathcal{R}(n)$. This shows that the cone $\mathcal{R}(n)$ is closed. □

Since $\mathcal{R}(n)$ is closed, we may ask which pairs of convex bodies or Steiner polynomials determine the boundary $\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$. We recall (cf. Proposition 1.1) that if $E\in\mathcal{K}_0^3$ is a cap-body of a planar convex body K, then (K,E) is a boundary-pair. We also notice that if $E\in\mathcal{K}_0^4$ is a cap-body of K with $\dim K=3$, then the condition for the boundary in Proposition 1.2 is satisfied, i.e., (K,E) is also a boundary-pair in dimension 4. However this is not the case for $n\geq 5$: in general, if $K\in\mathcal{K}^n$ with $\dim K=n-1$ and $E\in\mathcal{K}_0^n$ is a cap-body of K, then $\operatorname{vol}(E)=\mathrm{W}_0(E;K)=\cdots=\mathrm{W}_{n-1}(E;K)\neq 0$ (see [18, proof of Theorem 6.6.16, p. 368]); so, since $\mathrm{W}_0(K;E)=0$ we get

$$f_{K;E}(z)=\sum_{i=1}^{n}\binom{n}{i}\mathrm{W}_i(K;E)z^i=\sum_{i=1}^{n}\binom{n}{i}\mathrm{W}_{n-i}(E;K)z^i=\operatorname{vol}(E)P_{1,n}^n(z).$$

Then it can be checked that all roots of the Steiner polynomial $P_{1,5}^5(z)$ lie in the interior of the cone determined by the complex number $-0.5000+0.8660\mathrm{i}$, which is a root of the Steiner polynomial $P_{1,4}^5(z)$ (cf. Table 1). Analogously for dimensions $n=6,7,8,9$. Finally, it can be easily seen (cf. also [8, Corollary 3.1]) that all roots of $P_{1,n}^n(z)$ have non-positive real part, and thus, because of the non-stability of the Steiner polynomial for $n\geq 10$ (Proposition 1.3), they cannot determine the boundary.

Remark 3.1. *Numerical computations suggest that for each n and suitable $0<j<k\leq n$, the Steiner polynomials*

$$P_{j,k}^n(z)=\sum_{i=j}^{k}\binom{n}{i}z^i$$

have a root on the boundary $\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$ (cf. Conjecture 1.1). Table 1 lists, for $n\leq 20$, the indices j and k of those Steiner polynomials $P_{j,k}^n(z)$ having a root γ of minimal angle α with the positive real axis.

Of particular interest is also the entry for dimension $n=10$ in Table 1. Here we have for the first time a root γ with positive real part and so $P_{3,8}^{10}(z)$ is a non-weakly stable Steiner polynomial. Together with known results this settles the question when Steiner polynomials are (weakly) stable.

Proof of Proposition 1.3. The (weak) stability of the Steiner polynomial was shown for all dimensions $n\leq 9$ in [8, Proposition 1.1], as well as its non-stability when $n\geq 12$ [7, Remark 3.2]. Thus just the cases $n=10,11$ remain to be considered, but Table 1 provides two non-weakly stable Steiner polynomials in these dimensions. □

4. On the monotonicity of the cones $\mathcal{R}(n)$

First we observe that it is easy to see that $\mathcal{R}(n)\subseteq\mathcal{R}(n+1)$. To this end, let $\gamma\in\mathcal{R}(n)$ and $K,E\in\mathcal{K}^n$ such that $f_{K;E}(\gamma)=0$. Identifying K

$n=3$	$j=1,\ k=3$	$\gamma=-1.5000+0.8660\mathrm{i}$	$\alpha=2.6179$
$n=4$	$j=1,\ k=4$	$\gamma=-1.0000+1.0000\mathrm{i}$	$\alpha=2.3561$
$n=5$	$j=1,\ k=4$	$\gamma=-0.5000+0.8660\mathrm{i}$	$\alpha=2.0943$
$n=6$	$j=1,\ k=5$	$\gamma=-0.3856+0.9226\mathrm{i}$	$\alpha=1.9667$
$n=7$	$j=2,\ k=6$	$\gamma=-0.3249+1.2279\mathrm{i}$	$\alpha=1.8294$
$n=8$	$j=2,\ k=6$	$\gamma=-0.1464+0.9892\mathrm{i}$	$\alpha=1.7177$
$n=9$	$j=2,\ k=7$	$\gamma=-0.0698+0.9975\mathrm{i}$	$\alpha=1.6406$
$n=10$	$j=3,\ k=8$	$\gamma=\ \ 0.0158+1.1903\mathrm{i}$	$\alpha=1.5574$
$n=11$	$j=3,\ k=8$	$\gamma=\ \ 0.0854+0.9963\mathrm{i}$	$\alpha=1.4852$
$n=12$	$j=4,\ k=9$	$\gamma=\ \ 0.1533+1.1549\mathrm{i}$	$\alpha=1.4388$
$n=13$	$j=4,\ k=10$	$\gamma=\ \ 0.2127+1.1256\mathrm{i}$	$\alpha=1.3840$
$n=14$	$j=4,\ k=10$	$\gamma=\ \ 0.2400+0.9707\mathrm{i}$	$\alpha=1.3284$
$n=15$	$j=5,\ k=11$	$\gamma=\ \ 0.3139+1.0864\mathrm{i}$	$\alpha=1.2895$
$n=16$	$j=5,\ k=11$	$\gamma=\ \ 0.3121+0.9500\mathrm{i}$	$\alpha=1.2533$
$n=17$	$j=5,\ k=12$	$\gamma=\ \ 0.3452+0.9384\mathrm{i}$	$\alpha=1.2182$
$n=18$	$j=6,\ k=13$	$\gamma=\ \ 0.4186+1.0258\mathrm{i}$	$\alpha=1.1833$
$n=19$	$j=6,\ k=13$	$\gamma=\ \ 0.4076+0.9131\mathrm{i}$	$\alpha=1.1509$
$n=20$	$j=7,\ k=14$	$\gamma=\ \ 0.4727+0.9917\mathrm{i}$	$\alpha=1.1259$

Table 1. Numerical computations for $\mathrm{bd}\,\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$, $n\leq 20$.

and E with their canonical embedding in the hyperplane $\{\mathrm{e}_{n+1}\}^{\perp}\subsetneq\mathbb{R}^{n+1}$, let $E'=E\times\mathrm{conv}\{0,\mathrm{e}_{n+1}\}$ be the prism over E of height 1 in the direction e_{n+1}. Then we observe that

$$\mathrm{vol}(K+\lambda\,E')=\mathrm{vol}\big((K+\lambda E)\times\lambda\,\mathrm{conv}\{0,\mathrm{e}_{n+1}\}\big)=\lambda\,\mathrm{vol}_n(K+\lambda E),$$

i.e., $f_{K;E'}(z)=zf_{K;E}(z)$ and thus $f_{K;E'}(\gamma)=0$. Hence $\gamma\in\mathcal{R}(n+1)$, which shows that $\mathcal{R}(n)\subseteq\mathcal{R}(n+1)$. Theorem 1.3 states that this inclusion is strict.

Proof of Theorem 1.3. Let $\gamma_1\in\mathrm{bd}\,\mathcal{R}(n)\backslash\mathbb{R}_{\leq 0}$. By Theorem 1.2 $\mathcal{R}(n)$ is closed, and hence γ_1 is a root of some Steiner polynomial $f_{K;E}(z)$ of degree $r\leq n$, with $K,E\in\mathcal{K}^n$, $\dim E=r$, $\dim K=s$, $\dim(K+E)=n$. Let $\gamma_2,\dots,\gamma_r$ be the remaining roots of the polynomial, where $\gamma_2=\overline{\gamma}_1$ is the complex conjugate of γ_1. We may assume that $\gamma_1,\dots,\gamma_{r+s-n}\neq 0$ and $\gamma_{r+s-n+1}=\dots=\gamma_r=0$. So, 0 is (exactly) an $(n-s)$-fold root.

In the following we will show that γ_1 lies in the interior of $\mathcal{R}(n+1)$, i.e., we will prove the existence of $\varepsilon_0>0$ such that for any $z\in\mathbb{C}$ with modulus $|z|=1$, the $r+1$ complex numbers $\rho_1=\gamma_1+\varepsilon_0 z,\rho_2=\gamma_2+\varepsilon_0\overline{z},\gamma_3,\dots,\gamma_r,0$ are the roots of a Steiner polynomial $f_{K';E'}(z)$ of degree $r+1$ with $K',E'\in\mathcal{K}^{n+1}$, $\dim E'=r+1$, $\dim K'=s$ and $\dim(K'+E')=n+1$. According to

Corollary 2.1 this is equivalent to show that

$$\text{I)}\quad (-1)^i\sigma_i\left(\rho_1,\rho_2,\gamma_3\ldots,\gamma_r,0\right)>0,\quad 0\le i\le r+s-n,$$
$$\sigma_i\left(\rho_1,\rho_2,\gamma_3\ldots,\gamma_r,0\right)=0,\quad r+s-n+1\le i\le r+1,$$

$$\text{II)}\quad c_{r+1-i,n+1}\,\sigma_i\left(\rho_1,\rho_2,\gamma_3\ldots,\gamma_r,0\right)^2$$
$$\ge\sigma_{i-1}\left(\rho_1,\rho_2,\gamma_3\ldots,\gamma_r,0\right)\sigma_{i+1}\left(\rho_1,\rho_2,\gamma_3\ldots,\gamma_r,0\right),\ 1\le i\le r.$$

To this end we note that, for $0\le i\le r$ and for any $\varepsilon>0$,

$$\begin{aligned}\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0\right)&=\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r\right),\\ \sigma_{r+1}\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0\right)&=0.\end{aligned}\tag{4.1}$$

Since $n+1-s$ of the $r+1$ numbers $\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0$ are zero, we also have that, for any $\varepsilon>0$,

$$\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0\right)=0\quad\text{for }\ i\ge r+s-n+1.\tag{4.2}$$

Obviously, the numbers $\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0$ are roots of a polynomial with real coefficients. Hence, in view of (4.1), (2.6) i) and the continuity of polynomials, there exists $\varepsilon_1>0$ such that for any $0<\varepsilon\le\varepsilon_1$

$$(-1)^i\,\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r,0\right)$$
$$=(-1)^i\,\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r\right)>0,\quad 0\le i\le r+s-n.$$

So, with (4.2) both conditions in I) are satisfied for $\varepsilon\le\varepsilon_1$.

Relation (4.2) also implies that the inequalities in II) are certainly satisfied for $r+s-n\le i\le r$. So it remains to consider $1\le i<r+s-n$. By (2.6) ii) we know that

$$c_{r-i,n}\,\sigma_i\left(\gamma_1,\ldots,\gamma_r\right)^2\ge\sigma_{i-1}\left(\gamma_1,\ldots,\gamma_r\right)\sigma_{i+1}\left(\gamma_1,\ldots,\gamma_r\right),$$

and since $c_{r+1-i,n+1}>c_{r-i,n}$ for all $1\le i\le r$ and $\sigma_i\left(\gamma_1,\ldots,\gamma_r\right)^2>0$ for $0\le i\le r+s-n$ (cf. (2.6) i)), we get that

$$c_{r+1-i,n+1}\,\sigma_i\left(\gamma_1,\ldots,\gamma_r\right)^2>\sigma_{i-1}\left(\gamma_1,\ldots,\gamma_r\right)\sigma_{i+1}\left(\gamma_1,\ldots,\gamma_r\right)$$

for all $1\le i<r+s-n$. Hence, as before, by continuity of polynomials, there exists $\varepsilon_2>0$ such that

$$c_{r+1-i,n+1}\,\sigma_i\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r\right)^2$$
$$>\sigma_{i-1}\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r\right)\sigma_{i+1}\left(\gamma_1+\varepsilon z,\gamma_2+\varepsilon\overline{z},\gamma_3,\ldots,\gamma_r\right)$$

for all $0<\varepsilon\le\varepsilon_2$ and $1\le i<r+s-n$. On account of (4.1) we obtain II) for $\varepsilon\le\varepsilon_2$, and the assertion follows with $\varepsilon_0=\min\{\varepsilon_1,\varepsilon_2\}$. □

As a corollary of the above proof we obtain a necessary condition for convex bodies forming a boundary-pair.

Proof of Corollary 1.1. For $\gamma\in\operatorname{bd}\mathcal{R}(n)\backslash\mathbb{R}_{\le 0}$, $n\ge 3$, let $K,E\in\mathcal{K}^n$ be such that $f_{K;E}(\gamma)=0$, and let $\overline{\gamma},\gamma_3,\ldots,\gamma_n$ be the remaining roots of $f_{K;E}(z)$.

If we assume that K, E are not extremal sets in any Aleksandrov-Fenchel inequality, i.e., if we have strict inequalities in (1.4), then for all $1 \leq i \leq n-1$ we get by Corollary 2.1

$$c_{r-i,n}\, \sigma_i\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right)^2 > \sigma_{i-1}\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right) \sigma_{i+1}\left(\gamma, \overline{\gamma}, \gamma_3, \dots, \gamma_n\right).$$

By the continuity of the elementary symmetric functions, for $\varepsilon > 0$ small enough, the numbers $\gamma + \varepsilon z, \overline{\gamma} + \varepsilon \overline{z}, \gamma_3, \dots, \gamma_n$ are roots of a polynomial with real coefficients, satisfying also conditions i) and ii) of Corollary 2.1 for any $z \in \mathbb{C}$ with $|z| = 1$. This implies that $\{\gamma + \varepsilon z : |z| = 1\} \subsetneq \mathcal{R}(n)$, contradicting that $\gamma \in \operatorname{bd} \mathcal{R}(n) \backslash \mathbb{R}_{\leq 0}$. □

We conclude this section studying the behavior of the cones for high dimensions, i.e., we prove Theorem 1.4.

Proof of Theorem 1.4. The proof is based on known results on the distribution of the roots of the truncated binomial polynomials $P^n_{0,k}(z) = \sum_{i=0}^k \binom{n}{i} z^i$, $0 < k \leq n$, which are also Steiner polynomials (cf. Proposition 2.1).

Let $\{k_n : n \in \mathbb{N}\}$ be any sequence of positive integer numbers such that $\alpha = \lim_{n\to\infty} k_n/n \in (0,1)$. By [15, Remark 1] we have that the set of accumulation points of $\bigcup_{n=1}^\infty \bigl\{z \in \mathbb{C} : P^n_{0,k_n}(z) = 0\bigr\}$ coincides with the set

$$\left\{z \in \mathbb{C} : |z| = \alpha\,(1-\alpha)^{1/\alpha - 1}\,|1+z|^{1/\alpha} \text{ and } \left|z - \frac{\alpha^2}{1-\alpha^2}\right| \leq \frac{\alpha}{1-\alpha^2}\right\}.$$

Hence, taking $k_n = \lfloor n/2 \rfloor$, it can be checked that 1 is contained in the above set of accumulation points, and so we know that there exists a sequence $\gamma_n \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$, $n \in \mathbb{N}$, such that for each $n \in \mathbb{N}$ there is $m_n \in \mathbb{N}$ with

$$\lim_{n\to\infty} \gamma_n = 1 \quad \text{and} \quad P^{m_n}_{0,\lfloor m_n/2 \rfloor}(\gamma_n) = 0. \tag{4.3}$$

Now let $\gamma \in \mathbb{C}^+ \setminus \mathbb{R}_{>0}$. By the choice of the sequence γ_n (cf. (4.3)) we can find an $n_\gamma \in \mathbb{N}$ such that γ is contained in the interior of the cone generated by the negative x-axis and γ_{n_γ}, which in particular implies, by the convexity of the cone $\mathcal{R}(n_\gamma)$ (cf. Theorem 1.1), that $\gamma \in \mathcal{R}(n_\gamma)$. By Theorem 1.3 we get the desired statement. □

5. The 4-dimensional cone

We conclude the paper by characterizing the cone of roots of 4-dimensional Steiner polynomials, for which it suffices to determine its boundary (cf. Theorem 1.1).

Proof of Proposition 1.2. First we notice that $\{x + y\mathrm{i} \in \mathbb{C}^+ : x + y \leq 0\} \subseteq \mathcal{R}(4)$. Indeed, since $\mathcal{R}(4)$ is a convex cone containing $\mathbb{R}_{\leq 0}$ (Theorem 1.1), it suffices to prove that $-1 + \mathrm{i} \in \mathcal{R}(4)$, which follows from the fact that $P^4_{1,4}(-1+\mathrm{i}) = 0$ and Proposition 2.1.

Next we determine conditions verified by a pair of convex bodies whose Steiner polynomial has $-1+\mathrm{i}$ as a root. We have to distinguish two cases. If $E \in \mathcal{K}_0^4$ then such a polynomial has to take the form

$$f_{K;E}(z) = \sum_{i=0}^{4} \binom{4}{i} \mathrm{W}_i(K;E) z^i = \mathrm{W}_4(K;E)(z^2+2z+2)(z^2+cz+d),$$

for certain $c, d \geq 0$ because it is weakly stable (cf. Proposition 1.3). Then we have the identities

$$\begin{aligned} 2+c &= 4\frac{\mathrm{W}_3(K;E)}{\mathrm{W}_4(K;E)}, \quad 2(c+1)+d = 6\frac{\mathrm{W}_2(K;E)}{\mathrm{W}_4(K;E)}, \\ 2(c+d) &= 4\frac{\mathrm{W}_1(K;E)}{\mathrm{W}_4(K;E)}, \quad 2d = \frac{\mathrm{W}_0(K;E)}{\mathrm{W}_4(K;E)}. \end{aligned} \tag{5.1}$$

Inequalities (1.4) for $i=3$, $i=2$ and $i=1$ yield, in terms of c, d, respectively,

$$\begin{aligned} 3c^2 - 4c - 8d - 4 &\geq 0, \\ c^2 + (d+2)c - 2(d^2 - 5d + 4) &\leq 0, \\ 3c^2 - 2dc - d^2 - 8d &\geq 0, \end{aligned}$$

which, since $c, d \geq 0$, are equivalent to

$$\begin{aligned} & c \geq \frac{2}{3}\left(1 + \sqrt{2}\sqrt{2+3d}\right), \\ & c \leq d - 4 \text{ if } d \geq 2 \quad \text{and} \quad c \leq 2(1-d) \text{ if } d \leq 2, \\ & c \geq \frac{1}{3}\left(d + 2\sqrt{d(d+6)}\right), \end{aligned}$$

respectively. A straightforward computation allows to conclude that the three above inequalities hold simultaneously if and only if $d = 0$ and $c = 2$. Then, $f_{K;E}(z) = \mathrm{W}_4(K;E)(z^4 + 4z^3 + 6z^2 + 4z) = \mathrm{W}_4(K;E)P_{1,4}^4(z)$. In particular, $\mathrm{W}_0(K;E) = 0$ (cf. (5.1)) and, in view of $\mathrm{W}_1(K;E) > 0$, this shows that $\dim K = 3$ and moreover, it holds $\mathrm{W}_1(K;E) = \mathrm{W}_2(K;E) = \mathrm{W}_3(K;E) = \mathrm{W}_4(K;E)$.

Now we suppose $\dim E < 4$. Then the polynomial has to take the form

$$f_{K;E}(z) = (z^2+2z+2)(cz+d) = cz^3 + (d+2c)z^2 + 2(c+d)z + 2d,$$

for certain $c, d \geq 0$ and applying Lemma 2.1 it is easy to check that it is a Steiner polynomial if and only if $c = d$. Notice that it cannot be $c = d = 0$. Hence $f_{K;E}(z) = cz^3 + 3cz^2 + 4cz + 2c$, implying that

$$\frac{1}{2}\mathrm{W}_0(K;E) = \mathrm{W}_1(K;E) = 2\mathrm{W}_2(K;E) = 4\mathrm{W}_3(K;E) = c \neq 0$$

and, in particular, that $\dim K = 4$. In both cases we get the required equalities $\mathrm{W}_i(K;E)^2 = \mathrm{W}_{i-1}(K;E)\mathrm{W}_{i+1}(K;E)$, for $i = 2, 3$.

Finally we prove that $\mathcal{R}(4) = \{x + y\mathrm{i} \in \mathbb{C}^+ : x + y \leq 0\}$. Thus we assume that $\gamma = -1 + (1+\varepsilon)\mathrm{i} \in \mathcal{R}(4)$ for $\varepsilon > 0$, i.e., that there exist $K, E \in \mathcal{K}^n$ such that $f_{K;E}(\gamma) = 0$, and we will get a contradiction. Then (see [8, Lemma 2.1])

$\gamma-\varepsilon$ is a root of $f_{K+\varepsilon E;E}(z)$. But since $\gamma-\varepsilon=-(1+\varepsilon)+(1+\varepsilon)\mathrm{i}$, the previous property implies that either $\dim(K+\varepsilon E)=3$ with $E\in\mathcal{K}_0^4$, which is clearly not possible, or $\dim E=3$ and $\mathrm{vol}(K+\varepsilon E)=\mathrm{W}_i(K+\varepsilon E;2E)\neq 0$, $i=1,2,3$, which also leads to a contradiction. Indeed, if

$$\mathrm{W}_0(K+\varepsilon E;2E)=\mathrm{W}_1(K+\varepsilon E;2E)=\mathrm{W}_2(K+\varepsilon E;2E)=\mathrm{W}_3(K+\varepsilon E;2E),$$

we find by the Steiner formulae for quermassintegrals (see [18, (5.1.27) and p. 212]) that

$$\mathrm{W}_2(K;E)=2(1-\varepsilon)\mathrm{W}_3(K;E) \quad\text{and}\quad \mathrm{W}_1(K;E)=(4+3\varepsilon^2-6\varepsilon)\mathrm{W}_3(K;E).$$

Notice that this implies $\varepsilon<1$. However, substitution of the above expressions in inequality (1.4) for $i=2$ leads to $\varepsilon\geq 2$, a contradiction. □

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