
On the Minkowski-Hlawka theorem

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Abstract

The well-known Minkowski-Hlawka theorem establishes that if $S \subset \mathbb{R}^n$, $n \geq 2$, is a 0-symmetric star body whose volume is bounded from above by (twice) the Riemann zeta function, then there exists a K -admissible lattice with determinant 1. This result has several equivalent versions due to its connection with both the Geometry of Numbers and the Packing Theory. Astonishingly, Minkowski-Hlawka's theorem in dimension $n = 1$ is not known in the literature. In this work we prove this result in the different settings it can be stated: for 0-symmetric star bodies, for star bodies not necessarily symmetric, and for arbitrary sets.

1 Introduction

As usual, $\mathbb{R}^n$ will represent the n -dimensional Euclidean space, and $\text{vol}(\cdot)$ will denote the volume functional, i.e., the Lebesgue measure in $\mathbb{R}^n$.

Given n linearly independent vectors $b_1, \dots, b_n \in \mathbb{R}^n$, the *lattice* L generated by them is the set of all integer linear combinations of these vectors, that is,

$$L = \{a_1 b_1 + \dots + a_n b_n : a_1, \dots, a_n \in \mathbb{Z}\}.$$

The set $\{b_1, \dots, b_n\}$ is called a *basis* of L . In particular, we represent by $\mathbb{Z}^n$ the integer lattice, i.e., the lattice of all points with integral coordinates in $\mathbb{R}^n$. Henceforth, we will denote by $\mathcal{L}^n$ the family of all lattices in $\mathbb{R}^n$.

Given a basis $\{b_1, \dots, b_n\}$ of a lattice $L \in \mathcal{L}^n$, we can identify it with the $(n \times n)$ -matrix $B = (b_1 \dots b_n)$ with columns the vectors $b_1, \dots, b_n$, and clearly, $L = B\mathbb{Z}^n$. The determinant of the lattice is defined as $\det L = |\det B|$, and it is well known that it does not depend on the selected basis. As general references for lattices we refer to [2, 3].

Relating now a subset of $\mathbb{R}^n$ and a lattice we find several important concepts. Thus, given a set $A \subset \mathbb{R}^n$, a lattice L is said to be *admissible for A* (or *A-admissible*) if it contains no interior point of the set, except, possibly, the origin. Finally, the *critical determinant* of the set A is defined as

$$\Delta(A) = \inf\{\det L : L \in \mathcal{L}^n \text{ is } A\text{-admissible}\},$$

with the following convention: $\Delta(A) = \infty$ if there are no lattices admissible for A .

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In the following, we define a particular family of sets that plays a key role in establishing the Minkowski-Hlawka theorem.

A *star body* is a compact subset $S \subset \mathbb{R}^n$ satisfying the following property: if $x \in S$ and $0 \leq \lambda < 1$, then $\lambda x \in \text{int } S$, where $\text{int } S$ represents the interior of the set S . Furthermore, we say that S is *0-symmetric* if $S = -S$ or, equivalently, if for any $x \in S$ and $-1 < \lambda < 1$, then $\lambda x \in \text{int } S$.

In particular, any star body has non-empty interior and any convex body (compact and convex set) containing the origin in its interior is a star body. For general information on the theory of star bodies we refer to [6].

Minkowski's fundamental theorem in the Geometry of Numbers [5] (cf. [2]) states that given a lattice $L \in \mathcal{L}^n$, if $K \subset \mathbb{R}^n$ is a 0-symmetric convex body with $(\text{int } K) \cap L = \{0\}$, then

$$\text{vol}(K) \leq 2^n \det L.$$

In other words, it says that any lattice L with sufficiently small determinant contains a point of $\text{int } K$ different from the origin.

The Minkowski-Hlawka theorem is another crucial result in the Geometry of Numbers, and yields the following counterpart of the above statement of Minkowski's fundamental theorem: if $n \geq 2$, there is a lattice L with determinant not too large, which contains no point of K other than the origin. Minkowski-Hlawka's result, originally conjectured by Minkowski, was proved by Hlawka in 1944 [4]. It can be stated in several equivalent ways due to its close connection with the Packing Theory.

One of its most classical statements involves the well-known Riemann zeta function. We recall that the *Riemann zeta function*, denoted by $\zeta(s)$, is a complex-valued function defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This series converges for all $s \in \mathbb{C}$ with real part $\text{Re}(s) > 1$. The following classical version of the Minkowski-Hlawka theorem can be found in [1].

Theorem 1 (Minkowski-Hlawka's theorem). *Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a 0-symmetric star body and let $\Delta > 0$ be such that $2\zeta(n)\Delta > \text{vol}(S)$. Then there is a lattice L with $\det L = \Delta$ which is S -admissible.*

An equivalent version of this result, better suited to our purposes, and which includes the notion of critical determinant, can be justified as follows. Observe that if S is a 0-symmetric star body such that

$$\text{vol}(S) \geq 2\zeta(n)\Delta(S),$$

and $\Delta > 0$ satisfies that $2\zeta(n)\Delta > \text{vol}(S)$ then, by the definition of infimum, if $\varepsilon = \Delta - \Delta(S) > 0$ there exists a S -admissible lattice L' such that $\det L' < \Delta(S) + \varepsilon = \Delta$. Taking

$$\alpha = \left(\frac{\Delta}{\det L'} \right)^{1/n},$$

it is clear that $L = \alpha L'$ satisfies $\det L = \Delta$. If we assume that L is not S -admissible, then there exists $z \in (\text{int } S) \cap L$, $z \neq 0$, which can be expressed as $z = \alpha y$ for some $y \in L' \setminus \{0\}$. Since $\alpha > 1$, it follows that $y = z/\alpha \in \text{int } S$, and hence $(\text{int } S) \cap L' \neq \{0\}$, contradicting the S -admissibility of L' .

Conversely, we know that if $\Delta > 0$ is such that $2\zeta(n)\Delta > \text{vol}(S)$, Theorem 1 guarantees the existence of a S -admissible lattice $L_\Delta \in \mathcal{L}^n$ with $\det L_\Delta = \Delta$. If we assume that

$$\text{vol}(S) < 2\zeta(n)\Delta(S)$$

and choose $\Delta > 0$ satisfying $\text{vol}(S) < 2\zeta(n)\Delta < 2\zeta(n)\Delta(S)$, then L_Δ will be an admissible lattice for S with $\det L_\Delta = \Delta < \Delta(S)$, which contradicts the definition of $\Delta(S)$.

In summary, we have shown that Theorem 1 is equivalent to the following statement: if $S \subset \mathbb{R}^n$, $n \geq 2$, is a 0-symmetric star body, then $\text{vol}(S) \geq 2\Delta(S)\zeta(n)$. Next, we establish this second (equivalent) formulation of the Minkowski-Hlawka theorem, also including the corresponding bounds for the volume in the more general setting of generic star bodies and arbitrary bounded Jordan measurable sets. It can be found in [3].

Theorem 2 (Minkowski-Hlawka's theorem). *Let $S \subset \mathbb{R}^n$, $n \geq 2$, be a bounded Jordan measurable set. Then, the following hold:*

1. $\text{vol}(S) \geq \Delta(S)$.
2. *If S is a star body, then $\text{vol}(S) \geq \Delta(S)\zeta(n)$.*
3. *If S is a 0-symmetric star body, then $\text{vol}(S) \geq 2\Delta(S)\zeta(n)$.*

As can be also seen in the above results, the Minkowski-Hlawka theorem is always stated for dimension $n \geq 2$ and, to the best of our knowledge, there is no analogue in the literature for $n = 1$. In this work we state and prove the one-dimensional case of the Minkowski-Hlawka result. We point out that this is not a direct consequence of the above theorems, since $\zeta(1)$ is not well defined. Furthermore, the techniques used to prove the higher-dimensional result rely heavily on the fact that $n > 1$, and thus the one-dimensional case cannot be derived from them. Therefore, a different approach is needed.

2 The Minkowski-Hlawka theorem in dimension $n = 1$

Observe, on the one hand, that a lattice in $\mathbb{R}$ is a set of integer multiples of a positive number, i.e., if $L \in \mathcal{L}^1$ then $L = \alpha\mathbb{Z}$ for some $\alpha > 0$, and it is clear that $\det L = \det(\alpha\mathbb{Z}) = \alpha$. On the other hand, a star body (respectively, a 0-symmetric star body) is a compact interval containing zero in its interior (respectively, a compact interval which is symmetric with respect to zero). However, we will work in the more general setting of an arbitrary subset $S \subset \mathbb{R}$. Thus, in this case, the one-dimensional volume cannot be used as a measure, and we must instead consider the outer Lebesgue measure, which is defined as follows: if $S \subset \mathbb{R}^n$ is an arbitrary subset of $\mathbb{R}^n$, the *outer Lebesgue measure* of S is given by

$$\text{vol}^*(S) = \inf \left\{ \sum_{k=1}^{\infty} \text{vol}(P_k) : P_k \text{ are open orthogonal boxes with } S \subset \bigcup_{k=1}^{\infty} P_k \right\}.$$

For the proof of our main result, we need the following lemma. Observe that, since we work with a family of pairwise disjoint intervals, it is necessarily countable, either finite or infinite. The lemma applies in both cases, and so we index by n , ranging over either $\mathbb{N}$ or a finite index set.

Lemma 3. *Let $S \subset \mathbb{R}$ and let $\{I_n\}_n$ be a family of pairwise disjoint intervals such that $S \subset \bigcup_n I_n$. Then*

$$\text{vol}^*(S) = \sum_n \text{vol}^*(S \cap I_n).$$

Proof. It is not restrictive to assume, by removing, if necessary a countable set of points from S (which does not affect its measure), that all the intervals I_n are open. We prove the lemma for the case where the family of intervals is infinite; the finite case follows analogously.

From the definition of outer Lebesgue measure we have that

$$\text{vol}^*(S) \leq \sum_n \text{vol}^*(S \cap I_n).$$

So, it suffices to prove the reverse inequality.

Fix $\varepsilon > 0$. Then there exists an open set $A \supset S$ such that $\text{vol}(A) < \text{vol}^*(S) + \varepsilon$. But this inequality is also satisfied by the (smaller) open set

$$A' = A \cap \left(\bigcup_n I_n \right) = \bigcup_n (A \cap I_n) \supset S.$$

Since the open sets $A' \cap I_n$ are pairwise disjoint for all n and $A' = \bigcup_n (A' \cap I_n)$, we can deduce that

$$\sum_n \text{vol}^*(S \cap I_n) \leq \sum_n \text{vol}(A' \cap I_n) = \text{vol}(A') < \text{vol}^*(S) + \varepsilon.$$

Notice that in the above equality we have used the fact that $A' \cap I_n$ is measurable for all n . Taking the limit as $\varepsilon \rightarrow 0$, the proof is completed. $\square$

We finally state and prove our main result, the Minkowski-Hlawka theorem in dimension $n = 1$. The case where S is a star body (items 2 and 3) is straightforward, but we include it for completeness and to emphasize the symmetry with Theorem 2.

Theorem 4 (Minkowski-Hlawka's theorem in $\mathbb{R}$). *Let $S \subset \mathbb{R}$. Then, the following hold:*

1. $\text{vol}^*(S) \geq \Delta(S)/2$, and the inequality is tight.
2. If S is a star body, then $\text{vol}(S) > \Delta(S)$, and the inequality is tight.
3. If S is a 0-symmetric star body, then $\text{vol}(S) = 2\Delta(S)$.

Proof. We start by showing the first item. Notice that if $S^+ = S \cap [0, \infty)$, $S^- = S \cap (-\infty, 0)$, and $S' = S^+ \cup (-S^-)$, then

$$\begin{aligned} \Delta(S') &= \Delta(S) \quad \text{and} \\ \text{vol}^*(S') &\leq \text{vol}^*(S^+) + \text{vol}^*(-S^-) = \text{vol}^*(S^+) + \text{vol}^*(S^-) = \text{vol}^*(S) \end{aligned}$$

by Lemma 3, and so we may assume without loss of generality that $S \subset [0, \infty)$.

We are going to prove that if $\text{vol}^*(S) < 1$ then $\Delta(S) < 2$, which, using the homogeneity of the critical determinant, will show the result. Let

$$I_{n,k} = \frac{1}{k} [n, n+1) = \left[\frac{n}{k}, \frac{n+1}{k} \right) \quad \text{for} \quad \left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n.$$

Clearly, all intervals $I_{n,k}$ are contained in $[1, 2)$. Moreover, if $k < n$, then $(n+1)/(k+1) < n/k$, which guarantees that, for a fixed n , the intervals $I_{n,k}$ are pairwise disjoint. Moreover, let

$$S_n = S \cap [n, n+1), \quad R_{n,k} = \frac{1}{k} S_n, \quad R_n = \bigcup_{k=\lceil \frac{n+1}{2} \rceil}^n R_{n,k} \quad \text{and} \quad R = \bigcup_{n=1}^{\infty} R_n.$$

Clearly, $R \subset [1, 2)$. Furthermore,

$$\begin{aligned} \text{vol}^*(R_n) &= \sum_{k=\lceil \frac{n+1}{2} \rceil}^n \text{vol}^*(R_{n,k}) = \sum_{k=\lceil \frac{n+1}{2} \rceil}^n \frac{1}{k} \text{vol}^*(S_n) \\ &= \text{vol}^*(S_n) \sum_{k=\lceil \frac{n+1}{2} \rceil}^n \frac{1}{k} = \text{vol}^*(S_n) \sum_{k=\lceil \frac{n+1}{2} \rceil}^n \text{vol}(I_{n,k}) \leq \text{vol}^*(S_n), \end{aligned}$$

and hence,

$$\text{vol}^*(R) \leq \sum_{n=1}^{\infty} \text{vol}^*(R_n) \leq \sum_{n=1}^{\infty} \text{vol}^*(S_n) = \text{vol}^*(S \cap [1, \infty)) \leq \text{vol}^*(S) < 1.$$

Recall that a lattice $L \in \mathcal{L}^1$ is of the form $L = \alpha\mathbb{Z}$ for some $\alpha > 0$, being $\det L = \alpha$. If we assume that there exist $\alpha \in [1, 2)$ and $k \in \mathbb{N}$ such that $x := k\alpha \in S$, then it will be $\alpha \in R$. Indeed, let $n \in \mathbb{N}$ with $x = k\alpha \in S_n$. Since $1 \leq \alpha = x/k < 2$ we have, on the one hand, that $k \leq x < n+1$, i.e., $k \leq n$, and on the other hand that $n \leq x < 2k$, this is, $k \geq \lceil (n+1)/2 \rceil$. Consequently, $\alpha \in R_{n,k} \subset R_n \subset R$.

Since $R \subset [1, 2)$ and $\text{vol}^*(R) < 1$, the above observation ensures the existence of $\beta \in [1, 2) \setminus R$ satisfying $k\beta \notin S$ for all $k \in \mathbb{N}$. Hence, $\Delta(S) \leq \beta < 2$.

Now we see that the inequality $\Delta(S) \leq 2\text{vol}^*(S)$ is tight. Indeed, let us consider the set $S = [1 - \varepsilon, 2 + \varepsilon]$ for $\varepsilon > 0$ sufficiently small. Clearly $\text{vol}^*([1 - \varepsilon, 2 + \varepsilon]) = 1 + 2\varepsilon$. Furthermore, if we consider a lattice $\alpha\mathbb{Z}$ with $\alpha < 2 + \varepsilon$, then it is easy to see that there exists $n \in \mathbb{N}$ such that $n\alpha \in (1 - \varepsilon, 2 + \varepsilon)$, and hence $\alpha\mathbb{Z}$ is not S -admissible. Therefore $\alpha \geq 2 + \varepsilon$, which implies that $\Delta([1 - \varepsilon, 2 + \varepsilon]) = 2 + \varepsilon$. Thus,

$$\frac{\Delta(S)}{2\text{vol}^*(S)} = \frac{2 + \varepsilon}{1 + 2\varepsilon},$$

which tends to 1 when $\varepsilon \rightarrow 0$. It shows the tightness of the inequality.

We conclude the proof of the theorem by showing items 2 and 3. If S is a star body, then $S = [-a, b]$ with $a, b > 0$. Let $L = \alpha\mathbb{Z} \in \mathcal{L}^1$ for some $\alpha > 0$. Clearly, L is admissible for S if and only if $\alpha > \max\{a, b\}$. We assume that $b > a$. Then $\Delta(S) = b \leq a + b = \text{vol}(S)$, which proves item 2. The case $b < a$ follows analogously, and if $a = b$ we are in the symmetric case, which we consider next.

Regarding the tightness of the inequality, we observe that the equality $\text{vol}(S) = \Delta(S)$ cannot be attained, but the inequality is tight; indeed, if we consider the star bodies $S_\varepsilon = [-\varepsilon, 1]$, $\varepsilon > 0$, one has $\text{vol}(S_\varepsilon) = 1 + \varepsilon$ and $\Delta(S_\varepsilon) = 1$. Therefore $\text{vol}(S_\varepsilon) - \Delta(S_\varepsilon) = \varepsilon$, and $\lim_{\varepsilon \rightarrow 0} (\text{vol}(S_\varepsilon) - \Delta(S_\varepsilon)) = 0$.

Finally, let S be a 0-symmetric star body, that is, $S = [-a, a]$ with $a > 0$. Let $L = \alpha\mathbb{Z}$ be an S -admissible lattice. Then $\alpha > a$, and hence $\Delta(S) = a = \text{vol}(S)/2$. It concludes the proof. $\square$

References

- [1] J. W. S. Cassels, *An introduction to the geometry of numbers*, Reprint of the 1971 edition, Springer, Berlin, 1997.
- [2] P. M. Gruber, *Convex and discrete geometry*, Springer, Berlin Heidelberg, 2007.
- [3] P. M. Gruber, C. G. Lekkerkerker, *Geometry of Numbers*, 2nd edition, North-Holland Mathematical Library, vol. 37, North-Holland, Amsterdam, 1987.
- [4] E. Hlawka, Zur Geometrie der Zahlen, *Math. Z.* **49** (1944), 285–312, Selecta 38–65.
- [5] H. Minkowski, *Geometrie der Zahlen*, Teubner, Leipzig-Berlin, 1896, reprinted by Johnson Reprint Corp., New York, 1968.
- [6] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*, 2nd expanded edition, Cambridge University Press, Cambridge, 2014.