ON DISCRETE LOG-BRUNN-MINKOWSKI TYPE INEQUALITIES

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ABSTRACT. The conjectured log-Brunn-Minkowski inequality says that the volume of centrally symmetric convex bodies $K, L \subset \mathbb{R}^n$ satisfies $\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}$, $\lambda \in (0,1)$, and is known to be true in the plane and for particular classes of n-dimensional symmetric convex bodies. In this paper we get some discrete log-Brunn-Minkowski type inequalities for the lattice point enumerator. Among others, we show that if $K, L \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then

$$\mathrm{G}_n\bigg((1-\lambda)\cdot\big(K+C_n\big) +_0 \lambda\cdot\big(L+C_n\big) + \Big(-\frac{1}{2},\frac{1}{2}\Big)^n\bigg) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda},$$

where $C_n = [-1/2, 1/2]^n$. Neither C_n nor $(-1/2, 1/2)^n$ can be removed. Furthermore, it implies the (volume) log-Brunn-Minkowski inequality for unconditional convex bodies. The corresponding results in the L_p setting for $0 < p < 1$ are also obtained.

1. Introduction

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, endowed with the (Euclidean) inner product $\langle\cdot,\cdot\rangle$. One of the cornerstones of convex geometry is the Brunn-Minkowski inequality, which, in its classical form, provides a relation between the notions of Minkowski addition and volume of a pair of measurable sets. In particular, it states that given any two non-empty compact convex sets $K, L \subset \mathbb{R}^n$ and any $\lambda \in (0,1)$, one has

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda\mathrm{vol}(L)^{1/n}. \tag{1.1}$$

The Minkowski sum is the pointwise vector addition, i.e., $A + B = \{x + y : x \in A, y \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$, whereas the volume is the standard Lebesgue measure. Furthermore, rA denotes $\{rx : x \in A\}$ for any $r > 0$. Despite the traditional formulation being for compact convex sets (i.e., *convex bodies*), this hypothesis can be relaxed, and the result holds true for arbitrary measurable sets.

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Generalizations and analogues of this inequality have proved to be a fruitful field of study, involving other operations, spaces and measures, as well as obtaining related inequalities and simpler proofs of already known ones, being the isoperimetric inequality one of the most remarkable examples in this respect. We refer the reader to [3, 12] for extensive survey articles on the topic, as well as to the updated monograph [31, Chapter 9] and the references therein.

Of particular interest for us is the L_p version of the Brunn-Minkowski inequality. This was proved by Firey in [10], and states that for any two convex bodies $K, L \subset \mathbb{R}^n$ containing the origin, and any $p \geq 1$, one has

$$\mathrm{vol}\big((1-\lambda)\cdot K +_p \lambda \cdot L\big)^{p/n} \geq (1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda \mathrm{vol}(L)^{p/n}, \tag{1.2}$$

where $K +_p L$ is the only convex body whose support function is given by $(h_K(\cdot)^p + h_L(\cdot)^p)^{1/p}$, and where $\cdot$ is the p-scalar product, i.e., $r\cdot K = r^{1/p}K$ for any $r > 0$ (defined so for the sake of simplicity). We recall that the support function of a convex body K is given by $h_K(u) = \max_{x\in K}\langle x, u\rangle$ for all $u \in \mathbb{R}^n$. It is easy to see that $+_1$ is the standard Minkowski sum, and thus, that this notion provides a uniparametric generalization of the Brunn-Minkowski inequality. Additionally, it can be seen that $K +_\infty L = \mathrm{conv}(K \cup L)$, i.e., the convex hull of the union of the two bodies.

We note that the hypothesis for the sets in the previous definition, unlike in the standard Brunn-Minkowski inequality, cannot be relaxed. Indeed, compactness and convexity are required so that the support functions characterize the sets (see, e.g., [31, Theorem 1.7.1]), and the bodies need to contain the origin for the support function to be non-negative.

In order to elude this inconvenience, Lutwak, Yang and Zhang (see [23]) introduced an alternative pointwise definition which is valid for arbitrary sets. Specifically, for any two non-empty bounded sets $K, L \subset \mathbb{R}^n$ and any $p \geq 1$, they defined

$$K +_p L = \Big\{(1-\mu)^{1/q}x + \mu^{1/q}y : x \in K, y \in L, \, \mu \in [0,1]\Big\},$$

where $q \in [1, +\infty]$ is the Hölder conjugate of p, i.e., such that $1/p + 1/q = 1$, and showed that when K and L are convex bodies containing the origin, the definition coincides with the one of Firey. We note that if $p = 1$ then $q = \infty$, and thus the above notion again reduces to the standard Minkowski addition. The authors also proved the corresponding L_p Brunn-Minkowski type inequality of the form (1.2) in this general setting.

It is desirable to extend the aforementioned notions to the case $0 \leq p < 1$, and, in particular, to the case $p = 0$. A strong reason for this is that the corresponding and recently conjectured Brunn-Minkowski inequality in this setting for symmetric convex bodies, known in the literature as the *log-Brunn-Minkowski* inequality, would be stronger than all other L_p versions of the form (1.2) (see [5]). We recall that a convex body K is said to be centrally symmetric if $K = -K$.

Conjecture 1.1 (The log-Brunn-Minkowski inequality)**.** *Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies, and let $\lambda \in (0,1)$. Then*

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}. \tag{1.3}$$

However, it is easy to see that both definitions for $K +_p L$ given above can be problematic when $p < 1$. Indeed, the former fails since the p-sum of support functions is no longer sublinear, and thus, is not the support function of any convex body; whereas for the latter, difficulties may arise due to the fact that q would be negative. Therefore, the extension is obtained, in the setting of convex bodies, by means of the so-called *Wulff shape* (see, e.g., [31, Section 7.5]) determined by the support functions of the sets. In particular, given two convex bodies $K, L \subset \mathbb{R}^n$ containing the origin and a fixed $\lambda \in (0,1)$,

$$(1-\lambda)\cdot K +_p \lambda\cdot L = \bigcap_{u\in\mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq \big((1-\lambda)h_K(u)^p + \lambda h_L(u)^p\big)^{1/p}\right\},$$

where $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$. It can be seen that this definition coincides with the one of Firey when $p \geq 1$. In the case $p = 0$, the previous notion translates into the limit case

$$(1-\lambda)\cdot K +_0 \lambda\cdot L = \bigcap_{u\in\mathbb{S}^{n-1}} \left\{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda}h_L(u)^{\lambda}\right\}.$$

Conjecture 1.1 was solved in the plane already in [5], both for $p = 0$ and $0 < p < 1$, and the corresponding equality cases were characterized. The authors also noted that the central symmetry hypothesis cannot be removed. The conjecture can be solved completely in the complex case as a consequence of a generalization of the Blaschke-Santaló inequality due to Cordero-Erausquin, as shown by Rotem in [26].

Since then, other symmetric scenarios have been considered. For the case of unconditional bodies (i.e., bodies that have orthogonal symmetry with respect to all the canonical hyperplanes) in general dimension, Conjecture 1.1 was solved by Saroglou in [29] for $p = 0$, whereas the corresponding analog when $0 < p < 1$ was shown by Marsiglietti in [24]. These results were generalized by Böröczky and Kalantzopoulos in [4] to the setting of bodies which have linear symmetry (not necessarily orthogonal) with respect to n hyperplanes with linearly independent normal vectors. In a more functional setting, Saroglou also showed in [30] that the conjecture implies the corresponding inequality for any log-concave measure.

Furthermore, the question of the stability of Brunn-Minkowski type inequalities of this form has also been studied, and local results (with respect to the Hausdorff topology) have been obtained, for instance, in [8, 9, 22, 25].

For further information on the log-Brunn-Minkowski, we refer the reader to the previous manuscripts and the references therein.

Here we are interested in studying the discretization of inequalities of the aforementioned type. As already stated, another common approach to extend the Brunn-Minkowski inequality is to consider alternative spaces and measures. In this regard, the integer lattice $\mathbb{Z}^n$ endowed with the cardinality measure has been extensively studied. Ruzsa obtained in [27, 28] some of the first strengthenings of the classical discrete Brunn-Minkowski-like inequality $|A+B| \geq |A| + |B| - 1$. These results were later improved by Gardner and Gronchi in [13]. Recently, Hernández Cifre, Iglesias and Yepes Nicolás obtained in [17] an inequality in this setting.

Another common approach is to work with arbitrary bounded sets and then intersect them with the integer lattice. As it is common in the literature, we shall denote by

$$\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$$

the *lattice point enumerator* of any bounded set $K \subset \mathbb{R}^n$. This approach allows one to work with dilations of sets in a sensible way, since dilating a finite set does not change its cardinality.

In this setting, Iglesias, Yepes Nicolás and Zvavitch showed in [21] that for any non-empty bounded sets $K, L \subset \mathbb{R}^n$ and any $\lambda \in (0,1)$, one has

$$\mathrm{G}_n\Big((1-\lambda)K + \lambda L + (-1,1)^n\Big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}. \tag{1.4}$$

A version with arbitrary linear coefficients does not follow immediately, unlike for the volume, due to the lack of homogeneity. However, similar arguments can show it (see [20]). This discrete inequality was extended to the L_p setting in [18] by Yepes Nicolás and the authors, obtaining

$$\mathrm{G}_n\Big((1-\lambda)\cdot K +_p \lambda \cdot L + (-1,1)^n\Big)^{p/n} \geq (1-\lambda)\mathrm{G}_n(K)^{p/n} + \lambda \mathrm{G}_n(L)^{p/n}, \tag{1.5}$$

for any $p \geq 1$. Again, a version for arbitrary linear coefficients can also be obtained with similar arguments. Further analogous discrete results were obtained in [14, 19, 32].

Apart from the above-mentioned discrete Brunn-Minkowski type inequalities, various discrete counterparts, for the lattice point enumerator $\mathrm{G}_n(\cdot)$, of classical results in Convex Geometry have been recently proven. Some examples of such results are *Koldobsky's slicing inequality* [1], *Meyer's inequality* [11] or an isoperimetric type inequality [20]. We refer the reader to these articles and the references therein for other connected problems, questions and results.

In this paper we obtain discrete versions of the log-Brunn-Minkowski inequality (1.3), with methods that can also be easily adapted to the $0 < p < 1$ setting (see Section 2), both for the lattice point enumerator, and for some alternative measures.

The paper is structured as follows. Section 2 is devoted to establishing the necessary notation and presenting the main results of the paper. In Section 3 we prove the results concerning different point enumerators. Finally, Section 4 deals with those results involving different measures.

2. Notation and main results

To begin with, let us establish some notation, for the sake of simplicity. For any $t \in \mathbb{R}$ and any non-empty set $A \subset \mathbb{R}^n$, $A_{\geq t}$ will represent the set

$$A_{\geq t} = \{x = (x_1, \dots, x_n) \in A : x_i \geq t, i = 1, \dots, n\}.$$

Given $x, y \in \mathbb{R}^n$, we will write $xy \in \mathbb{R}^n$ to denote the point with coordinates $(xy)_i = x_i y_i$ for all $i = 1, \dots, n$, while, if $x \in \mathbb{R}^n_{\geq 0}$, x^λ will be the point such that $(x^\lambda)_i = x_i^\lambda$ for any given $\lambda > 0$, $i = 1, \dots, n$. Analogously, we will denote by $AB = \{ab \in \mathbb{R}^n : a \in A, b \in B\}$ for any $A, B \subset \mathbb{R}^n$, as well as $A^\lambda = \{a^\lambda : a \in A\}$ for any set $A \subset \mathbb{R}^n_{\geq 0}$ and any scalar $\lambda > 0$. Finally, for any bounded set $K \subset \mathbb{R}^n$ and any discrete set Λ, we will use the functional $\mathrm{G}_\Lambda(K) = |K \cap \Lambda|$, and for short we will write $\mathrm{G}_n(K) = \mathrm{G}_{\mathbb{Z}^n}(K)$.

Our initial result provides a discretization of Saroglou's result, i.e., a discrete version of the conjectured inequality (1.3) for the lattice point enumerator of unconditional convex bodies, as well as of (1.3) for centrally-symmetric planar convex bodies. For the sake of brevity, the (closed) centrally symmetric unit cube will be denoted as $C_n := [-1/2, 1/2]^n$.

Theorem 2.1. *Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies and let $\lambda \in (0,1)$. If either K, L are unconditional convex bodies or $n = 2$, then*
(2.1)
$$\mathrm{G}_n\left((1-\lambda)\cdot\big(K + C_n\big) +_0 \lambda \cdot \big(L + C_n\big) + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^\lambda.$$
Furthermore, it implies the log-Brunn-Minkowski inequality (1.3) *both for unconditional convex bodies or when $n = 2$.*

We observe that the Minkowski addition of the cube C_n in the left-hand side of the latter inequality in each body cannot be, in general, avoided, not even summing up a bigger cube instead of $(-1/2, 1/2)^n$; similarly, the Minkowski addition of $(-1/2, 1/2)^n$ is necessary (see Remark 3.1).

Next we introduce an operation closely related to the standard p-sum of convex bodies, which was utilized in [29] and [24] for the L_p Brunn-Minkowski inequalities discussed in the introduction. Given two non-empty sets $K, L \subset \mathbb{R}^n_{\geq 0}$ and $\lambda \geq 0$,

(2.2)
$$(1-\lambda)\cdot K \oplus_p \lambda \cdot L = \left\{ \left(\big((1-\lambda)x_1^p + \lambda y_1^p\big)^{1/p}, \dots, \big((1-\lambda)x_n^p + \lambda y_n^p\big)^{1/p} \right) : (x_1, \dots, x_n) \in K, (y_1, \dots, y_n) \in L \right\}.$$

Again, the case $p = 0$ must be understood as its limit case, and thus,

$$(1-\lambda)\cdot K \oplus_0 \lambda \cdot L = K^{1-\lambda}L^\lambda.$$

It was proved in [24] that $(1-\lambda)\cdot K \oplus_p \lambda \cdot L \subset (1-\lambda)\cdot K +_p \lambda \cdot L$ for any $p \in [0,1]$, which implies that, in order to obtain an L_p Brunn-Minkowski type inequality for $p \in [0,1]$, it suffices to consider the set $(1-\lambda)\cdot K \oplus_p \lambda \cdot L$ in the left-hand side (cf. (1.2)). Using this approach, Marsiglietti proved that if $K, L \subset \mathbb{R}^n$ are unconditional convex bodies, $\lambda \in (0,1)$ and $0 < p < 1$, then

$$\mathrm{vol}\big((1-\lambda)\cdot K \oplus_p \lambda \cdot L\big) \geq \mathcal{M}^{\lambda}_{\frac{p}{n}}\big(\mathrm{vol}(K), \mathrm{vol}(L)\big). \tag{2.3}$$

Here, $\mathcal{M}^{\lambda}_{p/n}(\cdot,\cdot)$ represents the (p/n)-mean of two non-negative numbers. We recall next the definition of α-mean, where α is always a parameter varying in $\mathbb{R} \cup \{\pm\infty\}$. We consider first the case $\alpha \in \mathbb{R}\backslash\{0\}$: given $a, b > 0$, let

$$\mathcal{M}^{\lambda}_{\alpha}(a,b) = \big((1-\lambda)a^{\alpha} + \lambda b^{\alpha}\big)^{1/\alpha}.$$

For $\alpha = \pm\infty$ we set $\mathcal{M}^{\lambda}_{\infty}(a,b) = \max\{a,b\}$ and $\mathcal{M}^{\lambda}_{-\infty}(a,b) = \min\{a,b\}$. Furthermore, if $ab = 0$, we define $\mathcal{M}^{\lambda}_{\alpha}(a,b) = 0$ for all $\alpha \in \mathbb{R} \setminus \{0\} \cup \{\pm\infty\}$. Finally, for $\alpha = 0$ we write $\mathcal{M}^{\lambda}_{0}(a,b) = a^{1-\lambda}b^{\lambda}$ (for a general reference for α-means of non-negative numbers, we refer the reader to the classic text of Hardy, Littlewood and Pólya [16] and to the handbook [6]).

Now, for any $p > 0$, we consider the change of variables $\varphi_p : \mathbb{R}^n_{\geq 0} \longrightarrow \mathbb{R}^n_{\geq 0}$ given by $\varphi_p(x)_i = x_i^{1/p}$ for every $i = 1, \dots, n$. Analogously, we will denote by $\psi_a : \mathbb{R}^n_{\geq 0} \longrightarrow \mathbb{R}^n_{\geq 1}$ the change of variables given by $\psi_a(x)_i = a^{x_i}$, for any $a > 1$ (so that ψ_a is bijective). These changes of variables will allow us to establish the spaces and functionals with which we will obtain our results. We will write $\Gamma_p = \varphi_p(\mathbb{Z}^n_{\geq 0})$ and $\Lambda_a = \psi_a(\mathbb{Z}^n_{\geq 0})$.

The following result for G_{Λ_a} can then be shown.

Proposition 2.1. *Let $a > 1$ and $\lambda \in (0,1)$, and let $K, L \subset \mathbb{R}^n_{\geq 1}$ be non-empty bounded sets with $\mathrm{G}_{\Lambda_a}(K)\mathrm{G}_{\Lambda_a}(L) > 0$. Then*

$$\mathrm{G}_{\Lambda_a}\Big((1,a^2)^n K^{1-\lambda} L^{\lambda}\Big)^{1/n} \geq (1-\lambda)\mathrm{G}_{\Lambda_a}(K)^{1/n} + \lambda \mathrm{G}_{\Lambda_a}(L)^{1/n}, \tag{2.4}$$

and the inequality is sharp.

In particular, a discrete log-Brunn-Minkowski type inequality for $\mathrm{G}_{\Lambda_a}(\cdot)$ is obtained as a direct consequence:

Corollary 2.1. *Let $a > 1$ and $\lambda \in (0,1)$, and let $K, L \subset \mathbb{R}^n_{\geq 1}$ be non-empty bounded sets. Then*

$$\mathrm{G}_{\Lambda_a}\Big((1,a^2)^n K^{1-\lambda} L^{\lambda}\Big) \geq \mathrm{G}_{\Lambda_a}(K)^{1-\lambda}\mathrm{G}_{\Lambda_a}(L)^{\lambda},$$

and the inequality is sharp.

Next, we define an alternative (discrete) measure for which a result in the same spirit as the previous one can also be proved. For any $a > 1$ and any bounded set $M \subset \mathbb{R}^n$, let

$$\mu_a(M) = \sum_{z \in M \cap \Lambda_a} \phi(z), \tag{2.5}$$

where the density function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is given by

$$\phi(z) = \prod_{i=1}^{n} z_i.$$

We note that μ_a coincides with G_{Λ_a} when the density function $\phi \equiv 1$. We will use a similar technique as the one used in [29] to approach the problem in the discrete setting.

Theorem 2.2. *Let $K, L \subset \mathbb{R}^n_{\geq 1}$ be non-empty bounded sets, and let $a > 1$ and $\lambda \in (0,1)$. Then*

$$a^n \mu_a \left(\left(a^{-1}, a\right)^n K^{1-\lambda} L^{\lambda} \right) \geq \mu_a(K)^{1-\lambda} \mu_a(L)^{\lambda}.$$

All the above results will be also extended to the case $0 < p < 1$ (see Theorem 3.1, Corollary 3.1 and Theorem 4.1).

3. Log-Brunn-Minkowski type inequalities for different point enumerators

The proof of Theorem 2.1 relies on the following relations between the volume and the lattice point enumerator of a convex body $K \subset \mathbb{R}^n$:

$$\begin{aligned} \mathrm{G}_n(K) &\leq \mathrm{vol}\left(K + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right), \\ \mathrm{vol}(K) &\leq \mathrm{G}_n\left(K + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right). \end{aligned} \tag{3.1}$$

The first inequality can be found in [15, (3.3)]. The second one, although well-known, does not appear (up to our knowledge) in any classical reference; a proof of it has been recently collected for completeness in [2].

Proof of Theorem 2.1. Clearly, if $K \subset \mathbb{R}^n$ is an unconditional convex body (or just centrally symmetric), so is $K + C_n$. Thus, using (3.1) and Saroglou's result (inequality (1.3) for unconditional convex bodies) we obtain

$$\begin{aligned} \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda} &\leq \mathrm{vol}(K + C_n)^{1-\lambda}\mathrm{vol}(L + C_n)^{\lambda} \\ &\leq \mathrm{vol}\Big((1-\lambda) \cdot \big(K + C_n\big) +_0 \lambda \cdot \big(L + C_n\big)\Big) \\ &\leq \mathrm{G}_n\left((1-\lambda) \cdot \big(K + C_n\big) +_0 \lambda \cdot \big(L + C_n\big) + \left(-\frac{1}{2}, \frac{1}{2}\right)^n\right), \end{aligned}$$

as required. The case of $n = 2$ is analogous but using the known log-Brunn-Minkowski inequality (1.3) for centrally symmetric planar convex bodies.

In order to conclude the proof, we show that (2.1) implies (1.3) when K and L are unconditional sets (respectively, when $n = 2$). It is a well-known fact that, roughly speaking, the volume and the lattice point enumerator are equivalent when the convex body K is large enough, i.e.,

$$\lim_{r \to \infty} \frac{\mathrm{G}_n(rK)}{r^n} = \mathrm{vol}(K) \tag{3.2}$$

(see e.g. [33, Lemma 3.22]). Moreover, it can be checked that, for any non-empty bounded set $M \subset \mathbb{R}^n$ containing the origin,

$$\lim_{r\to\infty} \frac{\mathrm{G}_n(rK+M)}{r^n} = \mathrm{vol}(K) \tag{3.3}$$

(see e.g. [2]). We also observe that, for any $K, L \in \mathcal{K}^n$ and any $r > 0$,

$$(1-\lambda)\cdot(rK) +_0 \lambda\cdot(rL) = r\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big).$$

Now, let $K, L \in \mathcal{K}^n$ be unconditional convex bodies (respectively, let $n = 2$), and fix $\varepsilon > 0$. Then, using (2.1), (3.2) and (3.3) we get, on one hand,

$$\begin{aligned}
\mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda} &= \lim_{r\to\infty} \frac{\mathrm{G}_n(rK)^{1-\lambda}\mathrm{G}_n(rL)^{\lambda}}{r^n} \\
&\le \lim_{r\to\infty} \frac{\mathrm{G}_n\Big((1-\lambda)\cdot(rK+C_n) +_0 \lambda\cdot(rL+C_n) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\Big)}{r^n} \\
&= \lim_{r\to\infty} \frac{\mathrm{G}_n\Big(r\Big((1-\lambda)\cdot\big(K+\frac{1}{r}C_n\big) +_0 \lambda\cdot\big(L+\frac{1}{r}C_n\big)\Big) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\Big)}{r^n} \\
&\le \lim_{r\to\infty} \frac{\mathrm{G}_n\Big(r\Big((1-\lambda)\cdot\big(K+[-\varepsilon,\varepsilon]^n\big) +_0 \lambda\cdot\big(L+[-\varepsilon,\varepsilon]^n\big)\Big) + \big(-\frac{1}{2},\frac{1}{2}\big)^n\Big)}{r^n} \\
&= \mathrm{vol}\Big((1-\lambda)\cdot\big(K+[-\varepsilon,\varepsilon]^n\big) +_0 \lambda\cdot\big(L+[-\varepsilon,\varepsilon]^n\big)\Big),
\end{aligned} \tag{3.4}$$

which is valid for all $\varepsilon > 0$. On the other hand, one clearly has that

$$h_{K+[-\varepsilon,\varepsilon]^n}(u)^{1-\lambda} h_{L+[-\varepsilon,\varepsilon]^n}(u)^{\lambda} \le \big(h_K(u)+\sqrt{n}\varepsilon\big)^{1-\lambda}\big(h_L(u)+\sqrt{n}\varepsilon\big)^{\lambda}$$

for any $\varepsilon > 0$ (for K and L fixed). Furthermore, since each sequence of functions $\big(h_K+\sqrt{n}\varepsilon\big)_\varepsilon$ and $\big(h_L+\sqrt{n}\varepsilon\big)_\varepsilon$ converges uniformly to h_K and h_L, respectively, and the function $(x,y)\mapsto x^{1-\lambda}y^{\lambda}$ is uniformly continuous in any closed rectangle $[0,a]\times[0,b]$, we get that $\big((h_K+\sqrt{n}\varepsilon)^{1-\lambda}(h_L+\sqrt{n}\varepsilon)^{\lambda}\big)_\varepsilon$, and so also $\big(h_{K+[-\varepsilon,\varepsilon]^n}^{1-\lambda} h_{L+[-\varepsilon,\varepsilon]^n}^{\lambda}\big)_\varepsilon$, converges uniformly to $h_K^{1-\lambda}h_L^{\lambda}$. Then [31, Lemma 7.5.2] ensures that the sequence of Wulff shapes associated to the functions $h_{K+[-\varepsilon,\varepsilon]^n}^{1-\lambda} h_{L+[-\varepsilon,\varepsilon]^n}^{\lambda}$, namely,

$$(1-\lambda)\cdot\big(K+[-\varepsilon,\varepsilon]^n\big) +_0 \lambda\cdot\big(L+[-\varepsilon,\varepsilon]^n\big),$$

converges to $(1-\lambda)\cdot K +_0 \lambda\cdot L$ in the Hausdorff metric. Finally, the continuity of the volume yields

$$\lim_{\varepsilon\to 0} \mathrm{vol}\Big((1-\lambda)\cdot\big(K+[-\varepsilon,\varepsilon]^n\big) +_0 \lambda\cdot\big(L+[-\varepsilon,\varepsilon]^n\big)\Big) = \mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big).$$

This, together with (3.4), shows (1.3) and concludes the proof. $\square$

Remark 3.1. *We note that the cube C_n cannot be removed in the left-hand side of* (2.1)*, not even summing up a bigger cube instead of $(-1/2,1/2)^n$; i.e., an inequality of the form*

$$\mathrm{G}_n\Big(\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) + (-\beta,\beta)^n\Big) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda} \tag{3.5}$$

for all $K,L\in\mathcal{K}^n$ does not hold for any constant $\beta>0$. Indeed, it suffices to consider the sets $K=[-a,a]$ and $L=[-b,b]$ in $\mathbb{R}$, where $0<b<1/2$ and $a\in\mathbb{N}$ is large enough in order for the inequality

$$a^{1-\lambda}\left(\left(1+\frac{1}{2a}\right)^{1-\lambda}\frac{1}{2^{\lambda}} - b^{\lambda}\right) > \beta + \frac{1}{2}$$

to hold. Then, the above expression rewrites as

$$2a^{1-\lambda}b^{\lambda} + 2\beta + 1 < (2a+1)^{1-\lambda}, \tag{3.6}$$

and since $\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) + (-\beta,\beta) = (-a^{1-\lambda}b^{\lambda}-\beta, a^{1-\lambda}b^{\lambda}+\beta)$, we have

$$\mathrm{G}_1\Big(\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) + (-\beta,\beta)\Big) \leq 2\left(a^{1-\lambda}b^{\lambda}+\beta\right)+1.$$

Furthermore, $\mathrm{G}_1(K)=2a+1$ and $\mathrm{G}_1(L)\geq 1$ and, consequently, (3.6) *contradicts* (3.5)*, as desired.*

Finally, we see that the Minkowski addition of $(-1/2,1/2)^n$ is also necessary i.e., an inequality of the form

$$\mathrm{G}_n\Big((1-\lambda)\cdot(K+C_n) +_0 \lambda\cdot(L+C_n) + (-\beta,\beta)^n\Big) \geq \mathrm{G}_n(K)^{1-\lambda}\mathrm{G}_n(L)^{\lambda} \tag{3.7}$$

does not hold, in general, if $0\leq\beta<1/2$. To show it, we consider the sets $K=[-a,a]$ and $L=[-b,b]$ in $\mathbb{R}$, for fixed $a,b>0$. Then, it is clear that

$$\begin{aligned}\mathrm{G}_1\Big((1-\lambda)\cdot(K+C_1) +_0 \lambda\cdot(L+C_1) + (-\beta,\beta)\Big)&\\ \leq 2\left\lfloor\left(a+\frac{1}{2}\right)^{1-\lambda}\left(b+\frac{1}{2}\right)^{\lambda}+\beta\right\rfloor + 1.&\end{aligned}$$

Note that, if $\beta<1/2$, we may choose $0<\lambda<1$ such that $(2b+1)^{\lambda}<2(1-\beta)$, because $2(1-\beta)>1$ and $\lim_{\lambda\to 0^+}(2b+1)^{\lambda}=1$. This condition is equivalent to $(1/2^{1-\lambda})(b+1/2)^{\lambda}+\beta<1$, and a simple continuity argument then shows that, for sufficiently small values of a, we also have $(a+1/2)^{1-\lambda}(b+1/2)^{\lambda}+\beta<1$. Consequently, in this case we have

$$\mathrm{G}_1\Big((1-\lambda)\cdot(K+C_1) +_0 \lambda\cdot(L+C_1) + (-\beta,\beta)\Big) = 1,$$

which contradicts (3.7) *when $b>1$ because, in that case, $\mathrm{G}_1(K)\geq 1$ and $\mathrm{G}_1(L)>1$.*

Following the same argument as the one in the proof of Theorem 2.1, but now using (2.3), one can get the L_p version of that theorem when $0<p<1$, i.e., a discrete version of Marsiglietti's result. Again, neither C_n nor $(-1/2,1/2)^n$ can be removed from the inequality.

Theorem 3.1. *Let $K,L\subset\mathbb{R}^n$ be two unconditional convex bodies and let $\lambda\in(0,1)$. Then, for any $0<p<1$,*

$$\mathrm{G}_n\left((1-\lambda)\cdot\bigl(K+C_n\bigr)\oplus_p\lambda\cdot\bigl(L+C_n\bigr)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)\geq\mathcal{M}^{\lambda}_{\frac{p}{n}}\bigl(\mathrm{G}_n(K),\mathrm{G}_n(L)\bigr).$$

Furthermore, it implies the L_p Brunn-Minkowski inequality (2.3) *for unconditional convex bodies.*

Next we will deal with the point enumerator $\mathrm{G}_{\Lambda_a}(\cdot)$ (and $\mathrm{G}_{\Gamma_p}(\cdot)$). In order to prove Proposition 2.1 (and the corollary afterwards) we need the following simple properties of the functions ψ_a and φ_p, which will be useful throughout the rest of the manuscript.

Lemma 3.1. *Let $K,L\subset\mathbb{R}^n_{\geq 1}$ be non-empty bounded sets and let $0<\lambda<1$. Then*

i) $\mathrm{G}_{\Lambda_a}(K)=\bigl|\psi_a^{-1}(K)\cap\mathbb{Z}^n\bigr|$ *and*
ii) $\psi_a^{-1}\bigl(K^{1-\lambda}L^{\lambda}\bigr)=(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)$.

Furthermore, if $K,L\subset\mathbb{R}^n_{\geq 0}$ then, for any $0<p<1$,

iii) $\mathrm{G}_{\Gamma_p}(K)=\bigl|\varphi_p^{-1}(K)\cap\mathbb{Z}^n\bigr|$ *and*
iv) $\varphi_p^{-1}\bigl((1-\lambda)\cdot K\oplus_p\lambda\cdot L\bigr)=(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)$.

Proof. On the one hand

$$\mathrm{G}_{\Lambda_a}(K)=|K\cap\Lambda_a|=\left|K\cap\psi_a\left(\mathbb{Z}^n_{\geq 0}\right)\right|=\bigl|\psi_a^{-1}(K)\cap\mathbb{Z}^n_{\geq 0}\bigr|=\bigl|\psi_a^{-1}(K)\cap\mathbb{Z}^n\bigr|.$$

On the other hand

$$\begin{aligned}\psi_a^{-1}\bigl(x^{1-\lambda}y^{\lambda}\bigr)_i&=\log_a\bigl(x_i^{1-\lambda}y_i^{\lambda}\bigr)=(1-\lambda)\log_a x_i+\lambda\log_a y_i\\&=(1-\lambda)\psi_a^{-1}(x)_i+\lambda\psi_a^{-1}(y)_i\end{aligned}$$

for all $x\in K,y\in L$ and all $i=1,\dots,n$.

Completely analogous arguments yield properties iii) and iv). $\square$

Now we can prove Proposition 2.1.

Proof of Proposition 2.1. We observe that the cube $(-1,1)^n$ in inequality (1.4) can be replaced by $(0,2)^n$ due to the invariance by integer translations, and so, we may apply it to the sets $\psi_a^{-1}(K)$ and $\psi_a^{-1}(L)$ to obtain

$$\begin{aligned}&\Bigl|\bigl[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(0,2)^n\bigr]\cap\mathbb{Z}^n\Bigr|^{1/n}\\&\qquad\geq(1-\lambda)\bigl|\psi_a^{-1}(K)\cap\mathbb{Z}^n\bigr|^{1/n}+\lambda\bigl|\psi_a^{-1}(L)\cap\mathbb{Z}^n\bigr|^{1/n}.\end{aligned}$$

Using both items i) and ii) of Lemma 3.1, and taking into account that $\psi_a\bigl((0,2)^n\bigr)=(1,a^2)^n$, we get (2.4).

To see that equality may be attained, we consider $K=L=[1,a^m]^n$ for any $m\in\mathbb{N}$, for which one has $(1,a^2)^nK^{1-\lambda}L^{\lambda}=(1,a^{m+2})^n$ and

$$\mathrm{G}_{\Lambda_a}\bigl([1,a^m]^n\bigr)=\mathrm{G}_{\Lambda_a}\bigl((1,a^{m+2})^n\bigr)=(m+1)^n. \qquad\square$$

Proposition 2.1 can be easily adapted to the $0 < p < 1$ setting, now using items iii) and iv) of Lemma 3.1:

Corollary 3.1. *Let $0 < p < 1$ and $\lambda \in (0,1)$, and let $K, L \subset \mathbb{R}^n_{\geq 0}$ be non-empty bounded sets with $\mathrm{G}_{\Gamma_p}(K)\mathrm{G}_{\Gamma_p}(L) > 0$. Then*

$$\mathrm{G}_{\Gamma_p}\Big((1-\lambda)\cdot K \oplus_p \lambda \cdot L \oplus_p \big(0, 2^{1/p}\big)^n\Big)^{1/n} \geq (1-\lambda)\mathrm{G}_{\Gamma_p}(K)^{1/n} + \lambda \mathrm{G}_{\Gamma_p}(L)^{1/n}.$$

4. A log-Brunn-Minkowski type inequality for an alternative discrete measure

In order to show Theorem 2.2, we need a discrete analogue of the Prékopa-Leindler inequality, a more general version of which was proved in [21]. In order to state it we need further notation.

In line with [21], for any real function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ we consider the extension given by

$$\phi^\diamond(z) = \sup_{u \in (-1,1)^n} \phi(z-u).$$

This extension coincides with the *Asplund product* $\star$ (also known as the *sup-convolution*) of ϕ and the characteristic function $\chi_{(-1,1)^n}$, since

$$\begin{aligned}\phi^\diamond(z) &= \sup_{u \in (-1,1)^n} \phi(z-u) = \sup_{u \in \mathbb{R}^n} \phi(z-u)\, \chi_{(-1,1)^n}(u) \\ &= \sup_{u_1+u_2=z} \phi(u_1)\, \chi_{(-1,1)^n}(u_2) = \big(\phi \star \chi_{(-1,1)^n}\big)(z).\end{aligned}$$

For more information on the Asplund product we refer the reader to [31, Section 9.5].

In [21, Theorem 2.2] the following discrete Borell-Brascamp-Lieb type inequality was proved. It will be exploited in the proofs of Theorems 2.2 and 4.1:

Theorem A. *Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, $\lambda \in (0,1)$ and $-1/n \leq \alpha \leq \infty$. If $f, g, h : \mathbb{R}^n \longrightarrow \mathbb{R}^n_{\geq 0}$ are such that*

$$h\big((1-\lambda)x + \lambda y\big) \geq \mathcal{M}^\lambda_\alpha\big(f(x), g(y)\big)$$

for all $x \in K$ and $y \in L$, then

$$\sum_{z \in M \cap \mathbb{Z}^n} h^\diamond(z) \geq \mathcal{M}^{\lambda}_{\frac{\alpha}{n\alpha+1}}\left(\sum_{x \in K \cap \mathbb{Z}^n} f(x), \sum_{y \in L \cap \mathbb{Z}^n} g(y)\right),$$

where $M = (1-\lambda)K + \lambda L + (-1,1)^n$.

The case $p = 0$ of the above theorem yields a discrete version of the classical Prékopa-Leindler inequality. With this tool we can prove our result for the measure μ_a.

Proof of Theorem 2.2. To begin with, we clearly have

$$\begin{aligned} a^n\mu_a\left(\big(a^{-1},a\big)^n K^{1-\lambda}L^\lambda\right) &= a^n \sum_{w\in[(a^{-1},a)^n K^{1-\lambda}L^\lambda]\cap\Lambda_a} \phi(w) \\ &\geq \sum_{w\in[(a^{-1},a)^n K^{1-\lambda}L^\lambda]\cap\Lambda_a} \sup_{v\in(a^{-1},a)^n} \phi(v)\phi(w). \end{aligned}$$

Applying the change of variables defined by ψ_a, and using the fact that $\psi_a^{-1}\big((a^{-1},a)^n\big)=(-1,1)^n$ and the symmetry of $(-1,1)^n$, the above expression rewrites into

$$\begin{aligned} &\sum_{z\in\psi_a^{-1}((a^{-1},a)^n K^{1-\lambda}L^\lambda)\cap\mathbb{Z}^n} \sup_{u\in(-1,1)^n} a^{\sum_{i=1}^n u_i} a^{\sum_{i=1}^n z_i} \\ &\qquad = \sum_{z\in\psi_a^{-1}((a^{-1},a)^n K^{1-\lambda}L^\lambda)\cap\mathbb{Z}^n} \sup_{u\in(-1,1)^n} a^{\sum_{i=1}^n u_i+z_i}. \end{aligned}$$

Next, we denote by $h(z)=a^{\sum_{i=1}^n z_i}$, and use Lemma 3.1 to get that the last sum equals to

$$\begin{aligned} \sum_{z\in\psi_a^{-1}((a^{-1},a)^n K^{1-\lambda}L^\lambda)\cap\mathbb{Z}^n} \sup_{u\in(-1,1)^n} h(u+z) &= \sum_{z\in\psi_a^{-1}((a^{-1},a)^n K^{1-\lambda}L^\lambda)\cap\mathbb{Z}^n} h^\diamond(z) \\ &= \sum_{z\in[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(-1,1)^n]\cap\mathbb{Z}^n} h^\diamond(z). \end{aligned}$$

Now, if we consider the functions $f=g=h$, it is straightforward to verify that they are under the conditions of the discrete Prékopa-Leindler inequality (Theorem A for $p=0$), that is, $h\big((1-\lambda)x+\lambda y\big)\geq f(x)^{1-\lambda}g(y)^\lambda$ for all $x\in\psi_a^{-1}(K)$ and $y\in\psi_a^{-1}(L)$, which yields

$$\begin{aligned} &\sum_{z\in[(1-\lambda)\psi_a^{-1}(K)+\lambda\psi_a^{-1}(L)+(-1,1)^n]\cap\mathbb{Z}^n} h^\diamond(z) \\ &\qquad \geq \left(\sum_{x\in\psi_a^{-1}(K)\cap\mathbb{Z}^n} f(x)\right)^{1-\lambda} \left(\sum_{y\in\psi_a^{-1}(L)\cap\mathbb{Z}^n} g(x)\right)^{\lambda}. \end{aligned}$$

Finally, performing the change of variables to f and g similarly to how we did it for h, and putting it all together, we can conclude the result:

$$\begin{aligned} a^n\mu_a\left(\big(a^{-1},a\big)^n K^{1-\lambda}L^\lambda\right) &\geq \left(\sum_{x\in K\cap\Lambda_a}\phi(x)\right)^{1-\lambda}\left(\sum_{y\in L\cap\Lambda_a}\phi(y)\right)^{\lambda} \\ &= \mu_a(K)^{1-\lambda}\mu_a(L)^\lambda. \end{aligned}$$

□

In order to extend the previous result to the $0 < p < 1$ setting, we need to consider the density function $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ given by

$$\phi(x) = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p},$$

and the measure ν_p defined as

$$\nu_p(A) = \sum_{z\in A\cap\Gamma_p} \phi(z),$$

for any non-empty bounded set $A \subset \mathbb{R}^n$.

Additionally, since φ_p can only be defined for points with non-negative coordinates, the definition of $\phi^\diamond$ must be adapted to

$$\phi^\diamond(z) = \sup_{u\in(0,2)^n} \phi(z-u),$$

which, due to the invariance by integer translations of the standard lattice point enumerator, still allows one to apply Theorem A.

With these ingredients, and using a similar argument to the one employed in the proof of Theorem 2.2, we can show the following result.

Theorem 4.1. *Let $K, L \subset \mathbb{R}^n_{\geq 0}$ be non-empty bounded sets and let $\lambda \in (0,1)$. Then, for any $0 < p < 1$,*

$$\nu_p\Big((1-\lambda)\cdot K \oplus_p \lambda\cdot L \oplus_p \left(0, 4^{1/p}\right)^n\Big) \geq \mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\nu_p(K), \nu_p(L)\big).$$

Proof. By definition we have

$$\begin{aligned}\nu_p\Big((1-\lambda)\cdot K &\oplus_p \lambda\cdot L \oplus_p \left(0, 4^{1/p}\right)^n\Big)\\ &= \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,4^{1/p})^n\right]\cap\Gamma_p} \left(\sum_{i=1}^{n} z_i^p\right)^{1/p},\end{aligned}$$

and since clearly

$$\left(0, 2^{1/p}\right)^n \oplus_p \left\{\left(2^{1/p},\dots,2^{1/p}\right)\right\} \subset \left(0, 4^{1/p}\right)^n,$$

the above expression can be bounded by

$$\begin{aligned}\nu_p\Big((1-\lambda)\cdot K &\oplus_p \lambda\cdot L \oplus_p \left(0, 4^{1/p}\right)^n\Big)\\ &\geq \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right]\cap\Gamma_p} \left(\sum_{i=1}^{n} z_i^p + 2\right)^{1/p}\\ &\geq \sum_{z\in\left[(1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right]\cap\Gamma_p} \sup_{u\in(0,2^{1/p})^n} \left(\sum_{i=1}^{n} z_i^p + u_i^p\right)^{1/p}.\end{aligned}$$

Next, applying the change of variables defined by φ_p, and denoting by $h(z) = \left(\sum_{i=1}^n z_i\right)^{1/p}$, the last term rewrites into

$$\begin{aligned}
&\sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n} \sup_{u\in(0,2)^n}\left(\sum_{i=1}^n z_i+u_i\right)^{1/p}\\
&\quad= \sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n} \sup_{u\in(0,2)^n} h(z+u)\\
&\quad= \sum_{z\in\left[\varphi_p^{-1}\left((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p(0,2^{1/p})^n\right)\right]\cap\mathbb{Z}^n} h^\diamond(z)\\
&\quad= \sum_{z\in\left[(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)+(0,2)^n\right]\cap\mathbb{Z}^n} h^\diamond(z),
\end{aligned}$$

where the last identity arises from Lemma 3.1 iv).

Now, if we consider the functions $f = g = h$, it is immediate that the condition $h\big((1-\lambda)x+\lambda y\big) = \mathcal{M}_p^\lambda\big(f(x), g(y)\big)$ holds, and thus, Theorem A yields

$$\begin{aligned}
&\sum_{z\in\left[(1-\lambda)\varphi_p^{-1}(K)+\lambda\varphi_p^{-1}(L)+(0,2)^n\right]\cap\mathbb{Z}^n} h^\diamond(z)\\
&\qquad\geq \mathcal{M}^\lambda_{\frac{p}{np+1}}\left(\sum_{x\in\varphi_p^{-1}(K)\cap\mathbb{Z}^n} f(x), \sum_{y\in\varphi_p^{-1}(L)\cap\mathbb{Z}^n} g(y)\right).
\end{aligned}$$

Finally, performing the change of variables defined by φ_p, it is easy to check that the above expression is equal to

$$\mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\nu_p(K), \nu_p(L)\big),$$

and so we can conclude that

$$\nu_p\Big((1-\lambda)\cdot K\oplus_p\lambda\cdot L\oplus_p\left(0,4^{1/p}\right)^n\Big) \geq \mathcal{M}^\lambda_{\frac{p}{np+1}}\big(\nu_p(K), \nu_p(L)\big),$$

as desired. □

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