Consequences and extensions of the Brunn-Minkowski theorem

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Abstract In this work we study some extensions and consequences of the fundamental Brunn-Minkowski inequality, using two different approaches: on one hand we deal with the so-called *Grünbaum inequality*, a beautiful consequence of the Brunn-Minkowski theorem which asserts, roughly speaking, that any hyperplane passing through the centroid divides any compact convex set into two not too small parts; on the other hand we study discrete versions of the Brunn-Minkowski inequality for the *lattice point enumerator*, this is, the functional counting how many points with integer coordinates are contained in a bounded set.

1 Introduction

As usual, we write $\mathbb{R}^n$ to represent the n-dimensional Euclidean space, endowed with the (Euclidean) inner product $\langle\cdot,\cdot\rangle$. One of the cornerstones of convex geometry is the Brunn-Minkowski inequality, which, in its classical form, provides a relation between the notions of Minkowski addition (of compact sets) and volume:

Theorem 1 *Let $K, L \subset \mathbb{R}^n$ be non-empty compact sets. Then, for all $\lambda \in (0,1)$,*

$$\mathrm{vol}\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{vol}(K)^{1/n} + \lambda \mathrm{vol}(L)^{1/n}, \tag{1}$$

with equality for some $\lambda \in (0,1)$, when $\mathrm{vol}(K)\mathrm{vol}(L) > 0$, if and only if K and L are homothetic compact convex sets.

Here $\mathrm{vol}(\cdot)$ denotes the n-dimensional Lebesgue measure and $+$ is used for the *Minkowski addition*, i.e., $A + B = \{a + b : a \in A, b \in B\}$ for any non-empty sets $A, B \subset \mathbb{R}^n$. Moreover, λA represents the set $\{\lambda a : a \in A\}$, for $\lambda \geq 0$.

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Despite its apparent simplicity, the Brunn-Minkowski inequality is one of the most powerful results in Convex Geometry and beyond: for instance, its equivalent analytic version (the Prékopa-Leindler inequality, see e.g. [12, Theorem 8.14]) and the fact that the compactness assumption can be weakened to Lebesgue measurability (see [21]), have allowed it to move to much wider fields. It implies very important inequalities such as the isoperimetric and Urysohn inequalities (see e.g. [32, page 382]), and it has been the starting point for new developments like the L_p-Brunn-Minkowski theory (see e.g. [22, 23]), or a reverse Brunn-Minkowski inequality (see e.g. [28]), among many others. It would not be possible to collect here all references regarding equivalent versions, applications and/or generalizations of the Brunn-Minkowski inequality. For extensive and beautiful surveys on them we refer to [2, 7].

The classical *Brunn concavity principle* (see e.g. [24, Theorem 12.2.1]) is one of the above mentioned equivalent versions of the Brunn-Minkowski inequality. It asserts that, for any non-empty compact and convex set $K \subset \mathbb{R}^n$ and a hyperplane H, the cross-sections volume function $f : H^{\perp} \longrightarrow \mathbb{R}_{\geq 0}$ defined by

$$f(x) = \mathrm{vol}_{n-1}\bigl(K \cap (x + H)\bigr)^{1/(n-1)}$$

is concave; here $H^{\perp}$ represents the orthogonal complement of H. Moreover, in the following we will denote by $M|H$ the orthogonal projection of a subset $M \subset \mathbb{R}^n$ onto H.

This result is the key fact in the classical proof of a celebrated theorem by Grünbaum [13]. In order to state it we need further notation: for any compact set $K \subset \mathbb{R}^n$ with non-empty interior, we write $\mathrm{g}(K)$ to represent its *centroid*, i.e., the affine-covariant point

$$\mathrm{g}(K) := \frac{1}{\mathrm{vol}(K)} \int_K x \,\mathrm{d}x.$$

Moreover, given $u \in \mathbb{S}^{n-1}$, we write $H_u := \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle = 0\bigr\}$ and $H_u^- := \bigl\{x \in \mathbb{R}^n : \langle x, u\rangle \leq 0\bigr\}$ to denote the (vector) hyperplane orthogonal to u and the corresponding closed halfspace with u as outer normal unit vector. Finally, we will say that K is a cone in the direction u if K is the convex hull of $\{x\} \cup \bigl(K \cap (y + H_u)\bigr)$, for some $x, y \in \mathbb{R}^n$.

Theorem 2 (Grünbaum)

Let $K \subset \mathbb{R}^n$ be a compact convex set, with non-empty interior, having its centroid at the origin. Then

$$\frac{\mathrm{vol}(K \cap H_u^-)}{\mathrm{vol}(K)} \geq \left(\frac{n}{n+1}\right)^n \tag{2}$$

for all $u \in \mathbb{S}^{n-1}$. Equality holds, for some $u \in \mathbb{S}^{n-1}$, if and only if K is a cone in the direction u.

In the last years Grünbaum's result has been extended to the case of sections [6, 29] and projections [34] of compact convex sets, and has been even generalized to the analytic setting of log-concave functions [27] and p-concave functions for

$p > 0$ [29] (we refer the reader to [7] for more information on log-concave and p-concave functions). Moreover, it has been also extended to the case of compact sets with a p-concave cross-sections volume function [25], for $p \geq 0$.

The original proof of Theorem 2 relies on exploiting the Brunn concavity principle to compare both the volume of the compact convex set K and of $K \cap H_u^-$ with those of a suitable cone C in the direction $u \in \mathbb{S}^{n-1}$ and $C \cap H_u^-$, respectively. In this paper we show, on the one hand, how one can derive Grünbaum's result as a direct application of the Brunn-Minkowski theorem (Theorem 1). Furthermore, the characterization of the equality given in Theorem 2 now will follow from the equality case of Theorem 1.

On the other hand, we devote this work to exploring discrete versions of the Brunn-Minkowski inequality. Nowadays there is a growing interest for studying discrete analogues of classical (continuous) results, which can be carried out from two points of view: either considering finite subsets $A, B \subset \mathbb{Z}^n$ of integer points and measuring with the cardinality $|\cdot|$, or working with compact sets $K, L \subset \mathbb{R}^n$ and using the so-called *lattice point enumerator* as measure, this is, $\mathrm{G}_n(K) = |K \cap \mathbb{Z}^n|$.

Regarding the cardinality, and besides the simple and classical inequality

$$|A + B| \geq |A| + |B| - 1 \tag{3}$$

for finite $A, B \subset \mathbb{Z}^n$, Gardner and Gronchi obtained in [8] a beautiful and powerful discrete Brunn-Minkowski inequality: they proved that if A, B are finite subsets of the integer lattice $\mathbb{Z}^n$, with dimension $\dim B = n$, then

$$|A + B| \geq \left|D^B_{|A|} + D^B_{|B|}\right|.$$

Here, for any $m \in \mathbb{N}$, D^B_m is a *B-initial segment*, i.e., the set of the first m points of $\mathbb{Z}^n_{\geq 0} = \left\{x \in \mathbb{Z}^n : x_i \geq 0\right\}$ in the so-called "B-order", which is a particular order defined on $\mathbb{Z}^n_{\geq 0}$ depending only on the cardinality of B. They also derive some inequalities that improve previous results obtained by Ruzsa in [30, 31]. For a proper definition and a deep study of it we refer the reader to [8].

Recently [9, 15, 20], different discrete analogues of the Brunn-Minkowski inequality have been obtained for the cardinality, including the case of its classical form (1): in [15] it is shown that if $A, B \subset \mathbb{Z}^n$ are non-empty finite sets, then

$$\left|\bar{A} + B\right|^{1/n} \geq |A|^{1/n} + |B|^{1/n},$$

where $\bar{A}$ is a suitably defined extension of A not depending on B.

In this paper we will focus on investigating discrete Brunn-Minkowski type inequalities for the lattice point enumerator, and will present the more recent advances in this respect.

2 Deriving Grünbaum's inequality as a consequence of the Brunn-Minkowski theorem

Before showing Theorem 2 we need to introduce some notation. Given a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior, and a vector $u \in \mathbb{S}^{n-1}$, we denote by $K_u(t) = K \cap (tu + H_u)$ and by $K_u^-(t) = K \cap (tu + H_u^-)$, for any $t \in \mathbb{R}$. Furthermore, we observe that if K has centroid at the origin then, using Fubini's theorem, we get

$$0 = \int_K \langle x, u \rangle \, \mathrm{d}x = \int_a^b t \mathrm{vol}_{n-1}\bigl(K_u(t)\bigr) \, \mathrm{d}t, \tag{4}$$

where $a, b \in \mathbb{R}$ are such that $K|H_u^{\perp} = [au, bu]$ (here, as usual, by $[x, y]$ we denote the segment with endpoints $x, y \in \mathbb{R}^n$).

Now we are in a position to prove Theorem 2. We will follow here the approach used in [26] to derive the functional version of Grünbaum's inequality.

Proof (of Theorem 2) Let $u \in \mathbb{S}^{n-1}$ be fixed and assume that $K|H_u^{\perp} = [au, bu]$ for some $a, b \in \mathbb{R}$ with $a < b$. First we observe that since K is a compact convex set with interior points we have that $\mathrm{vol}_{n-1}\bigl(K_u(t)\bigr) > 0$ for all $t \in (a, b)$ and so the condition (4) yields $a < 0 < b$. In particular we have $\mathrm{vol}_{n-1}\bigl(K_u(0)\bigr) > 0$.

On the one hand, from the convexity of K we get

$$K_u^-\bigl((1-\lambda)t_1 + \lambda t_2\bigr) \supset (1-\lambda)K_u^-(t_1) + \lambda K_u^-(t_2)$$

for all $t_1, t_2 \in [a, b]$ and all $\lambda \in [0, 1]$. Then the Brunn-Minkowski inequality (1) applied to the equation above implies that $\mathrm{vol}\bigl(K_u^-(\cdot)\bigr)^{1/n}$ is a concave function on $[a, b]$, and further we have $\mathrm{vol}\bigl(K_u^-(t)\bigr) = 0$ for all $t \leq a$ and $\mathrm{vol}\bigl(K_u^-(t)\bigr) = \mathrm{vol}(K)$ for all $t \geq b$.

On the other hand, since $\mathrm{vol}_{n-1}\bigl(K_u(\cdot)\bigr)$ is continuous in (a, b) (due to the fact that every concave function is continuous in the interior of its domain and $\mathrm{vol}_{n-1}\bigl(K_u(\cdot)\bigr)^{1/(n-1)}$ is so), from the fundamental theorem of calculus and Fubini's theorem we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}\bigl(K_u^-(t)\bigr) = \mathrm{vol}_{n-1}\bigl(K_u(t)\bigr) \tag{5}$$

for all $t \in (a, b)$. Thus $\mathrm{vol}\bigl(K_u^-(\cdot)\bigr)^{1/n}$ is concave and differentiable on (a, b), and then its tangent at $t = 0$, which is given by the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$h(t) = \frac{1}{n}\mathrm{vol}\bigl(K_u^-(0)\bigr)^{1/n}(mt + n)$$

for

$$m = \frac{\mathrm{vol}_{n-1}\bigl(K_u(0)\bigr)}{\mathrm{vol}\bigl(K_u^-(0)\bigr)} > 0,$$

lies above its graph. Then $0 \leq \operatorname{vol}\bigl(K_u^-(t)\bigr)^{1/n} \leq h(t)$ for all $t \in [a,b]$ and further, taking into account that h is negative on $(-\infty, -n/m)$ and $\operatorname{vol}\bigl(K_u^-(t)\bigr)^{1/n}$ is constant for all $t \geq b$, we have

$$\operatorname{vol}\bigl(K_u^-(t)\bigr)^{1/n} \leq h(t) \qquad \text{for all } t \in \left[-\frac{n}{m}, \infty\right). \tag{6}$$

Moreover, applying integration by parts (jointly with (5)) and using (4) we get

$$\int_a^b \operatorname{vol}\bigl(K_u^-(t)\bigr)\,\mathrm{d}t = b\operatorname{vol}(K) - \int_a^b t\operatorname{vol}_{n-1}\bigl(K_u(t)\bigr)\,\mathrm{d}t = b\operatorname{vol}(K). \tag{7}$$

Hence, noticing on one hand that $\operatorname{vol}\bigl(K_u^-(\cdot)\bigr)$ is strictly increasing on $[a,b]$ and that $\operatorname{vol}\bigl(K_u^-(t)\bigr) = \operatorname{vol}(K)$ for all $t \geq b$ on the other hand, by (7) and (6) we have

$$\begin{aligned}
b\operatorname{vol}(K) &= \int_a^b \operatorname{vol}\bigl(K_u^-(t)\bigr)\,\mathrm{d}t = \int_{-n/m}^b \operatorname{vol}\bigl(K_u^-(t)\bigr)\,\mathrm{d}t \\
&= \int_{-n/m}^{1/m} \operatorname{vol}\bigl(K_u^-(t)\bigr)\,\mathrm{d}t + \int_{1/m}^b \operatorname{vol}\bigl(K_u^-(t)\bigr)\,\mathrm{d}t \\
&\leq \int_{-n/m}^{1/m} h(t)^n\,\mathrm{d}t + \left(b - \frac{1}{m}\right)\operatorname{vol}(K) \\
&= \frac{\operatorname{vol}\bigl(K_u^-(0)\bigr)}{m}\left(\frac{n+1}{n}\right)^n + \left(b - \frac{1}{m}\right)\operatorname{vol}(K).
\end{aligned}$$

Therefore

$$\operatorname{vol}\bigl(K \cap H_u^-\bigr) = \operatorname{vol}\bigl(K_u^-(0)\bigr) \geq \left(\frac{n}{n+1}\right)^n \operatorname{vol}(K),$$

and so (2) follows. Furthermore, equality holds, for such a fixed vector $u \in \mathbb{S}^{n-1}$, if and only if

$$\operatorname{vol}\bigl(K_u^-(t)\bigr) = h(t)^n \tag{8}$$

for all $t \in [a,b]$, with $a = -n/m$ and $b = 1/m$.

First, we assume that the above conditions hold (for such $u \in \mathbb{S}^{n-1}$ fixed). Hence $\operatorname{vol}\bigl(K_u^-(\cdot)\bigr)^{1/n}$ is affine on $[a,b]$, which implies, from the equality case of the Brunn-Minkowski theorem (see Theorem 1), that $K_u^-(t_1)$ and $K_u^-(t_2)$ are homothetic for all $t_1, t_2 \in [a,b]$. Then, for every $t \in [a,b]$, we have $K_u^-(t) = r(t)K_u^-(b) + y_t = r(t)K + y_t$ for some $r(t) \geq 0$ and some $y_t \in \mathbb{R}^n$, from where we further get

$$K_u(t) = r(t)K_u(b) + y_t \tag{9}$$

for all $t \in [a,b]$. Moreover, for the suitable constants $A, B > 0$, we have

$$\begin{aligned}
\operatorname{vol}(K)r(t)^n = \operatorname{vol}\bigl(r(t)K + y_t\bigr) = \operatorname{vol}\bigl(K_u^-(t)\bigr) = h(t)^n = A(mt+n)^n \\
= B(t-a)^n,
\end{aligned}$$

where in the last equality we have used that $a = -n/m$. Thus $r(t) = C(t - a)$ for some $C > 0$ and since $r(b) = 1$ we get

$$r(t) = \frac{t-a}{b-a} \tag{10}$$

for all $t \in [a, b]$. Now, for every fixed $t \in [a, b]$, if we set $\lambda = (b-t)/(b-a) \in [0, 1]$ then $t = (1 - \lambda)b + \lambda a$ and so, from the convexity of K, we have

$$K_u(t) \supset \left(\frac{t-a}{b-a}\right) K_u(b) + \left(\frac{b-t}{b-a}\right) K_u(a). \tag{11}$$

Since $K_u(a) = r(a)K_u(b) + y_a = y_a$, taking volumes in (11) and using (9) and (10) we obtain that

$$\begin{aligned}\mathrm{vol}_{n-1}\big(K_u(t)\big) &\geq \mathrm{vol}_{n-1}\left[\left(\frac{t-a}{b-a}\right) K_u(b) + \left(\frac{b-t}{b-a}\right) K_u(a)\right] \\ &= \left(\frac{t-a}{b-a}\right)^{n-1} \mathrm{vol}_{n-1}\big(K_u(b)\big) = \mathrm{vol}_{n-1}\big(r(t)K_u(b)\big) \\ &= \mathrm{vol}_{n-1}\big(K_u(t)\big),\end{aligned}$$

and thus (11) holds with equality, for all $t \in [a, b]$. We conclude that K is the convex hull of $K_u(b)$ and the point $K_u(a)$, that is, K is a cone in the direction u.

Finally, if K is a cone in a direction $u \in \mathbb{S}^{n-1}$, then $K_u(t) = r(t)K_u(b) + y_t$ (cf. (11)), where r is given by (10) and y_t is the point $(1 - r(t))K_u(a)$. So, using the well-known formula for the volume of a cone, we get

$$\frac{\mathrm{vol}(K \cap H_u^-)}{\mathrm{vol}(K)} = \frac{-a\mathrm{vol}_{n-1}\big(K_u(0)\big)}{(b-a)\mathrm{vol}_{n-1}\big(K_u(b)\big)} = \left(\frac{-a}{b-a}\right)^n, \tag{12}$$

where in the last equality we have used that

$$\mathrm{vol}_{n-1}\big(K_u(0)\big) = r(0)^{n-1}\mathrm{vol}_{n-1}\big(K_u(b)\big) = \left(\frac{-a}{b-a}\right)^{n-1} \mathrm{vol}_{n-1}\big(K_u(b)\big).$$

Now, (4) implies that

$$0 = \int_a^b t\, r(t)^{n-1}\, \mathrm{d}t = (b-a)\frac{nb+a}{n(n+1)},$$

which is equivalent to $nb = -a$. Replacing the latter in (12) we conclude that (2) indeed holds with equality when K is a cone in the direction u. This finishes the proof. □

Given a non-negative measurable function with finite positive integral, its centroid is the point defined by

$$\mathrm{g}(f) := \frac{1}{\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x} \int_{\mathbb{R}^n} x f(x)\,\mathrm{d}x.$$

In [26] (see also the references therein) it is shown that one can obtain the functional analogue of Grünbaum's inequality (2) by exploiting the functional counterpart of the Brunn-Minkowski inequality, the so-called *Borell-Brascamp-Lieb inequality* (see [3] and [4]). More precisely, given a p-concave function $f : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$ for some $p \in [0, \infty]$ with compact support and centroid at the origin, and any hyperplane H, one has

$$\int_{H^-} f(x)\,\mathrm{d}x \geq \left(\frac{np+1}{(n+1)p+1}\right)^{(np+1)/p} \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x. \tag{13}$$

As usual, if $p = 0$ or $p = \infty$, the constant appearing in the right-hand side of the above inequality is the value that is obtained "by continuity", that is, the limit as $p \to 0^+$ or $p \to \infty$, respectively. We notice that Grünbaum's inequality (2) is then recovered from (13) by taking f the characteristic function of the n-dimensional compact convex set K with centroid at the origin, which is ∞-concave.

3 Discrete Brunn-Minkowski type inequalities for the lattice point enumerator

We note that the known discrete Brunn-Minkowski inequalities for the cardinality in its classical form involve the Minkowski addition of two finite subsets $A, B \subset \mathbb{Z}^n$, but not its *convex combination*. Indeed, if one aims to get a discrete analog of (1), one should observe the following: for any pair of non-empty finite sets $A, B \subset \mathbb{R}^n$, using (3) and the convexity of the function $t \mapsto t^n$ for $t \geq 0$, one gets

$$\begin{aligned}\big|(1-\lambda)A + \lambda B\big| &\geq \big|(1-\lambda)A\big| + |\lambda B| - 1 = |A| + |B| - 1\\ &= (1-\lambda)|A| + \lambda|B| + \lambda|A| + (1-\lambda)|B| - 1\\ &\geq (1-\lambda)|A| + \lambda|B| \geq \Big((1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}\Big)^n;\end{aligned}$$

this inequality is however meaningless from a geometric point of view, because while the quantities $|A|, |B|$ on the right-hand side are reduced by the factors $1-\lambda$ and λ, the sets $(1-\lambda)A$ and λB on the left-hand side have the same cardinality as A and B, respectively.

So, one needs to involve a way of "counting points" for which dilations affect, and a perfect candidate for this is the lattice point enumerator G_n (for compact subsets of $\mathbb{R}^n$). However, and as in the case of the cardinality, one cannot expect to obtain a discrete Brunn-Minkowski inequality in the classical form for the lattice point enumerator, namely, the relation

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

is in general not true. In fact, just taking $K = \{0\}$ and the cube $L = [0, m]^n$ with $m \in \mathbb{N}$ odd, then it is

$$\mathrm{G}_n\left(\frac{1}{2}K + \frac{1}{2}L\right)^{1/n} = \frac{m+1}{2} < \frac{m+2}{2} = \frac{1}{2}\mathrm{G}_n(K)^{1/n} + \frac{1}{2}\mathrm{G}_n(L)^{1/n}.$$

So, the question arises what is the "best" way to define a set M, for given compact sets $K, L \subset \mathbb{R}^n$, such that $(1-\lambda)K + \lambda L \subset M$ and

$$\mathrm{G}_n(M)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}$$

holds *for all* $\lambda \in (0, 1)$.

In [20] the authors answered this question by proving that if $K, L \subset \mathbb{R}^n$ are non-empty bounded sets and $\lambda \in (0, 1)$, then

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1, 1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n},$$

the inequality being sharp. Furthermore, the cube cannot be reduced in the latter inequality and it implies the classical Brunn-Minkowski inequality (1) for bounded convex sets.

The latter inequality was obtained as a direct consequence of a (more general) functional discrete inequality: indeed, the authors proved a discrete version of the Borell-Brascamp-Lieb inequality. Furthermore, they showed that their discrete Borell-Brascamp-Lieb type inequality implies the classical functional one (under mild assumptions on the functions there involved), which makes it a powerful result in the field.

Here we provide a new proof of the above Brunn-Minkowski type inequality for the lattice point enumerator, using a completely geometrical approach, and we show that it implies the continuous version in the more general case of Jordan measurable sets. More precisely, we show the following result.

Theorem 3 *Let* $K, L \subset \mathbb{R}^n$ *be non-empty bounded sets and let* $\lambda \in (0, 1)$. *Then*

$$\mathrm{G}_n\big((1-\lambda)K + \lambda L + (-1, 1)^n\big)^{1/n} \geq (1-\lambda)\mathrm{G}_n(K)^{1/n} + \lambda \mathrm{G}_n(L)^{1/n}. \tag{14}$$

Moreover, it implies the Brunn-Minkowski inequality (1) *for bounded Jordan measurable sets.*

Before showing this result, we need some additional notation and an auxiliary property: we will represent by B_0 the n-dimensional Euclidean open unit ball and we denote by $\mathrm{cl}\, M$ the closure of a set $M \subset \mathbb{R}^n$.

The proof of the theorem relies on the following relations between the volume and the lattice point enumerator of a non-empty bounded measurable set $A \subset \mathbb{R}^n$:

$$\begin{aligned} \mathrm{G}_n(A) &\leq \mathrm{vol}\left(A + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)^n\right), \\ \mathrm{vol}(A) &\leq \mathrm{G}_n\left(A + \left(-\tfrac{1}{2}, \tfrac{1}{2}\right)^n\right). \end{aligned} \tag{15}$$

The first inequality can be found in [10, (3.3)], whereas the second one is gathered in [11, p. 877] (see also [1]).

Moreover, if A is further Jordan measurable, it is a well-known fact that, roughly speaking, the volume and the lattice point enumerator are equivalent when A is large enough, i.e.,

$$\lim_{r\to\infty}\frac{\mathrm{G}_n(rA)}{r^n}=\mathrm{vol}(A) \tag{16}$$

(see e.g. [12, Formula (3), p.120]). Furthermore, the following property holds:

Lemma 1 *Let $A\subset\mathbb{R}^n$ be a non-empty bounded Jordan measurable set and let $M\subset\mathbb{R}^n$ be a non-empty bounded set containing the origin. Then*

$$\lim_{r\to\infty}\frac{\mathrm{G}_n(rA+M)}{r^n}=\mathrm{vol}(A). \tag{17}$$

Proof Given $m\in\mathbb{N}$, it follows that for any $r>0$ large enough one has $(1/r)M\subset(1/m)B_0$ and thus

$$\begin{aligned}\mathrm{vol}(A)=\lim_{r\to\infty}\frac{\mathrm{G}_n(rA)}{r^n}&\leq\liminf_{r\to\infty}\frac{\mathrm{G}_n(rA+M)}{r^n}\leq\limsup_{r\to\infty}\frac{\mathrm{G}_n(rA+M)}{r^n}\\&\leq\limsup_{r\to\infty}\frac{\mathrm{G}_n\Big(r\big(\mathrm{cl}\,A+\frac{1}{m}B_0\big)\Big)}{r^n}\leq\lim_{r\to\infty}\frac{\mathrm{G}_n\Big(r\big(F_m+\frac{2}{m}B_0\big)\Big)}{r^n},\end{aligned} \tag{18}$$

where F_m is some finite subset of $\mathrm{cl}\,A$ such that $\mathrm{cl}\,A\subset F_m+(1/m)B_0$ (which exists from the compactness of $\mathrm{cl}\,A$). Now, since $F_m+(2/m)B_0$ is a finite union of open balls and so it is Jordan measurable, we have, by (16), that

$$\lim_{r\to\infty}\frac{\mathrm{G}_n\Big(r\big(F_m+\frac{2}{m}B_0\big)\Big)}{r^n}=\mathrm{vol}\left(F_m+\frac{2}{m}B_0\right)\leq\mathrm{vol}\left(\mathrm{cl}\,A+\frac{2}{m}B_0\right). \tag{19}$$

Moreover, since $\mathrm{cl}\,A$ is compact, a standard straightforward computation shows that

$$\mathrm{cl}\,A=\bigcap_{m=1}^{\infty}\left(\mathrm{cl}\,A+\frac{2}{m}B_0\right).$$

Since the boundary of A has null Lebesgue measure (because A is Jordan measurable), the latter identity together with the fact that

$$\mathrm{vol}\left(\bigcap_{m=1}^{\infty}\left(\mathrm{cl}\,A+\frac{2}{m}B_0\right)\right)=\lim_{m\to\infty}\mathrm{vol}\left(\mathrm{cl}\,A+\frac{2}{m}B_0\right),$$

which holds because $\mathrm{cl}\,A+(2/m)B_0$ is a decreasing sequence (see e.g. [5, Proposition 1.2.5 (b)]), allows us to deduce that

$$\mathrm{vol}(A)=\mathrm{vol}(\mathrm{cl}\,A)=\lim_{m\to\infty}\mathrm{vol}\left(\mathrm{cl}\,A+\frac{2}{m}B_n\right).$$

Therefore, since m was arbitrary in (18) and (19), (17) holds. □

Now we are in a position to prove Theorem 3.

Proof (of Theorem 3) Noticing that $M+(-1/2,1/2)^n$ is open (and thus measurable) for any non-empty subset $M\subset\mathbb{R}^n$, from (1) and (15) we get

$$\begin{aligned}
\mathrm{G}_n\big((1-\lambda)K+\lambda L+(-1,1)^n\big)^{1/n} &\geq \mathrm{vol}\left((1-\lambda)K+\lambda L+\left(-\tfrac{1}{2},\tfrac{1}{2}\right)^n\right)^{1/n}\\
&= \mathrm{vol}\left((1-\lambda)\left[K+\left(-\tfrac{1}{2},\tfrac{1}{2}\right)^n\right]+\lambda\left[L+\left(-\tfrac{1}{2},\tfrac{1}{2}\right)^n\right]\right)^{1/n}\\
&\geq (1-\lambda)\mathrm{vol}\left(K+\left(-\tfrac{1}{2},\tfrac{1}{2}\right)^n\right)^{1/n}+\lambda\mathrm{vol}\left(L+\left(-\tfrac{1}{2},\tfrac{1}{2}\right)^n\right)^{1/n}\\
&\geq (1-\lambda)\mathrm{G}_n(K)^{1/n}+\lambda\mathrm{G}_n(L)^{1/n}.
\end{aligned}$$

In order to conclude the proof, we show that (14) implies (1) when K and L are non-empty bounded Jordan measurable sets. Then, using (14), (16) and Lemma 1 we get

$$\begin{aligned}
&(1-\lambda)\mathrm{vol}(K)^{1/n}+\lambda\mathrm{vol}(L)^{1/n}\\
&\qquad= (1-\lambda)\left(\lim_{r\to\infty}\frac{\mathrm{G}_n(rK)}{r^n}\right)^{1/n}+\lambda\left(\lim_{r\to\infty}\frac{\mathrm{G}_n(rL)}{r^n}\right)^{1/n}\\
&\qquad= \lim_{r\to\infty}\frac{(1-\lambda)\mathrm{G}_n(rK)^{1/n}+\lambda\mathrm{G}_n(rL)^{1/n}}{r}\\
&\qquad\leq \lim_{r\to\infty}\frac{\mathrm{G}_n\big((1-\lambda)(rK)+\lambda(rL)+(-1,1)^n\big)^{1/n}}{r}\\
&\qquad= \left(\lim_{r\to\infty}\frac{\mathrm{G}_n\Big(r\big((1-\lambda)K+\lambda L\big)+(-1,1)^n\Big)}{r^n}\right)^{1/n}\\
&\qquad= \mathrm{vol}\big((1-\lambda)K+\lambda L\big)^{1/n},
\end{aligned}$$

as desired. □

Other discrete analogues of the Brunn-Minkowski inequality for the lattice point enumerator can be found in [14, 19, 20, 33]. We conclude by highlighting the following nice result obtained by Halikias, Klartag and Slomka in [14]: for non-empty bounded sets $K,L\subset\mathbb{R}^n$ one has

$$\mathrm{G}_n\left(\frac{K+L}{2}+(-1,0]^n\right)\mathrm{G}_n\left(\frac{K+L}{2}+[0,1)^n\right)\geq \mathrm{G}_n(K)\mathrm{G}_n(L),$$

which yields the discrete multiplicative Brunn-Minkowski type inequality

$$\mathrm{G}_n\left(\frac{K+L}{2}+[0,1]^n\right) \geq \sqrt{\mathrm{G}_n(K)\mathrm{G}_n(L)}.$$

In this line, in [20] it is also shown that

$$\mathrm{G}_n\left(\frac{K+L}{2}+[0,1]^n\right)^{1/n} \geq \frac{\mathrm{G}_n(K)^{1/n}+\mathrm{G}_n(L)^{1/n}}{2}, \tag{20}$$

provided that K, L contain some integer point. More recently, some other discrete Brunn-Minkowski type inequalities have been considered. We emphasize some extensions of (14) to the L_p setting, both in the case of $p \geq 1$ [17] and for $p \in [0,1)$ [16].

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