ON DIFFERENTIABILITY OF QUERMASSINTEGRALS

M. A. HERNÁNDEZ CIFRE AND E. SAORÍN

Abstract. In this paper we study the problem of classifying the convex bodies in $\mathbb{R}^n$, depending on the differentiability of their associated quermassintegrals with respect to the one-parameter-depending family given by the inner and outer parallel bodies. This problem was originally posed by Hadwiger in the 3-dimensional space. We characterize one of the non-trivial classes and give necessary conditions for a convex body to belong to the others. We also consider particular families of convex bodies, e.g. polytopes and tangential bodies.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$. The subset of $\mathcal{K}^n$ consisting of all convex bodies with non-empty interior is denoted by $\mathcal{K}^n_0$. Let B_n be the n-dimensional unit ball, and $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$, its closure by $\mathrm{cl}\, M$ and its boundary by $\mathrm{bd}\, M$.

For two convex bodies $K \in \mathcal{K}^n$ and $E \in \mathcal{K}^n_0$ and a non-negative real number ρ the *outer parallel body of K (relative to E) at distance ρ* is the Minkowski sum $K + \rho E$. On the other hand, for $0 \leq \rho \leq \mathrm{r}(K;E)$ the *inner parallel body of K (relative to E) at distance ρ* is the set

$$K \sim \rho E = \{x \in \mathbb{R}^n : \rho E + x \subset K\},$$

where the *relative inradius* $\mathrm{r}(K;E)$ of K with respect to E is defined by $\mathrm{r}(K;E) = \sup\{r : \exists\, x \in \mathbb{R}^n \text{ with } x + r\,E \subset K\}$. When $E = B_n$, $\mathrm{r}(K;B_n) = \mathrm{r}(K)$ is the classical inradius (see [3, p. 59]). Clearly if $\rho = 0$ the original body K is obtained. Notice that $K \sim \mathrm{r}(K;E)E$ is the set of (relative) incenters of K, usually called *kernel* of K with respect to E and denoted by $\ker(K;E)$. The dimension of $\ker(K;E)$ is strictly less than n (see [3, p. 59]). The inner parallel bodies and their properties were studied mainly by Bol

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[2], Dinghas [4] (see also [7] and [8]) and later by Sangwine-Yager [13]. The *full system of (relative) parallel bodies* of K is defined by

$$K_\rho := \begin{cases} K \sim (-\rho)E & \text{for } -\mathrm{r}(K;E) \le \rho \le 0, \\ K + \rho E & \text{for } 0 \le \rho < \infty, \end{cases} \tag{1.1}$$

and it is a concave family, i.e., it satisfies

$$(1-\lambda)K_\rho + \lambda K_\sigma \subset K_{(1-\lambda)\rho+\lambda\sigma} \tag{1.2}$$

for $\lambda \in [0,1]$ and $\rho, \sigma \in \big[-\mathrm{r}(K;E), \infty\big)$ (see [16, p. 135]).

The so called *Minkowski-Steiner formula* (or *relative Steiner formula*) states that the volume of the outer parallel body $K + \rho E$ is a polynomial of degree n in ρ,

$$\mathrm{V}(K + \rho E) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K;E)\rho^i. \tag{1.3}$$

The coefficients $\mathrm{W}_i(K;E)$ are called the *relative quermassintegrals* of K, and they are just a special case of the more general *mixed volumes* for which we refer to [16, s. 5.1] and [6, s. 6.2, 6.3] (see Section 3). In particular, we have $\mathrm{W}_0(K;E) = \mathrm{V}(K)$ and $\mathrm{W}_n(K;E) = \mathrm{V}(E)$. If $E = B_n$, the polynomial in the right hand side of (1.3) becomes the classical *Steiner polynomial* [17]. For the study of the roots of the relative Steiner polynomial as well as related geometric polynomials see [1, 9, 10].

Analogous formulae give the value of the relative i-th quermassintegral of the outer parallel body $K + \rho E$, namely

$$\mathrm{W}_i(K + \rho E; E) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K;E)\rho^k, \tag{1.4}$$

for $\rho \ge 0$ and $i = 0, \dots, n$. There is however no general formula for the volume (quermassintegrals) of the inner parallel bodies of a body K (see [11] for a detailed study of this question).

From (1.2) and the general Brunn-Minkowski theorem for relative quermassintegrals, which states that the $(n-i)$-th root of the i-th relative quermassintegral W_i, $i = 0, \dots, n$, is a concave function (see e.g. [16, p. 339]), it is easy to see that

$${}'\mathrm{W}_i(\rho) \ge \mathrm{W}_i'(\rho) \ge (n-i)\mathrm{W}_{i+1}(\rho) \tag{1.5}$$

for $i = 0, \dots, n-1$, where ${}'\mathrm{W}_i$ and W_i' denote, respectively, the left and right derivatives of the function $\mathrm{W}_i(\rho) := \mathrm{W}_i(K_\rho; E)$. It is well known (see e.g. [2, 12]) that the volume is always differentiable and $\mathrm{V}'(\rho) = n\mathrm{W}_1(\rho)$. Moreover, if $\rho \ge 0$ then it is clear from (1.4) that all quermassintegrals are differentiable at ρ (notice that in the case $\rho = 0$ we speak about differentiability from the right) and $\mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho)$. The question arises for which convex bodies equalities hold in (1.5) for the full range $-\mathrm{r}(K;E) \le \rho < \infty$. With the previous notation we introduce the following definition.

Definition 1.1. *Let $E \in \mathcal{K}_0^n$. A convex body $K \in \mathcal{K}^n$* belongs to the class *$\mathcal{R}_p$, $0 \leq p \leq n-1$, if for all $0 \leq i \leq p$, and for $-\mathrm{r}(K;E) \leq \rho < \infty$ it holds*

$$ {}'\mathrm{W}_i(\rho) = \mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho). \tag{1.6}$$

Since $\mathrm{V}'(\rho) = n\mathrm{W}_1(\rho)$ the class $\mathcal{R}_0 = \mathcal{K}^n$ is the family of all convex bodies in $\mathbb{R}^n$. Moreover $\mathcal{R}_{i+1} \subset \mathcal{R}_i$, $i = 0, \dots, n-2$, and all these inclusions are strict, as follows from Theorem 1.3 and the fact that there exist i-tangential bodies of E which are not $(i+1)$-tangential bodies of E (see Section 4 for the definition).

The problem of determining the convex bodies belonging to the class $\mathcal{R}_p$ was originally posed and studied by Hadwiger [7] in the 3-dimensional case and for $E = B_n$. Here we consider the general n-dimensional problem. From now on $E \in \mathcal{K}_0^n$ will be a fixed convex body (with interior points). First we determine the convex bodies belonging to the smallest class, i.e., $\mathcal{R}_{n-1}$:

Theorem 1.1. *The only sets in $\mathcal{R}_{n-1}$ are the outer parallel bodies of k-dimensional convex bodies, for $0 \leq k \leq n-1$, i.e.,*

$$\mathcal{R}_{n-1} = \bigl\{K = L + \rho E : L \in \mathcal{K}^n,\ \dim L \leq n-1,\ \rho \geq 0\bigr\}.$$

For each of the remaining classes $\mathcal{R}_p$, $p = 1, \dots, n-2$, necessary conditions for a convex body to lie in it are stated in terms of the support function h of the so called relative form body of K_ρ, denoted by K_ρ^*, the mixed area measures and the set of p-extreme normal vectors of K_ρ, $\mathcal{U}_p(K_\rho)$; see Section 2 for precise definitions.

Theorem 1.2. *Let $K \in \mathcal{K}^n$ and let $E \in \mathcal{K}_0^n$ be a regular and strictly convex body. If $K \in \mathcal{R}_p \backslash \mathcal{R}_{n-1}$, $0 \leq p \leq n-2$, then for all $\rho \in \bigl(-\mathrm{r}(K;E), 0\bigr]$ the following holds:*

- i.a) *$h(K_\rho^*, u) = h(E,u)$ for all $u \in \operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1], E[i]; \cdot\bigr)$ and $i = 0, \dots, p$.*
- i.b) *$h(K_\rho^*, u) = h(E,u)$ for all $u \in \operatorname{cl}\mathcal{U}_p(K_\rho)$.*
- ii) *If $p \neq 0$, then $\mathrm{S}\bigl(K_\rho[n-i-1], E[i]; \cdot\bigr) = \mathrm{S}\bigl(K_\rho^*, K_\rho[n-i-1], E[i-1]; \cdot\bigr)$ for $i = 1, \dots, p$.*
- iii) *$\operatorname{supp}\mathrm{S}\bigl(K_\rho^*[n-1]; \cdot\bigr) \cup \Bigl(\bigcup_{i=0}^p \operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1], E[i]; \cdot\bigr)\Bigr) \subset \operatorname{cl}\mathcal{U}_0(K_\rho)$.*
- iv) *$\operatorname{cl}\mathcal{U}_0(K_\rho) = \operatorname{cl}\mathcal{U}_1(K_\rho) = \cdots = \operatorname{cl}\mathcal{U}_p(K_\rho)$.*

Moreover, we prove that all the above conditions are equivalent for any convex body $K \in \mathcal{K}_0^n$, see Lemma 2.2 in Section 2.

Theorem 1.1 and Theorem 1.2 are proved in Section 3. On the other hand, Theorem 1.2 allows to exclude convex sets from the classes $\mathcal{R}_p$; e.g., we prove that there are no polytopes lying in $\mathcal{R}_p$, $p = 1, \dots, n-1$, when $E \in \mathcal{K}_0^n$ is a regular and strictly convex body, see Corollary 3.2.

Finally, Section 4 is devoted to study the so called tangential bodies; for a definition see also Section 4. First we determine the tangential bodies lying in each class. Then we get a new necessary condition for a convex body K to lie in $\mathcal{R}_p$, now in terms of the quermassintegrals of K and its form body.

Theorem 1.3. *A tangential body $K\in\mathcal{K}^n$ of E lies in the class $\mathcal{R}_p$ if and only if K is an $(n-p-1)$-tangential body of E.*

Theorem 1.4. *Let $K\in\mathcal{K}^n_0$ and write $\mathrm{r}=\mathrm{r}(K;E)$. If $K\in\mathcal{R}_p$, $0\le p\le n-1$, then*

(1.7)

$$\begin{aligned}&\mathrm{W}_p(K;E)-\mathrm{W}_p(K_{-\mathrm{r}};E)\le \mathrm{W}_{p+1}(K^*;E)^{\frac{-1}{n-p-1}}\\ &\quad\left(\mathrm{W}_{p+1}(K;E)^{\frac{n-p}{n-p-1}}-\left[\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}}-\mathrm{r}\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\right]^{n-p}\right).\end{aligned}$$

Equality holds if and only if K is homothetic to an $(n-p-1)$-tangential body of E.

We remark that this result was already proved for the class $\mathcal{R}_0$ in [14, Theorem 19].

2. Preliminary results

In this section we state some preliminary lemmas which will be needed in the proof of Theorem 1.2. First we need some additional definitions and notation. As usual in the literature, we denote by $h(K,u)=\sup\bigl\{\langle x,u\rangle : x\in K\bigr\}$, $u\in\mathbb{R}^n$, the *support function* of $K\in\mathcal{K}^n$ (see e.g. [16, s. 1.7]). For convex bodies $K_1,\dots,K_m\in\mathcal{K}^n$ and real numbers $\rho_1,\dots,\rho_m\ge 0$, the volume of the linear combination $\rho_1K_1+\cdots+\rho_mK_m$ is expressed as a polynomial of degree n in the variables $\rho_1,\dots,\rho_m$,

$$\mathrm{V}\bigl(\rho_1K_1+\cdots+\rho_mK_m\bigr)=\sum_{i_1=1}^m\cdots\sum_{i_n=1}^m\mathrm{V}(K_{i_1},\dots,K_{i_n})\rho_{i_1}\cdot\ldots\cdot\rho_{i_n}.$$

The coefficients $\mathrm{V}(K_{i_1},\dots,K_{i_n})$ are the *mixed volumes* of $K_1,\dots,K_m$. This formula (the mixed volumes) extends the relative Steiner formula (1.3) (the relative quermassintegrals). Moreover, if $K_1,\dots,K_{n-1}\in\mathcal{K}^n$, the *mixed surface area measure* $\mathrm{S}(K_1,\dots,K_{n-1};\cdot)$ is the unique finite Borel measure on $\mathbb{S}^{n-1}$ such that

$$\mathrm{V}(K,K_1,\dots,K_{n-1})=\frac{1}{n}\int_{\mathbb{S}^{n-1}}h(K,u)\,d\mathrm{S}(K_1,\dots,K_{n-1};u). \tag{2.1}$$

For the sake of brevity we will use the abbreviation $\bigl(K_1[r_1],\dots,K_m[r_m]\bigr)\equiv\bigl(K_1,\overset{(r_1)}{\ldots},K_1,\dots,K_m,\overset{(r_m)}{\ldots},K_m\bigr)$. For a deep study of mixed volumes and mixed surface area measures we refer to [16, s. 5.1].

We write $N(K,x)$ to denote the *normal cone* of K at $x\in\operatorname{bd}K$, i.e., the set of all outer normal vectors of K at x (with the zero vector). A vector $u\in\mathbb{S}^{n-1}$ is an *r-extreme normal vector* of K if we cannot write $u=u_1+\cdots+u_{r+2}$, with u_i linearly independent normal vectors at one and the same boundary point of K. We denote the set of r-extreme normal vectors of K by $\mathcal{U}_r(K)$. Clearly each r-extreme normal vector is also an s-extreme one for $r<s\le n-1$. This notion admits a generalization that will

be used later, namely, the one of $(K_1,\dots,K_{n-1})$-*extreme* normal vector, for $K_1,\dots,K_{n-1}\in\mathcal{K}^n$. Since this definition is a bit more involved we omit it here. See [16, pp. 74–77] for precise definitions and properties. We just will need the following property: for $K\in\mathcal{K}^n$ and E regular, u is an r-extreme normal vector of K if and only if u is $\bigl(K[n-1-r],E[r]\bigr)$-extreme.

Finally the (relative) *form body* of a convex body $K\in\mathcal{K}_0^n$ with respect to E, denoted by K^*, is defined as

$$K^*=\bigcap_{u\in\mathcal{U}_0(K)}\bigl\{x:\langle x,u\rangle\le h(E,u)\bigr\}.$$

We start by proving a relation between the 0-extreme normal vectors of a convex body K and its form body K^* with respect to E. In [13, Lemma 4.6, p. 75] it is shown that

$$\mathcal{U}_0(K^*)\subset\operatorname{cl}\mathcal{U}_0(K) \tag{2.2}$$

for any convex body $E\in\mathcal{K}_0^n$. Here we prove that equality holds under certain restrictions.

Lemma 2.1. *Let $E\in\mathcal{K}_0^n$ be a regular convex body. Then for any $K\in\mathcal{K}^n$ it holds*

$$\mathcal{U}_0(K^*)=\operatorname{cl}\mathcal{U}_0(K). \tag{2.3}$$

Proof. First we prove that $\mathcal{U}_0(K)\subset\mathcal{U}_0(K^*)$. We will use the following characterization of 0-extreme normal vectors (see [13, Lemma 2.3, p. 20]): $u\in\mathcal{U}_0(K)$ if and only if for

$$u_1,u_2\in\mathbb{S}^{n-1}\ \text{ and }\ \alpha,\beta>0\ \text{ such that }\ u=\alpha u_1+\beta u_2 \tag{2.4}$$

it holds $h(K,u)<\alpha h(K,u_1)+\beta h(K,u_2)$.

Thus, let $u\in\mathcal{U}_0(K)$ and let $u_1,u_2\in\mathbb{S}^{n-1}$ and $\alpha,\beta>0$ as in (2.4). Since $u\in\mathcal{U}_0(K)$, by the definition of form body (with respect to E) it holds that $h(E,u)=h(K^*,u)$. On the other hand, since E is regular, $\mathcal{U}_0(E)=\mathbb{S}^{n-1}$, and then u is also a 0-extreme normal vector of E. Hence

$$h(K^*,u)=h(E,u)<\alpha h(E,u_1)+\beta h(E,u_2)\le\alpha h(K^*,u_1)+\beta h(K^*,u_2),$$

where the last inequality follows from $E\subset K^*$. Using the above characterization we get $u\in\mathcal{U}_0(K^*)$.

Now we prove (2.3). By (2.2) we just have to see that $\mathcal{U}_0(K^*)\supset\operatorname{cl}\mathcal{U}_0(K)$. Thus, let $u\in\operatorname{cl}\mathcal{U}_0(K)$ and suppose that $u\notin\mathcal{U}_0(K^*)$.

We take a sequence $(u_k)_{k\in\mathbb{N}}\subset\mathcal{U}_0(K)$ with $u_k\to u$ for $k\to\infty$. Since we know that $\mathcal{U}_0(K)\subset\mathcal{U}_0(K^*)$ then $u_k\in\mathcal{U}_0(K^*)$ for all $k\in\mathbb{N}$ and hence, by definition of form body, we get $h(E,u_k)=h(K^*,u_k)$ for all $k\in\mathbb{N}$. Therefore $h(E,u)=h(K^*,u)$ by the continuity of the support function. It assures that there exists $x\in\operatorname{bd}K^*\cap\operatorname{bd}E$ such that $u\in N(K^*,x)\cap N(E,x)$.

Since we suppose that $u\notin\mathcal{U}_0(K^*)$ then by definition of 0-extreme normal vector, u can be written as $u=u_1+u_2$ with $u_1,u_2\neq u$ linearly independent normal vectors at the same boundary point $x\in\operatorname{bd}K^*$, which implies that

$\dim N(K^*,x)\geq 2$. Notice however that $\dim N(E,x)=1$ since $u\in\mathcal{U}_0(E)$ (by the regularity of E, $\mathcal{U}_0(E)=\mathbb{S}^{n-1}$ and then $u\in\mathcal{U}_0(E)$). On the other hand it is clear that $\dim N(E,x)\geq\dim N(K^*,x)$ because $E\subset K^*$. Hence we get $\dim N(E,x)\geq 2$, a contradiction. It shows that $u\in\mathcal{U}_0(K^*)$. □

Remark 2.1. *Using an analogous argument as in the proof of Lemma 2.1 it is shown that any $u\in\mathbb{S}^{n-1}$ such that $h(K^*,u)=h(E,u)$ is a 0-extreme normal vector of K^*.*

The following lemma states the equivalence between all conditions in Theorem 1.2 for any convex body $K\in\mathcal{K}_0^n$.

Lemma 2.2. *Let $K\in\mathcal{K}_0^n$ and let $E\in\mathcal{K}_0^n$ be a regular and strictly convex body. For $0\leq p\leq n-2$, the following conditions are equivalent:*

- i.a) $h(K_\rho^*,u)=h(E,u)$ *for all* $u\in\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)$ *and* $i=0,\dots,p$.
- i.b) $h(K_\rho^*,u)=h(E,u)$ *for all* $u\in\operatorname{cl}\mathcal{U}_p(K_\rho)$.
- ii) *If* $p\neq 0$, *then* $\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)=\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr)$ *for* $i=1,\dots,p$.
- iii) $\operatorname{supp}\mathrm{S}\bigl(K_\rho^*[n-1];\cdot\bigr)\cup\Bigl(\bigcup_{i=0}^p\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)\Bigr)\subset\operatorname{cl}\mathcal{U}_0(K_\rho)$.
- iv) $\operatorname{cl}\mathcal{U}_0(K_\rho)=\operatorname{cl}\mathcal{U}_1(K_\rho)=\cdots=\operatorname{cl}\mathcal{U}_p(K_\rho)$.

Proof. Property (i.b) is just a reformulation of (i.a). In fact, since E is regular and strictly convex, $\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)$ is the closure of the set of $\bigl(K_\rho[n-i-1],E[i]\bigr)$-extreme normal vectors (see [15, pp. 135–136]), which is the set $\operatorname{cl}\mathcal{U}_i(K_\rho)$, i.e.,

$$\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)=\operatorname{cl}\mathcal{U}_i(K_\rho),\tag{2.5}$$

$i=0,\dots,p$. Since $\mathcal{U}_i(K_\rho)\subset\mathcal{U}_p(K_\rho)$ for all $i=0,\dots,p-1$ we get the equivalence between properties (i.a) and (i.b).

Now we prove that (i.a) is equivalent to (ii). Since $K,E\in\mathcal{K}_0^n$, then the mixed volumes $\mathrm{V}\bigl(K_\rho^*[2],K_\rho[n-i-1],E[i-1]\bigr),\mathrm{V}\bigl(K_\rho[n-i-1],E[i+1]\bigr)>0$ for $i=1,\dots,p$. Under this assumption, since E is a regular and strictly convex body and $i\geq 1$, results by Schneider [15, pp. 134–135] show that (i.a) implies that $\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr)=\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)$. This proves property (ii).

Conversely, we now assume that for $i=1,\dots,p$, $\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)=\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr)$. Then using the formula for the mixed volumes given in (2.1) we get

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}h(K_\rho^*,u)\,d\mathrm{S}\bigl(K_\rho[n-i-1],E[i];u\bigr)&=n\mathrm{V}\bigl(K_\rho^*,K_\rho[n-i-1],E[i]\bigr)\\&=\int_{\mathbb{S}^{n-1}}h(E,u)\,d\mathrm{S}\bigl(K_\rho^*K_\rho[n-i-1],E[i-1];u\bigr)\\&=\int_{\mathbb{S}^{n-1}}h(E,u)\,d\mathrm{S}\bigl(K_\rho[n-i-1],E[i];u\bigr)\end{aligned}$$

and therefore

$$\int_{\mathbb{S}^{n-1}} \big[h(K_\rho^*,u)-h(E,u)\big]\,d\mathrm{S}\big(K_\rho[n-i-1],E[i];u\big)=0. \tag{2.6}$$

Since $E\subset K_\rho^*$ and hence $h(K_\rho^*,u)\geq h(E,u)$, we get that (2.6) is equivalent to $h(K_\rho^*,u)=h(E,u)$ for all $u\in\operatorname{supp}\mathrm{S}\big(K_\rho[n-i-1],E[i];\cdot\big)$ and $i=0,\dots,p$. This proves (i.a).

Now we prove that (i.a) implies (iii). In [13, p. 48, Lemma 3.5] it is shown that for each $0\leq p\leq n-1$ the derivative of W_p exists almost everywhere in $-\mathrm{r}(K;E)<\rho\leq 0$ and

$$\mathrm{W}_p'(\rho)\geq (n-p)\mathrm{V}\big(K_\rho[n-p-1],K_\rho^*,E[p]\big). \tag{2.7}$$

In particular, when $p=0$ we get

$$\mathrm{V}'(\rho)\geq n\mathrm{V}\big(K_\rho[n-1],K_\rho^*\big)\geq n\mathrm{V}\big(K_\rho[n-1],E\big), \tag{2.8}$$

where the second inequality follows from the monotonicity of the mixed volumes (cf. e.g. [6, p. 97]) and $E\subset K_\rho^*$. Since the volume is differentiable and $\mathrm{V}'(\rho)=n\mathrm{W}_1(\rho)=n\mathrm{V}\big(K_\rho[n-1],E\big)$, we have equalities in (2.8), and hence we can assure that any convex body $K\in\mathcal{K}_0^n$ satisfies

$$\mathrm{V}\big(K[n-1],E\big)=\mathrm{V}\big(K[n-1],K^*\big).$$

In particular, the above relation applied to the form body K_ρ^* assures that $\mathrm{V}\big(K_\rho^*[n-1],E\big)=\mathrm{V}(K_\rho^*)$. Using again the formula for the mixed volumes given by (2.1) we get that for any $K\in\mathcal{K}_0^n$ it holds that $h(K_\rho^*,u)=h(E,u)$ for all $u\in\operatorname{supp}\mathrm{S}\big(K_\rho^*[n-1];\cdot\big)$. This condition joined to (i.a) gives

$$\begin{aligned} &h(K_\rho^*,u)=h(E,u)\quad\text{ for all}\\ &\qquad u\in\operatorname{supp}\mathrm{S}\big(K_\rho^*[n-1];\cdot\big)\cup\left(\bigcup_{i=0}^{p}\operatorname{supp}\mathrm{S}\big(K_\rho[n-i-1],E[i];\cdot\big)\right).\end{aligned} \tag{2.9}$$

On the other hand it is clear that since E is regular then $h(K_\rho^*,u)=h(E,u)$ if and only if $u\in\mathcal{U}_0(K_\rho^*)$. Moreover by using Lemma 2.1 we know that $\mathcal{U}_0(K_\rho^*)=\operatorname{cl}\mathcal{U}_0(K_\rho)$ and hence we have $h(K_\rho^*,u)=h(E,u)$ if and only if $u\in\operatorname{cl}\mathcal{U}_0(K_\rho)$. From here using (2.9) we get the required property (iii).

In order to prove (iii) implies (iv), notice that since E is regular and strictly convex then $\operatorname{supp}\mathrm{S}\big(K_\rho[n-i-1],E[i];\cdot\big)=\operatorname{cl}\mathcal{U}_i(K_\rho)$, for $i=0,\dots,p$ (cf. (2.5)). Hence we get from (iii) that in particular $\operatorname{cl}\mathcal{U}_p(K_\rho)\subset\operatorname{cl}\mathcal{U}_0(K_\rho)$. Since it always holds that $\mathcal{U}_0(K_\rho)\subset\cdots\subset\mathcal{U}_p(K_\rho)$, we obtain (iv).

It remains to show that (iv) implies (i.a). Using again the identity $\operatorname{supp}\mathrm{S}\big(K_\rho[n-i-1],E[i];\cdot\big)=\operatorname{cl}\mathcal{U}_i(K_\rho)$ for $i=0,\dots,p$ (cf. (2.5)), then we get from (iv) that for all $i=1,\dots p$

$$\operatorname{supp}\mathrm{S}\big(K_\rho[n-i-1],E[i];\cdot\big)=\operatorname{supp}\mathrm{S}\big(K_\rho[n-1];\cdot\big).$$

On the other hand, since E is regular we know that $h(K_\rho^*,u)=h(E,u)$ if and only if $u\in\mathcal{U}_0(K_\rho^*)=\operatorname{cl}\mathcal{U}_0(K_\rho)$ (cf. (2.3)). Thus if for $i\in\{0,\dots,p\}$

$u \in \operatorname{supp} \mathrm{S}\bigl(K_\rho[n-i-1], E[i]; \cdot\bigr)$, then $u \in \operatorname{supp} \mathrm{S}\bigl(K_\rho[n-1]; \cdot\bigr) = \operatorname{cl} \mathcal{U}_0(K_\rho)$ (cf. (2.5)), which implies that $h(K_\rho^*, u) = h(E, u)$ and shows (i.a). □

3. Convex bodies lying in the classes $\mathcal{R}_p$

We start characterizing the smallest class, $\mathcal{R}_{n-1}$.

Proof of Theorem 1.1. For the sake of brevity we write $\mathrm{r} = \mathrm{r}(K; E)$. If K is a k-dimensional convex body, $k \leq n-1$, then $\mathrm{r} = 0$ and the full system of parallel bodies is reduced to the family of outer parallel sets. Hence the equalities in (1.6) trivially hold for all $i = 0, \dots, n-1$ and we have that $K \in \mathcal{R}_{n-1}$. Thus we suppose that $K \in \mathcal{K}_0^n$, which implies that $\mathrm{r} > 0$.

If $K = L + \rho_0 E$ with $L \in \mathcal{K}^n$, $\dim L \leq n-1$, and $\rho_0 > 0$, then clearly $\mathrm{r} = \rho_0$ and the inner parallel body $K_\rho = L + \bigl(\rho_0 - |\rho|\bigr)E$ for $-\rho_0 \leq \rho \leq 0$. Moreover $K_{-\mathrm{r}} = L$. Then it is not difficult to see that

$$\mathrm{W}_i(\rho) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E)(-|\rho|)^k = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K; E)\rho^k$$

for all $i = 0, \dots n$ and $-\rho_0 \leq \rho \leq 0$, and clearly all the quermassintegrals are differentiable and $\mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho)$, for $i = 0, \dots, n-1$. Hence $K \in \mathcal{R}_{n-1}$.

Conversely, if $K \in \mathcal{R}_{n-1}$ we have in particular that W_{n-1} is differentiable and $\mathrm{W}_{n-1}'(\rho) = \mathrm{W}_n(K; E) = \mathrm{V}(E)$, for all $\rho \in [-\mathrm{r}, 0]$. Then integration with respect to ρ yields

$$\mathrm{W}_{n-1}(\rho) - \mathrm{W}_{n-1}(-\mathrm{r}) = \int_{-\mathrm{r}}^{\rho} \mathrm{W}_{n-1}'(s)\, ds = \int_{-\mathrm{r}}^{\rho} \mathrm{V}(E)\, ds = \mathrm{V}(E)(\rho + \mathrm{r}),$$

for all $\rho \in [-\mathrm{r}, 0]$. In particular, when $\rho = 0$ we get $\mathrm{W}_{n-1}(0) - \mathrm{W}_{n-1}(-\mathrm{r}) = \mathrm{r}\mathrm{V}(E)$, i.e.,

$$\mathrm{W}_{n-1}(K; E) = \mathrm{W}_{n-1}(K_{-\mathrm{r}}; E) + \mathrm{r}\mathrm{W}_{n-1}(E; E) = \mathrm{W}_{n-1}(K_{-\mathrm{r}} + \mathrm{r}E; E), \tag{3.1}$$

where the last equality follows from the (Minkowski) linearity of $\mathrm{W}_{n-1}(K; E)$ in its first variable (see [16, p. 279]). Since $K_{-\mathrm{r}} + \mathrm{r}E \subset K$ we get from (3.1) that $K_{-\mathrm{r}} + \mathrm{r}E = K$, which proves the required statement. □

For the rest of this section $E \in \mathcal{K}_0^n$ will be a regular and strictly convex body. We prove Theorem 1.2 by showing that $K \in \mathcal{R}_p$ implies property (i.a). From Lemma 2.2 we get the remaining statements.

Proof of Theorem 1.2. First notice that by hypothesis $K \notin \mathcal{R}_{n-1}$. Since all k-dimensional convex bodies, $k \leq n-1$, are contained in $\mathcal{R}_{n-1}$, we have $K \in \mathcal{K}_0^n$. Hence we can apply Lemma 2.2 and prove just property (i.a).

Since $E \subset K_\rho^*$, the monotonicity of the mixed volumes implies that

$$\mathrm{V}\bigl(K_\rho[n-p-1], K_\rho^*, E[p]\bigr) \geq \mathrm{W}_{p+1}(\rho) \tag{3.2}$$

and hence we get from (2.7) that

$$\mathrm{W}_p'(\rho) \geq (n-p)\mathrm{W}_{p+1}(\rho).$$

Thus, if $K\in\mathcal{R}_p$ then we have equality in the previous inequality and also in (3.2). Moreover since $K\in\mathcal{R}_p\subset\cdots\subset\mathcal{R}_0$ we get $\mathrm{V}\bigl(K_\rho[n-i-1],K_\rho^*,E[i]\bigr)=\mathrm{W}_{i+1}(\rho)$ for all $i=0,\dots,p$. Using the formula for the mixed volumes given by (2.1) we can write

$$\begin{aligned}0&=n\mathrm{V}\bigl(K_\rho[n-i-1],K_\rho^*,E[i]\bigr)-n\mathrm{V}\bigl(K_\rho[n-i-1],E[i+1]\bigr)\\&=\int_{\mathbb{S}^{n-1}}\bigl[h(K_\rho^*,u)-h(E,u)\bigr]\,d\mathrm{S}\bigl(K_\rho[n-i-1],E[i];u\bigr),\end{aligned}$$

which is equivalent to $h(K_\rho^*,u)=h(E,u)$ for all $u\in\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)$, for $i=0,\dots,p$. This proves (i.a) and the theorem. $\square$

Remark 3.1. *If $K\in\mathcal{R}_p\backslash\mathcal{R}_{n-1}$, $1\le p\le n-2$, then property (ii) of Theorem 1.2 assures that the surface area measures $\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)$ and $\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr)$ coincide. Hence we can rewrite* (i.a) *as*

$$h(K_\rho^*,u)=h(E,u)\qquad\text{for }\ u\in\operatorname{supp}\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr)$$

for $i=1,\dots,p$. We also can write that

$$\begin{aligned}\mathrm{cl}\,\mathcal{U}_i(K_\rho)&=\mathrm{cl}\Bigl\{\bigl(K_\rho[n-i-1],E[i]\bigr)\text{-extreme normal vectors}\Bigr\}\\&=\operatorname{supp}\mathrm{S}\bigl(K_\rho[n-i-1],E[i];\cdot\bigr)=\operatorname{supp}\mathrm{S}\bigl(K_\rho^*,K_\rho[n-i-1],E[i-1];\cdot\bigr).\end{aligned}$$

Using (2.1) the following corollary is an immediate consequence of (ii) in Theorem 1.2.

Corollary 3.1. *If $K\in\mathcal{R}_p\backslash\mathcal{R}_{n-1}$, $1\le p\le n-2$, then for any convex body $L\in\mathcal{K}^n$ and for all $\rho\in\bigl(-\mathrm{r}(K;E),0\bigr]$ it holds*

$$\mathrm{V}\bigl(L,K_\rho^*,K_\rho[n-i-1],E[i-1]\bigr)=\mathrm{V}\bigl(L,K_\rho[n-i-1],E[i]\bigr),\quad i=1,\dots,p.$$

Replacing L by K_ρ, K_ρ^* and E in the previous expression we get that the relations

$$\mathrm{W}_{i+1}(\rho)=\mathrm{V}\bigl(K_\rho^*,K_\rho[n-i-1],E[i]\bigr)=\mathrm{V}\bigl(K_\rho^*[2],K_\rho[n-i-1],E[i-1]\bigr)\tag{3.3}$$

hold for all $i=1,\dots,p$. In particular, we have equality in the Aleksandrov-Fenchel inequality for the convex bodies K_ρ^* and E:

$$\begin{aligned}&\mathrm{V}\bigl(K_\rho^*,K_\rho[n-i-1],E[i]\bigr)^2\\&\qquad=\mathrm{V}\bigl(K_\rho^*[2],K_\rho[n-i-1],E[i-1]\bigr)\mathrm{V}\bigl(K_\rho[n-i-1],E[i+1]\bigr).\end{aligned}$$

Using Theorem 1.2 the family of polytopes can be excluded from all the classes $\mathcal{R}_p$, $p=1,\dots,n-1$.

Corollary 3.2. *There are no polytopes in $\mathcal{R}_p$, for all $1\le p\le n-1$.*

Proof. Since $\mathcal{R}_{n-1}\subset\cdots\subset\mathcal{R}_1$ it is enough to show the assertion for the class $\mathcal{R}_1$. Let $P\in\mathcal{K}^n$ be a convex polytope lying in the class $\mathcal{R}_1$ and let $\rho\in\bigl(-\mathrm{r}(P;E),0\bigr]$. Theorem 1.2, item (i.b), assures that

$$h(P_\rho^*,u)=h(E,u)\qquad\text{for all }\ u\in\mathrm{cl}\,\mathcal{U}_1(P_\rho).\tag{3.4}$$

On the other hand it is clear that P_ρ is also a polytope, and moreover P_ρ^* is a polytope all whose $(n-1)$-faces touch E. Then $h(P_\rho^*,u)=h(E,u)$ if and only if u is a 0-extreme normal vector of P_ρ^*. Hence from (3.4) we can assure that $\mathcal{U}_1(P_\rho)\subset\mathcal{U}_0(P_\rho^*)$.

Let $u\in\mathcal{U}_1(P_\rho)\backslash\mathcal{U}_0(P_\rho)$. Notice that such a vector u exists since P_ρ is a polytope. By definition of 0-extreme normal vector u can be written as $u=u_1+u_2$ with $u_1,u_2\neq u$ linearly independent normal vectors at the same boundary point of P. Then the 2-dimensional cone determined by u_1 and u_2 contains u in its relative interior and provides a 1-dimensional neighborhood $\mathcal{V}\subset\mathbb{S}^{n-1}$ of u. Moreover $\mathcal{V}\subset\mathcal{U}_1(P_\rho)\subset\mathcal{U}_0(P_\rho^*)$. This leads to a contradiction, since we have shown that there exists a relative 1-dimensional open set $\mathcal{V}\subset\mathbb{S}^{n-1}$ of 0-extreme normal vectors of the polytope P_ρ^*. □

Remark 3.2. *Notice that if we remove the hypothesis of regularity and strict convexity for E then Corollary 3.2 is not true, since trivially there are polytopes in the classes $\mathcal{R}_p$; in fact, just taking E a polytope then $E\in\mathcal{R}_{n-1}$.*

4. Tangential bodies in $\mathcal{R}_p$

A convex body $K\in\mathcal{K}^n$ containing the convex body $E\in\mathcal{K}_0^n$ is called a *p-tangential body* of E, $p\in\{0,\dots,n-1\}$, if each $(n-p-1)$-extreme support plane of K supports E, $p=0,\dots,n-1$ [16, pp. 75–76]. Here a supporting hyperplane is said to be p-extreme if its outer normal vector is a p-extreme direction. For further characterizations and properties of p-tangential bodies we refer to [16, Section 2.2].

So a 0-tangential body of E is just the body E itself and each p-tangential body of E is also a q-tangential body for $p<q\leq n-1$. A 1-tangential body is usually called *cap-body*, and it can be seen as the convex hull of E and countably many points such that the line segment joining any pair of those points intersects E. An $(n-1)$-tangential body will be briefly called a *tangential body*.

The following theorem shows the close relation existing between the inner parallel bodies and the tangential bodies.

Theorem 4.1 (Schneider [16, pp. 136–137])**.** *Let $K\in\mathcal{K}_0^n$ and $-\mathrm{r}(K;E)<\rho<0$. Then K_ρ is homothetic to K if, and only if, K is homothetic to a tangential body of E.*

We will make also use of the following result, which gives a characterization of n-dimensional p-tangential bodies in terms of the quermassintegrals.

Theorem 4.2 (Favard [5], [16, p. 367])**.** *Let $K,E\in\mathcal{K}_0^n$, $E\subset K$, and let $p\in\{0,\dots,n-1\}$. Then $\mathrm{W}_0(K;E)=\mathrm{W}_1(K;E)=\cdots=\mathrm{W}_{n-p}(K;E)$ if and only if K is a p-tangential body of E.*

Now we prove Theorem 1.3.

Proof of Theorem 1.3. Since K is a tangential body of E we have $\mathrm{r}(K;E)=1$ and we know from the proof of Theorem 4.1 that $K_\rho=\big(1-|\rho|\big)K$. Hence

$\mathrm{W}_i(\rho) = \bigl(1-|\rho|\bigr)^{n-i}\mathrm{W}_i(K;E) = (1+\rho)^{n-i}\mathrm{W}_i(K;E)$ for all $i = 0,\dots,n$ and then

$$\mathrm{W}_i'(\rho) = (n-i)(1+\rho)^{n-i-1}\mathrm{W}_i(K;E). \tag{4.1}$$

We suppose first that $K \in \mathcal{R}_p$. Then $\mathrm{W}_i'(\rho) = (n-i)\mathrm{W}_{i+1}(\rho)$, for $i = 0,\dots,p$ and thus

$$\begin{aligned}\mathrm{W}_i'(\rho) &= (n-i)\mathrm{W}_{i+1}(K_\rho;E) = (n-i)\mathrm{W}_{i+1}\bigl((1+\rho)K;E\bigr)\\ &= (n-i)(1+\rho)^{n-i-1}\mathrm{W}_{i+1}(K;E)\end{aligned}$$

for $i = 0,\dots,p$. The last two expressions for the derivative $\mathrm{W}_i'(\rho)$ give $\mathrm{W}_i(K;E) = \mathrm{W}_{i+1}(K;E)$ for $i = 0,\dots,p$. This proves (see Theorem 4.2) that K is a $(n-p-1)$-tangential body of E.

Conversely, if K is an $(n-p-1)$-tangential body of E we have $\mathrm{W}_i(K;E) = \mathrm{W}_{i+1}(K;E)$ for $i = 0,\dots,p$. Then we get from (4.1) that

$$\mathrm{W}_i'(\rho) = (n-i)(1+\rho)^{n-i-1}\mathrm{W}_{i+1}(K;E) = (n-i)\mathrm{W}_{i+1}(\rho)$$

for $i = 0,\dots,p$, which shows that $K \in \mathcal{R}_p$. □

We finish the section with proving inequality (1.7).

Proof of Theorem 1.4. Let $\rho \in [-\mathrm{r},0]$ with $\mathrm{r} = \mathrm{r}(K;E)$. It is known [13, Lemma 2.10, p. 36] that $K_\rho + |\rho|K^* \subset K$. Then for all $0 \le i \le n$

$$\begin{aligned}\mathrm{W}_i(K;E)^{\frac{1}{n-i}} \ge \mathrm{W}_i\bigl(K_\rho + |\rho|K^*;E\bigr)^{\frac{1}{n-i}} &\ge \mathrm{W}_i(K_\rho;E)^{\frac{1}{n-i}} + |\rho|\mathrm{W}_i(K^*;E)^{\frac{1}{n-i}}\\ &= \mathrm{W}_i(\rho)^{\frac{1}{n-i}} - \rho\mathrm{W}_i(K^*;E)^{\frac{1}{n-i}},\end{aligned}$$

where the last inequality comes from Brunn-Minkowski's inequality for relative quermassintegrals. Since $K \in \mathcal{R}_p$ we have $\mathrm{W}_p'(\rho) = (n-p)\mathrm{W}_{p+1}(\rho)$, and taking $i = p+1$ in the previous inequality we can integrate from $-\mathrm{r}$ to 0 with respect to ρ:

$$\begin{aligned}\frac{1}{n-p}\bigl[\mathrm{W}_p(0) - \mathrm{W}_p(-\mathrm{r})\bigr] &= \int_{-\mathrm{r}}^0 \mathrm{W}_{p+1}(\rho)\,d\rho\\ &\le \int_{-\mathrm{r}}^0 \Bigl(\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}} + \rho\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\Bigr)^{n-p-1}\\ &= \frac{1}{n-p}\left.\frac{\Bigl(\mathrm{W}_{p+1}(K;E)^{\frac{1}{n-p-1}} + \rho\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}\Bigr)^{n-p}}{\mathrm{W}_{p+1}(K^*;E)^{\frac{1}{n-p-1}}}\right|_{-\mathrm{r}}^{0},\end{aligned}$$

from which we get directly (1.7). Equality holds in this inequality if and only if both $K = K_\rho + |\rho|K^*$ and equality holds in Brunn-Minkowski's inequality for every $0 \le i \le p+1$ and $-\mathrm{r} \le \rho \le 0$.

We suppose first that equality holds in (1.7). From the Brunn-Minkowski equality case we know that K and K^* are homothetic, and this is the case if and only if K is homothetic to a tangential body of E (see [16, p. 321]). Since

$K \in \mathcal{R}_p$ Theorem 1.3 assures that then K is homothetic to an $(n-p-1)$-tangential body of E. It just remains to see that any $(n-p-1)$-tangential body K of E satisfies $K = K_\rho + |\rho| K^*$. In fact, if K is such a set then $K_\rho = \bigl(1-|\rho|/\mathrm{r}\bigr)K$ and clearly $K^* = (1/\mathrm{r})K$. Therefore $K = \bigl(1-|\rho|/\mathrm{r}\bigr)K + \bigl(|\rho|/\mathrm{r}\bigr)K = K_\rho + |\rho|K^*$.

Conversely, if K is homothetic to an $(n-p-1)$-tangential body of E, we know that $K = K_\rho + |\rho|K^*$ and that K, K^* are homothetic, which implies equality in Brunn-Minkowski's inequality. So we get equality in (1.7). □

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Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100-Murcia, Spain

E-mail address: mhcifre@um.es, esaorin@um.es